25.05.09

Basic properties of 1D Shrödinger Equation

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi$$

1. Discrete states are nongenerate

For discrete states $\psi \to 0$ at $\pm\infty$.

Assume, there $\exists$ ψ_1 & ψ_2, $\psi_1 \neq \psi_2$. Then:

$$\psi_1''/\psi_1 = \frac{2m}{\hbar^2}(U-E) = \psi_2''/\psi_2 \to \psi_2\psi_1'' = \psi_1\psi_2''$$

Integrate both sides by parts:

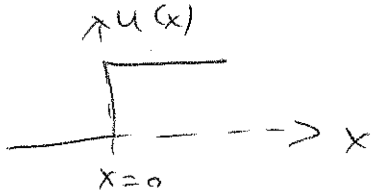
$$\psi_2\psi_1'\Big|_{-\infty}^{x} - \int_{-\infty}^{x}\psi_2'\psi_1'\,dx = \psi_1\psi_2'\Big|_{-\infty}^{x} - \int_{-\infty}^{x}\psi_1'\psi_2'\,dx$$

$\hookrightarrow$ $\psi_2\psi_1' = \psi_1\psi_2'$ at x $\quad(\psi_2(-\infty)\to 0,\ \psi_1(-\infty)\to 0)$

$$\psi_2'/\psi_2 = \psi_1'/\psi_1$$

Integrate again: $\ln\psi_2 = \ln\psi_1 + \ln C \to \psi_2 = C\psi_1$, so both states are basically the same.

2. ψ' is continuous at the jump of $U(x)$



$$-\psi'' = (E-U)\frac{2m}{\hbar^2}\psi, \quad \psi'' = (U-E)\frac{2m}{\hbar^2}\psi$$

Integrate around the jump at $x=0$:

$$\int_{-\delta}^{\delta}\psi''\,dx = \psi'(\delta) - \psi'(-\delta) = \frac{2m}{\hbar^2}\int_{-\delta}^{\delta}(U(x)-E)\psi(x)\,dx$$

The integral in the RHS $\to 0$ as $\delta \to 0$ $\Rightarrow$ $\boxed{\psi'(\delta) = \psi'(-\delta)}$

3. (General for any Shr. Eq.). Energies $\boxed{E > \min U(x)}$

Indeed, for discrete states $\psi(\pm\infty) = 0$,

$$E = \langle\psi| -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}|\psi\rangle + \langle\psi|U|\psi\rangle \geq \langle\psi| -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}|\psi\rangle + U_{min}$$

since $\langle\psi|U|\psi\rangle \geq \langle\psi|U_{min}|\psi\rangle = U_{min}\langle\psi|\psi\rangle = U_{min}$

Then, $\langle\psi| -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}|\psi\rangle = \int_{-\infty}^{\infty}\left(-\frac{\hbar^2}{2m}\right)\psi\psi''\,dx = -\frac{\hbar^2}{2m}\left[\underbrace{\psi\psi'\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty}(\psi')^2\,dx\right]$

$= +\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}(\psi')^2\,dx$ and is positive, i.e. $E \geq U_{min}$. Continuous spectrum