Risken's book

One random variable

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- **Probability density** ξ - random variable

$$P(\xi \le x) = \langle \theta(x-\xi) \rangle$$

$$W_\xi(x) = \frac{dP(\xi \le x)}{dx} = \frac{d}{dx}\langle \theta(x-\xi)\rangle = \langle \delta(x-\xi)\rangle$$

or

$$P(\xi \le x+dx) - P(\xi \le x) = \underbrace{W_\xi(x)}_{dP(\xi\le x)/dx}\, dx$$

$$\int_{x_{min}}^{x_{max}} W_\xi(x)\,dx = 1$$ from any of the definitions above.

For any $f(\xi)$:

$$f(\xi) = \int f(x)\,\delta(x-\xi)\,dx$$

$$\langle f(\xi)\rangle = \int f(x)\langle \delta(x-\xi)\rangle dx \equiv \int f(x)\, W_\xi(x)\,dx$$

Again, taking $f \equiv 1$ normalisation of $W_\xi(x)$ follows.

- **Transformation of variables**

$$\xi \to \eta = g(\xi)$$

Then, $W_\eta(y) = \langle \delta(y-\eta)\rangle = \langle \delta(y-g(\xi))\rangle$

$$= \int W_\xi(x)\,\delta(y-g(x))\,dx$$

If $y-g(x)=0 \to$ roots $x_n \equiv x_n(y)$, then

$$\delta(y-g(x)) = \sum_n \frac{1}{|-g'(x_n)|}\,\delta(x-x_n) = \sum_n \frac{\delta(x-x_n)}{|g'(x_n)|}$$

and

$$W_\eta(y) = \sum_n \frac{W_\xi(x_n)}{|g'(x_n)|}$$

- As an example, consider the Maxwell's distribution for the velocity

$$W_\xi(V) = \sqrt{\frac{m}{2\pi kT}}\exp\left(-\frac{mV^2}{2kT}\right)$$

and ~~let~~ us derive its analog for the energy $E = \frac{mV^2}{2}$.
Here E stands for η, V for ξ, $g(x) = \frac{1}{2}mx^2$. ~~Equation~~:

$E - \frac{1}{2}mV^2 = 0 \rightarrow$ roots $\quad V_{1,2} = \pm\sqrt{\frac{2E}{m}}$

and $g'(x_{1,2}) = m x_{1,2} = \pm\sqrt{2Em}$, so that

$$W(E) = \sqrt{\frac{m}{2\pi kT}} \sum_{n=1,2} \frac{W_\xi(x_n)}{|\pm\sqrt{2Em}|} = \frac{1}{\sqrt{2Em}} \cdot 2\sqrt{\frac{m}{2\pi kT}}\, e^{-E/kT}$$

$$= \frac{1}{\sqrt{2Em\pi}}\, e^{-E/kT}.$$

- <u>Characteristic function</u>

$$C_\xi(u) = \langle e^{iu\xi}\rangle = \int_{x_{min}}^{x_{max}} W_\xi(x)\, e^{iux}\, dx \qquad (2.19)$$

It is a FT if $x_{min} = -\infty$, $x_{max} = +\infty$.
Moments of the distribution:

$$M_n = \langle \xi^n \rangle = \int x^n W_\xi(x)\, dx$$

$$\left(\frac{\partial}{\partial u}\right)^n C_\xi(u) = \langle (i\xi)^n e^{iu\xi}\rangle = i^n \int x^n e^{iux}\, W_\xi(x)\, dx$$

$$\left(\frac{\partial}{\partial u}\right)^n C_\xi(u)\Big|_{u=0} = i^n \int x^n W_\xi(x)\, dx \equiv i^n M_n,$$

$$\hookrightarrow \quad M_n = \frac{1}{i^n}\left(\frac{\partial}{\partial u}\right)^n C_\xi(u)\Big|_{u=0} \qquad (2.20)$$

Taylor expansion of $C_\xi(u)$:

$$C_\xi(u) = \sum_{n=0}^{\infty} \frac{u^n}{n!}\left[\left(\frac{\partial}{\partial u}\right)^n C_\xi(u)\right]_{u=0} \equiv \sum_{n=0}^{\infty}\frac{(iu)^n}{n!} M_n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} M_n \qquad (2.21)$$

- <u>Cumulants</u> K_n

$$C_\xi(u) = 1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} M_n \overset{\text{Def}}{\equiv} \exp\left[\sum_{n=1}^{\infty} \frac{(iu)^n}{n!} K_n\right] \qquad (2.24)$$

or

$$\ln C_\xi(u) = \ln\left[1 + \sum_{n=1}^{\infty}\frac{(iu)^n}{n!} M_n\right] \equiv \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} K_n \qquad (2.25)$$

Expanding on both sides allows obtaining relationships between M_n and K_n. From (2.24):

$$\cancel{u^1:\ \frac{iu}{1}M_1} = 1 + \sum_{n=1}^{\infty}\frac{(iu)^n}{n!}M_n = 1 + \sum_{n=1}\frac{(iu)^n}{n!}K_n + \frac{1}{2}\left(\sum_{n=1}\frac{(iu)^n}{n!}K_n\right)^2$$
$$+ \frac{1}{6}\left(\sum_{n=1}\frac{(iu)^n}{n!}K_n\right)^3 + \dots$$

from which:

$$u^1:\quad \frac{iu}{1}M_1 = \frac{iu}{1}K_1 \to \quad M_1 = K_1 .$$

$$u^2:\quad \left(\frac{(iu)^2}{2}\right)M_2 = \frac{(iu)^2}{2}K_2 + \frac{1}{2}\left(\frac{iu}{1}\right)^2 K_1^2 \to M_2 = K_2 + K_1^2$$

$$u^3:\quad \frac{(iu)^3}{6}M_3 = \frac{(iu)^3}{6}K_3 + \frac{1}{2}\frac{(iu)^3}{2\cdot 1}(K_1K_2 + K_2K_1) + \frac{1}{6}\frac{(iu)^3}{1}\cdot K_1^3$$

$$\hookrightarrow M_3 = K_3 + 3K_1K_2 + K_1^3 \quad , \text{etc.}$$

Reversely, from (2.25):

$$\sum_{n=1}\frac{(iu)^n}{n!}M_n - \frac{1}{2}\left(\sum_{n=1}\frac{(iu)^n}{n!}M_n\right)^2 + \frac{1}{3}\left(\sum_{n=1}\frac{(iu)^n}{n!}M_n\right)^3 + \dots = \sum_{n=1}\frac{(iu)^n}{n!}K_n ,$$

giving:

$$u^1:\quad \frac{(iu)}{1}M_1 = \frac{iu}{1}K_1 \to \quad M_1 = K_1$$

u^2: $\frac{(iu)^2}{2} M_2 - \frac{1}{2}\frac{(iu)^2}{1} M_1^2 = \frac{(iu)^2}{2} K_2 \rightarrow K_2 = M_2 - M_1^2$.

u^3: $\frac{(iu)^3}{6} M_3 - \frac{1}{2}\frac{(iu)^3}{2 \cdot 1} \cdot 2 M_1 M_2 + \frac{1}{3}\frac{(iu)^3}{1} M_1^3 = \frac{(iu)^3}{6} K_3$

$\hookrightarrow K_3 = M_3 - 3 M_1 M_2 + 2 M_1^3$, etc.

There is a general relationship between the two via determinants

- The 2nd cumulant

$$K_2 = M_2 - M_1^2 = \langle \xi^2 \rangle - \langle \xi \rangle^2 = \langle (\xi^2 - \langle \xi \rangle^2) \rangle$$

$$\equiv \langle (\xi - \langle \xi \rangle)^2 \rangle \geq 0 \qquad (2.10)$$

while $\cancel{K_1} = M_1 = \langle \xi \rangle$,

- All moments cannot be zero, but only a few $\neq 0$. But for cumulants this may work.

Consider several cases:

(a) $\underline{K_1 \neq 0, \text{ but } K_2 = K_3 = \dots = 0}$

Then $C_\xi(u) = e^{iuK_1}$, $W_\xi(u) = \delta(\cancel{x} - K_1)$ (2.11)

as, from (2.19),

$$\int W_\xi(x) e^{iux} dx = e^{iuK_1} \text{ if } W_\xi(x) \equiv \delta(x - K_1).$$

(b) $\underline{K_1 \neq 0, K_2 \neq 0, \text{ but } K_3 = K_4 = \dots = 0}$

In that case

$C_\xi(u) = \exp\left[iuK_1 - \frac{1}{2}u^2 K_2\right]$

To derive $W_\xi(x)$, we assume $-\infty < x < \infty$, then

$$C_\xi(u) = \int_{-\infty}^{\infty} W_\xi(x) e^{iux} dx \rightarrow W_\xi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_\xi(u) e^{-iux} du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu(x-K_1)} e^{-\frac{K_2}{2}u^2} du \equiv \frac{1}{\sqrt{2\pi K_2}} e^{-\frac{(x-K_1)^2}{2K_2}} \quad (2.32)$$

is the Gaussian distribution.

If higher order $K_n \neq 0$, the $W_\xi(x)$ may not always be positive.

Several variables

$\xi_1, \dots, \xi_r$ ← random variables

$$\langle f(\xi_1, \dots, \xi_r) \rangle = \int dx_1 \dots dx_r \, f(x_1, \dots, x_r) \, W_{\xi_1 \dots \xi_r}(x_1, \dots, x_n) \quad (2.35)$$

$$W_{\xi_1 \dots \xi_r}(x_1, \dots, x_r) \equiv W_r(x_1, \dots, x_n) = \langle \delta(x_1 - \xi_1) \dots \delta(x_r - \xi_r) \rangle \quad (2.34)$$

Reduced distribution:

$$W_i(x_1, \dots, x_i) = \int dx_{i+1} \dots dx_r \quad W_r(x_1, \dots, x_i, x_{i+1}, \dots, x_r) \quad (2.36)$$

Characteristic function:

$$C_r(u_1, \dots, u_r) = \langle e^{i(u_1\xi_1 + \dots + u_r\xi_r)} \rangle$$

$$= \int e^{i(u_1 x_1 + \dots + u_r x_r)} W_r(x_1, \dots, x_r) \, dx_1 \dots dx_r \quad (2.37)$$

and the moments

$$M_{n_1, \dots n_r} = \langle \xi_1^{n_1} \dots \xi_r^{n_r} \rangle \equiv \int x_1^{n_1} \dots x_r^{n_r} W_r(x_1, \dots, x_r) \, dx_1 \dots dx_r$$

$$\equiv \left(\frac{\partial}{\partial(iu_1)}\right)^{n_1} \dots \left(\frac{\partial}{\partial(iu_r)}\right)^{n_r} C_r(u_1, \dots, u_r) \Big|_{u_1 = \dots = u_r = 0} \quad (2.38)$$

$$\hookrightarrow C_r(u_1, \dots, u_r) = \sum_{n_1 \dots n_r = 0}^{\infty} M_{n_1 \dots n_r} \frac{(iu_1)^{n_1}}{n_1!} \dots \frac{(iu_r)^{n_r}}{n_r!} \quad (2.39)$$

If any $x \in (-\infty, \infty)$, the $W_r(x)$ is given via the Inverse FT of $C_r(u)$:

$$W_r(x_1,\dots,x_r) = \frac{1}{(2\pi)^r}\int du_1\dots du_r\, C_r(u_1,\dots,u_r)\, e^{-i(u_1x_1+\dots+u_rx_r)} \qquad (2.40)$$

Cumulants are introduced via

$$\ln C_r(u_1,\dots,u_r) \equiv \sum_{n_1\dots n_r=1}^{\infty} \frac{(iu_1)^{n_1}}{n_1!}\dots\frac{(iu_r)^{n_r}}{n_r!}\, K_{n_1\dots n_r} \qquad (2.42)$$

so that

$$K_{n_1\dots n_r} = \left(\frac{\partial}{\partial(iu_1)}\right)^{n_1}\dots\left(\frac{\partial}{\partial(iu_r)}\right)^{n_r} \ln C_r(u_1,\dots,u_r) \qquad (2.41)$$

Conditional probability density

If $\xi_2,\dots,\xi_r$ take fixed values, then ~~PEs~~ $W_r(\xi_1,\xi_{20},\xi_{30},\dots)$ is called conditional probability density and denoted

$$P(x_1|\, x_2,\dots,x_r)$$

Obviously,

$$W_r(x_1,\dots,x_r) = P(x_1|x_2,\dots,x_r)\,W_{r-1}(x_2,\dots,x_r) \qquad (2.43)$$

$$\hookrightarrow\; P(x_1|x_2,\dots,x_r) = \frac{W_r(x_1,\dots,x_r)}{W_{r-1}(x_2,\dots,x_r)} = \frac{W_r(x_1,\dots,x_r)}{\int W_r(x_1,\dots,x_r)\,dx_1} \qquad (2.44)$$

For two random variables:

$$P(x_1/x_2) = \frac{W_2(x_1,x_2)}{\int W_2(x_1,x_2)\,dx_1} \qquad (2.45)$$

Cross correlation

If $P(x_1|x_2)$ does not depend on x_2, ξ_1 and ξ_2 are uncorrelated:

$$W_2(x_1,x_2) = \underbrace{P(x_1|x_2)}_{P(x_1)\text{ only}}\int W_2(x_1,x_2)\,dx_1 = \underbrace{P(x_1)}_{W_1^{(1)}(x_1)}\cdot\underbrace{P(x_2)}_{W_1^{(2)}(x_2)},$$

is the product of two probabilities.

In another extreme, $\xi_1 = f(\xi_2)$, i.e. ξ_1 is completely determined by ξ_2, and hence

$$P(x_1|x_2) = \delta(x_1 - f(x_2)) \qquad (2.47)$$

$$W_2(x_1, x_2) = \delta(x_1 - f(x_2))\, W_1(x_2) \qquad (2.48)$$

To study intermediate cases of correlation of random variables the degree of correlation may be measured by

$$R = \frac{\langle \xi_1 \xi_2 \rangle - \langle \xi_1 \rangle \langle \xi_2 \rangle}{\sqrt{\langle \xi_1^2 \rangle - \langle \xi_1 \rangle^2}\,\sqrt{\langle \xi_2^2 \rangle - \langle \xi_2 \rangle^2}} \qquad (2.50)$$

If ξ_1 and ξ_2 are uncorrelated, $\langle \xi_1 \xi_2 \rangle = \langle \xi_1 \rangle \langle \xi_2 \rangle$ and $R = 0$.

In the general case of r random variables:

$$\varkappa(\xi_1, \dots, \xi_r) = K_{11\dots 1} = \frac{\partial^r}{\partial(iu_1)\,\partial(iu_2)\dots\partial(iu_r)} \ln C_r(u_1, \dots, u_r)$$

is introduced. It is $= 0$ only if all variables are independent. It also vanishes if only one is independent on the others, e.g.

$$W_r(x_1, \dots, x_r) = W_1^{(1)}(x_1)\, W_{r-1}(x_2, \dots, x_r)$$

The $C_r(u_1, \dots, u_r) = \int e^{i(u_1 x_1 + \dots + u_r x_r)}\, W_r(x_1, \dots, x_r)\, dx_1 \dots dx_r$

$$= \int e^{iu_1 x_1} W_1^{(1)}(x_1)\, dx_1 \int \dots\dots \, dx_2 \dots dx_r$$

$$= C_1^{(1)}(u_1)\, C_{r-1}(u_2, \dots, u_r)$$

$\hookrightarrow \ln C_r = \ln C_1^{(1)} + \ln C_{r-1}$

If $r \geq 2$, then $\partial^r / \partial iu_1 \partial iu_2 \dots$ is zero, since $\ln C_1^{(1)}$ does not depend on $u_2, \dots$, but $\ln C_{r-1}$ does not depend on u_1.

Gaussian distribution

All cumulants except those with $n_1+n_2+\dots+n_r \le 2$ vanish.

$$C_r(u_1,\dots,u_r) = \exp\left[\sum_{j=1}^{r} a_j\, iu_j + \frac{1}{2}\sum_{j,k=1}^{r} \underbrace{\sigma_{jk}}_{=\sigma_{kj}}\, iu_j\, iu_k\right] \quad (2.55)$$

$$\langle \xi_j \rangle = M_{00..1...0} = \frac{\partial}{\partial iu_j} C_r\Big|_{u=0} = a_j \quad (2.56)$$

$$\langle \xi_i \xi_k \rangle = M_{0...\underset{i}{1}...\underset{k}{1}...0} = \frac{\partial}{\partial iu_i}\frac{\partial}{\partial iu_k} C_r\Big|_{u=0} =$$

$$= \frac{\partial}{\partial iu_i}\left\{ C_r \left\{ a_k + \sum_{j=1}^{r} \sigma_{jk}\, iu_j \right\}\right\}\Big|_{u=0} = C_r\{\sigma_{ik}\}_{u=0} + C_r\left\{a_i + \sum_j \sigma_{ji}\, iu_j\right\}\times$$

$$\times\left\{ a_k + \sum_j \sigma_{jk}\, iu_j \right\}_{u=0} \equiv \sigma_{ik} + a_i a_k \quad (2.57)$$

$$\to \langle (\xi_j - \langle \xi_j\rangle)(\xi_k - \langle \xi_k \rangle)\rangle = \langle \xi_j \xi_k \rangle - \langle \xi_j \rangle \langle \xi_k \rangle$$

$$= (\sigma_{jk} + a_j a_k) - a_j a_k \equiv \sigma_{jk} \quad (2.58)$$

Probability density is the inverse FT:

$$W_r(x_1,\dots,x_r) = \frac{1}{(2\pi)^r}\int du_1\dots du_r\, C_r(u_1,\dots,u_r)\, e^{-i(u_1x_1+\dots+u_rx_r)}$$

$$= \frac{1}{(2\pi)^r}\int du_1\dots du_r \exp\left\{ \sum_{j=1}^{r}(a_j - x_j)\, iu_j + \frac{1}{2}\sum_{jk}\sigma_{jk}\, iu_j\, iu_k \right\} \quad (2.59)$$

The matrix $\sigma = \|\sigma_{jk}\|$ is assumed to be positively definite. Then

$$\{\dots\} = iA^T u - \frac{1}{2} u^T S u,$$

$$A = \|a_j - x_j\|, \quad u = \|u_j\|, \quad S = \|\sigma_{jk}\|$$

$Se_\lambda = \sigma_\lambda e_\lambda$, $V = \|e_\lambda\|$ – modal matrix of S, $V^{-1} = V^T$

and hence $S = \sum_\lambda \sigma_\lambda e_\lambda e_\lambda^T$

then $u^T S u = \sum_\lambda \sigma_\lambda (u^T e_\lambda)(e_\lambda^T u) = \sum_\lambda \sigma_\lambda (e_\lambda^T u)^T (e_\lambda^T u)$

If $y_\lambda = e_\lambda^T u$ is a new set of ~~vectors, then~~ numbers, then

$$u^T S u = \sum_\lambda \sigma_\lambda y_\lambda^2 .$$

The old (~~ass~~) ($U = \|u_j\|$) and new ($Y = \|y_\lambda\|$) "coordinates" are related via

$$y_\lambda = e_\lambda^T u \equiv \sum_j \underbrace{e_{\lambda j}}_{\text{unitary } E} u_j \ , \quad u_j = \sum_\lambda \underbrace{e_{\lambda j}}_{E} y_\lambda$$

The Jacobian

$$J = \frac{\partial(u_1, \dots, u_r)}{\partial(y_1, \dots, y_r)} = |\underbrace{e_{\lambda j}}_{E}| = 1 \text{ as it is a unitary matrix}$$

and hence

$$\{\dots\} \text{ in } (2.59) = i A^T E Y - \frac{1}{2} \sum_\lambda \sigma_\lambda y_\lambda^2$$

$$= -\frac{1}{2} \sum_\lambda \sigma_\lambda y_\lambda^2 + i \sum_\lambda \underbrace{\Big[\sum_j (a_j - x_j) \cdot e_{\lambda j}\Big]}_{\varkappa_\lambda} y_\lambda$$

$$= \sum_\lambda \left\{ -\frac{1}{2}\sigma_\lambda y_\lambda^2 + i \varkappa_\lambda y_\lambda \right\} = \sum_\lambda \left\{ -\frac{1}{2}\sigma_\lambda \Big(y_\lambda - \frac{i\varkappa_\lambda}{\sigma_\lambda}\Big)^2 - \frac{\varkappa_\lambda^2}{2\sigma_\lambda} \right\}$$

and hence

$$W_r = \prod_\lambda \frac{1}{2\pi} \Big[\underbrace{\int dy_\lambda \, e^{-\frac{1}{2}\sigma_\lambda (y_\lambda - \frac{i\varkappa_\lambda}{\sigma_\lambda})^2}}_{\sqrt{\frac{\pi 2}{\sigma_\lambda}} = \sqrt{\frac{2\pi}{\sigma_\lambda}}} \Big] e^{-\frac{\varkappa_\lambda^2}{2\sigma_\lambda}} = \prod_\lambda \frac{1}{\sqrt{2\pi\sigma_\lambda}} e^{-\frac{\varkappa_\lambda^2}{2\sigma_\lambda}}$$

$$= \frac{1}{(2\pi)^{r/2}} \cdot \frac{1}{\sqrt{\underbrace{\prod_\lambda \sigma_\lambda}_{\det S}}} \exp\left\{ -\frac{1}{2} \sum_\lambda \frac{\varkappa_\lambda^2}{\sigma_\lambda} \right\},$$

where

$$\sum_\lambda \frac{\varkappa_\lambda^2}{\sigma_\lambda} = \sum_\lambda \frac{1}{\sigma_\lambda} \sum_{ij} (a_j - x_j) e_{\lambda j} (a_i - x_i) e_{\lambda i})$$

$$= \sum_{ij} (a_j - x_j)\, S^{-1}_{ji}\, (a_i - x_i),$$

i.e.,

$$W_r(x_1, \dots, x_r) = \frac{1}{(2\pi)^{r/2}} \cdot \frac{1}{\sqrt{\det S}} \cdot \exp\left\{ -\frac{1}{2} \sum_{jk} (a_j - x_j)\, S^{-1}_{jk}\, (a_k - x_k) \right\} \quad (2.62)$$

• Introduce random variables (mean zero):

$$\eta_i = \xi_i - \underbrace{a_i}_{\langle \xi_i \rangle}, \quad \langle \eta_i \rangle = 0 \quad (2.63)$$

Then, it is (2.55) with $a_j = 0$.

$$\underbrace{\langle \eta_{j_1} \eta_{j_2} \cdots \eta_{j_{2n}} \rangle}_{\text{odd } \cancel{\text{number of variables in the product}}} = \frac{\partial}{\partial i u_{j_1}} \cdots \frac{\partial}{\partial i u_{j_{2n}}} C_r \Big|_{u=0}$$

$$= \frac{\partial}{\partial (i u_{j_1})} \cdots \frac{\partial}{\partial (i u_{j_{2n+1}})} \exp\left\{ \sum_{jk} \sigma_{jk}\, i u_j\, i u_k \right\} \Big|_{u=0}$$

$$= 0 \text{ if } n = \text{odd and } \neq 0 \text{ if } n = \text{even}.$$

• Linear transformation

A linear transformation of stoch. variables

$$\xi \to \xi' \quad \text{via} \quad \xi_i' = \sum_j \alpha_{ij} \xi_j + \beta_i$$

is still a Gaussian, see (2.62).

- Here we shall consider all moments of the zero-centred Gaussian distribution (2.62):

$$W_r(x_1,\dots,x_r) = (2\pi)^{-r/2}(\det S)^{-1/2}\exp\left[-\frac{1}{2}\sum_{jk} x_j S^{-1}_{jk} x_k\right]$$

The characteristic function

$$C_r(u_1,\dots,u_r) = \exp\left(\frac{1}{2}\sum_{jk}\sigma_{jk}\, iu_j\, iu_k\right)$$

and the moments

$$M_{n_1\dots n_r} = \left(\frac{\partial}{\partial i u_1}\right)^{n_1}\cdots\left(\frac{\partial}{\partial i u_r}\right)^{n_r} C_r(u_1,\dots,u_r)\Big|_{u=0}$$

In particular,

$$\underbrace{M_{01100\dots1\dots}}_{\langle \eta_{j_1}\eta_{j_2}\dots\eta_{j_n}\rangle} = \left\{\frac{\partial}{\partial i u_{j_1}}\frac{\partial}{\partial i u_{j_2}}\cdots\frac{\partial}{\partial i u_{j_n}}\exp\left[\frac{1}{2}\sum_{jk}\sigma_{jk}\, iu_j\, iu_k\right]\right\}\Big|_{u=0}$$

$\to$ all j_k are <u>different</u>.

Again, consider $S e_\lambda = \sigma_\lambda e_\lambda$, $e_\lambda = \| e_{\lambda i}\|$,

$$u_j = \sum_\lambda e_{\lambda i} y_\lambda, \quad y_\lambda = \sum_j e_{\lambda j} u_j$$

$$\frac{\partial}{\partial u_i} = \sum_\lambda e_{\lambda i}\frac{\partial}{\partial y_\lambda} \quad \text{and} \quad \sum_{jk}\sigma_{jk}u_j u_k \equiv \sum_\lambda \sigma_\lambda y_\lambda^2$$

This gives:

$$\langle \eta_{j_1}\cdots\eta_{j_n}\rangle = \sum_{\lambda_1}\cdots\sum_{\lambda_n} e_{\lambda_1 j_1}\cdots e_{\lambda_n j_n}\left[\frac{\partial}{\partial i y_{\lambda_1}}\frac{\partial}{\partial i y_{\lambda_2}}\cdots\frac{\partial}{\partial i y_n}\prod_{\lambda'} e^{+\frac{i^2}{2}\sigma_{\lambda'}y_{\lambda'}^2}\right]_{y=0} \quad (*)$$

Here $\{\lambda_i\}$ may be different or the same (repeat)! We shall now consider various cases which may appear:

(i) single repetition:

$$\frac{\partial}{\partial i y_\lambda} e^{\frac{i^2}{2}\sigma_\lambda y_\lambda^2}\Big|_{y_\lambda=0} = e^{\frac{i^2}{2}\sigma_\lambda y_\lambda^2}\, i\sigma_\lambda y_\lambda\Big|_{y_\lambda=0} = 0$$

(ii) double:

$$\left(\frac{\partial}{\partial i y_\lambda}\right)^2 e^{i^2\sigma_\lambda y_\lambda^2/2}\Big|_{y_\lambda=0} = \frac{\partial}{\partial i y_\lambda} i\sigma_\lambda y_\lambda e^{i^2\sigma_\lambda y_\lambda^2/2}\Big|_{y_\lambda=0} =$$

$$= \left[(i\sigma_\lambda y_\lambda)^2 + \sigma_\lambda \right] e^{\cdots} \to \sigma_\lambda$$

It follows, if to continue, ~~that~~

$$\left(\frac{\partial}{\partial i y_\lambda}\right)^{2n} e^{\sigma_\lambda (i y_\lambda)^2/2} \Big|_{y_\lambda = 0} = \underbrace{\frac{(2n)!}{2^n\, n!}}_{\#\text{ of pairs}} \sigma_\lambda^n \qquad (*)$$

This means that we can simply divide all z_i terms in $(*)$ into pairs of equil $\lambda_i = \lambda_j$, each pair giving exactly σ_{λ_i} (for n = even):

$$\langle y_{j_1} \cdots y_{j_n} \rangle = \sum_{S(airs)} \underbrace{\left(\sum_{\lambda_1} e_{\lambda_1 j_i} e_{\lambda_1 j_j} \sigma_{\lambda_1} \right)}_{\text{from one pair}} \underbrace{(\ldots)(\ldots)}_{\text{other pairs}}$$

$$= \sum_S \underbrace{\sigma_{j_i j_j} \sigma_{j_{i'} j_{j'}} \cdots}_{\text{for all possible pairs.}} \qquad (2.64)$$

For n = odd the result is zero (one incomplete pair $\Rightarrow 0$).

<u>Proof of $(*)$:</u> $\left(\frac{\partial}{\partial x}\right)^{2n} e^{\sigma x^2/2} \Big|_{x=0} = \frac{(2n)!}{2^n n!} \sigma^n.$

Expand the exponent ~~and consider~~:

$$\left(\frac{\partial}{\partial x}\right)^{2n} \sum_{k=0}^{\infty} \left(\frac{\sigma}{2}\right)^k \frac{x^{2k}}{k!} \Big|_{x=0} = \sum_{k=n}^{\infty} \left(\frac{\sigma}{2}\right)^k \frac{\boxed{\frac{(2k)!}{(2k-2n)!}}}{k!} x^{2k-2n} \Big|_{x=0} = \left(\frac{\sigma}{2}\right)^n \frac{(2n)!}{n!}$$

since only $k = n$ term remains after $x = 0$.

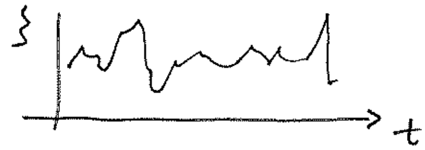
which is the required result.

If, however, the power is odd:

$$\left(\frac{\partial}{\partial x}\right)^{2n+1} e^{\sigma x^2/2} \Big|_{x=0} = \sum_{k=n+1}^{\infty} \left(\frac{\sigma}{2}\right)^k \frac{(2k)!\; x^{2k-2n-1}}{(2k-2n-1)!\, k!} \Big|_{x=0} = 0.$$

Hence, if there are y_λ the same, their contribution is exactly the same as breaking all $\{y_\lambda\}$ into pairs.

Time dependent random variables

$\xi = \xi(t)$  ← for a single system in the ensemble

Probability density at time t for ξ:

$$W_1(x_1, t) = \langle \delta(x_1 - \xi(t_1)) \rangle \qquad (2.66)$$

$$W_1(x_1, t)\, dx_1 = ~~prob.~~\ \mathrm{P}(x_1 \le \xi(t) < x_1 + dx_1; t)$$

~~For many variables at different times:~~

$\cancel{W_n(x_n t_n; x_{n-1} t_{n-1}; \ldots; x_1 t_1) = \langle \delta(x_1 - \xi(}$

To find ξ at diff. times being within dx_1 (at t_1), dx_2 (at t_2), etc. leads to:

$$W_n(x_n, t_n; \ldots; x_1, t_1) = \langle \delta(x_1 - \xi(t_1))\, \delta(x_2 - \xi(t_2)) \ldots \delta(x_n - \xi(t_n)) \rangle \qquad (2.66a)$$

$W_n\, dx_1 \ldots dx_n$ – probability

The hierarchy

$$\begin{aligned} &W_1(x_1, t_1) \\ &W_2(x_2 t_2; x_1 t_1) \qquad (t_0 \le t_i \le t_0 + T) \\ &W_3(x_3 t_3; x_2 t_2; x_1 t_1) \end{aligned}$$

defines completely the process in time, between t_0 and $t_0 + T$.

- The correlation function

$$\langle \xi(t_1)\, \xi(t_2) \rangle = \int dx_1 dx_2\, W_2(x_2 t_2; x_1 t_1)\, x_1 x_2 \qquad (2.68)$$

- From W_2 one can obtain W_1 by integration (reduction):

$$W_1(x_1, t_1) = \int dx_2\, W_2(x_2 t_2; x_1 t_1), \text{ etc.}$$

- Stationary processes:

$$W_n(\ldots t_i + T \ldots) \equiv W_n(\ldots t_i \ldots)$$

for all i at the same time, T – arbitrary

Then $W_1(x_1, t_1 + T) \equiv W_1(x_1, t_1)$ for any $T \to W_1(x_1, t_1) \equiv W_1(x_1)$

does not depend on time,

$$W_2(x_2, t_2+T; x_1, t_1+T) \equiv W_2(x_2, t_2; x_1, t_1)$$

for any T, $\Rightarrow$ W_2 only depends on $t_2 - t_1$.

- <u>Classification of random processes</u>

$$P(x_n t_n | x_{n-1} t_{n-1}; \dots; x_1 t_1) = \langle \delta(x_n - \xi(t_n)) \rangle \Big|_{\substack{\xi(t_{n-1}) \equiv x_{n-1} \\ \dots \\ \xi(t_1) \equiv x_1}}$$

↳ at times $t_1, \dots, t_{n-1}$ ξ has fixed values.

This is conditional probability; we may have

$$t_n > t_{n-1} > \dots > t_1 ,$$

then P gives the cond. prob. that at time t_n $\xi = x_n$ provided that at previous times $t_{n-1}, \dots, t_1$ ξ was $x_{n-1}, \dots, x_1$.

From (2.44):

$$P(x_n t_n | x_{n-1} t_{n-1}; \dots; x_1 t_1) = \frac{W_n(x_n t_n; \dots; x_1 t_1)}{\int dx_n W_n(x_n t_n; \dots; x_1 t_1)} = \frac{W_n(\dots)}{W_{n-1}(x_{n-1} t_{n-1}; \dots; x_1 t_1)} \quad (2.70)$$

ⓐ <u>Purely random processes</u>

If P does not depend on values at previous times:

$$P(x_n t_n | x_{n-1} t_{n-1}; \dots; x_1 t_1) = P(x_n, t_n)$$

Then:

$$W_n(x_n t_n; \dots; x_1 t_1) = P(x_n; t_n) W_{n-1}(x_{n-1} t_{n-1}; \dots; x_1 t_1)$$

$$= \dots = P(x_n t_n) P(x_{n-1} t_{n-1}) \dots P(x_1 t_1) \quad (2.72)$$

Purely random process cannot describe a physical system as at close enough times the values of ξ should somehow correlate.

(b) Markov processes

$$P(x_n t_n | x_{n-1} t_{n-1}; \ldots; x_1 t_1) \equiv P(x_n t_n | x_{n-1} t_{n-1}) \qquad (2.73)$$

only depends on a previous time, but not on all others before it.

$$W_n(x_n t_n; \ldots; x_1 t_1) = P(x_n t_n | x_{n-1} t_{n-1}) W_{n-1}(x_{n-1} t_{n-1}; \ldots; x_1 t_1)$$

$$= \underbrace{P(x_n t_n | x_{n-1} t_{n-1})}_{\text{also called transition probability}} P(x_{n-1} t_{n-1} | x_{n-2} t_{n-2}) \ldots P(x_2 t_2 | x_1 t_1) W_1(x_1 t_1). \qquad (2.74)$$

The transition probability

$$P(x_2 t_2 | x_1 t_1) = \frac{W_2(x_2 t_2; x_1 t_1)}{W_1(x_1 t_1)} = \frac{W_2(x_2 t_2; x_1 t_1)}{\int dx_2\, W_2(x_2 t_2; x_1 t_1)} \qquad (2.75)$$

↳ all information is contained in $W_2(x_2 t_2; x_1 t_1)$.

When $t_2 \to t_1$ we should have $x_2 = x_1$:

$$\lim_{t_2 \to t_1} P(x_2 t_2 | x_1 t_1) = \delta(x_2 - x_1) \qquad (2.76)$$

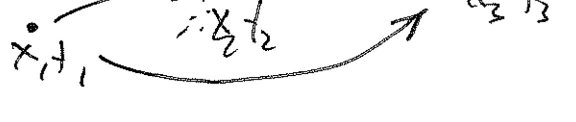
(c) General processes

More previous times could be used, but it is more complex this way. Generalised Fokker-Planck eq. could be better suited (with memory) to describe non-Markovian behaviour.

- Chapman-Kolmogorov eq.

$$\underbrace{W_2(x_3 t_3; x_1 t_1)}_{P(x_3 t_3 | x_1 t_1) W_1(x_1 t_1)} = \int \underbrace{W_3(x_3 t_3; x_2 t_2; x_1 t_1)}_{P(x_3 t_3 | x_2 t_2) P(x_2 t_2 | x_1 t_1) W_1(x_1 t_1)} dx_2$$

$$\hookrightarrow P(x_3 t_3 | x_1 t_1) = \int P(x_3 t_3 | x_2 t_2) P(x_2 t_2 | x_1 t_1)\, dx_2 \qquad (2.78)$$



$x_1 \to x_3$ is the same as taking via all possible x_2 at $t_1 < t_2 < t_3$

Wiener-Khintchine theorem

$$\xi(t) \to \tilde{\xi}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \xi(t)\, dt \leftarrow \underline{\text{also random}} \qquad (2.79)$$

Stationary processes:

Jacobian = 1

$$\langle \tilde{\xi}(\omega) \tilde{\xi}^*(\omega') \rangle = \int dt\, dt'\, e^{-i\omega t} e^{i\omega' t'} \underbrace{\langle \xi(t)\, \xi^*(t') \rangle}_{\langle \xi(t-t')\, \xi^*(0)\rangle} = \left| \begin{array}{l} \tau = t - t' \\ t_0 = \frac{t+t'}{2} \end{array} \right|$$

(where $t = t_0 + \tau/2$, $t' = t_0 - \tau/2$)

$$= \int d\tau \int dt_0 \langle \xi(\tau)\, \xi^*(0) \rangle e^{-i(\omega-\omega')t_0} e^{-i(\omega+\omega')\tau/2}$$

$$= 2\pi \delta(\omega - \omega') \int d\tau\, e^{-i(\omega+\omega')\tau/2} \langle \xi(\tau)\, \xi^*(0) \rangle$$

$$= 2\pi \delta(\omega - \omega') \int d\tau\, e^{-i\omega\tau} \langle \xi(\tau)\, \xi^*(0) \rangle$$

$$= \pi \delta(\omega - \omega')\, S(\omega), \qquad (2.85)$$

$$S(\omega) = 2 \int d\tau\, e^{-i\omega\tau} \langle \xi(\tau)\, \xi^*(0) \rangle \leftarrow \underline{\text{spectral density}} \qquad (2.86)$$

Several time dependent r. variables

Similarly.