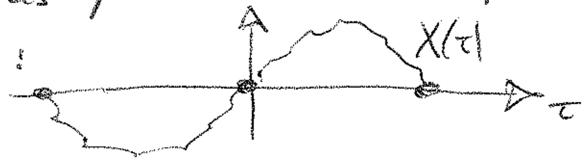
1.2.7 Fourier decomposition

• $X(\tau) = X_c(\tau) + X(\tau)$, $X_c(0) = X_0$, $X_c(t) = X_t$ (1.2.191) (1.2.192)

$X(0) = X(t) = 0$ (1.2.193)

$X(t)$ between 0 and t is expressed as periodic with the period of $2t$ and odd ($X(-\tau) = -X(\tau)$):



~~$X(\tau) = \sum$~~

$$X(\tau) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \sin\frac{\pi n \tau}{t} + b_n \cos\frac{\pi n \tau}{t}\right]$$

$$b_n = \frac{1}{t}\int_{-t}^{t} X(\tau)\cos\frac{\pi n \tau}{t}\,d\tau = 0 \quad \text{as } X(-\tau) = X(\tau)$$

$$a_n = \frac{1}{t}\int_{-t}^{t} X(\tau)\sin\frac{\pi n \tau}{t}\,d\tau = \frac{2}{t}\int_{0}^{t} X(\tau)\sin\frac{\pi n \tau}{t}\,d\tau$$

$$a_0 = \frac{1}{t}\int_{-t}^{t} X(\tau)\,d\tau = 0 \quad \text{as } X(\tau) \text{ is odd.}$$

$$\hookrightarrow \boxed{X(\tau) = \underbrace{\sqrt{\frac{2}{t}}}_{\text{for convenience}}\sum_{n=1}^{\infty} a_n \sin\left(\pi n \frac{\tau}{t}\right)} \qquad (1.2.194)$$

• $\boxed{\psi_n(\tau) = \sqrt{\frac{2}{t}}\sin\left(\pi n \frac{\tau}{t}\right)}$ – forms an orthogonal & complete set of functions over $\tau \in [0, t]$:

$$\int_0^t \psi_n \psi_m\,dx = \frac{2}{t}\int_0^t \underbrace{\sin\left(\pi n \frac{\tau}{t}\right)\sin\left(\pi m \frac{\tau}{t}\right)}_{-\frac{1}{2}\cos\left(\pi(n+m)\frac{\tau}{t}\right) + \frac{1}{2}\cos\left(\pi(n-m)\frac{\tau}{t}\right)} d\tau$$

If $n \neq m$, then

$$\hookrightarrow = \frac{2}{t}\left\{-\frac{1}{2}\frac{t}{\pi(n+m)}\sin\left(\pi(n+m)\frac{\tau}{t}\right) + \frac{1}{2}\frac{t}{\pi(n-m)}\sin\left(\pi(n-m)\frac{\tau}{t}\right)\right\}_0^t = 0$$

If $n = m$, then $\int_0^t \psi_n \psi_n\,d\tau = \frac{2}{t}\int_0^t \left\{\cancel{-\frac{1}{2}\frac{t}{\pi 2n}\sin\left(2\pi n\frac{\tau}{t}\right)} - \frac{1}{2}\cos\left(\pi 2n \frac{\tau}{t}\right) + \frac{1}{2}\right\}d\tau$

$$= \frac{2}{t}\cdot\frac{1}{2}\cdot t = 1$$

$$\boxed{\int_0^t \psi_n \psi_m \, d\tau = \delta_{nm}}$$

- So, instead of $X(\tau)$, coefficients a_n are used as random variables.
- Finite ~~d~~ difference calculus

$$\boxed{\begin{cases} \partial^{(\varepsilon)} x(t) = \frac{1}{\varepsilon}\left[x(t+\varepsilon) - x(t)\right] \to \partial_t x(t) \\ \bar{\partial}^{(\varepsilon)} x(t) = \frac{1}{\varepsilon}\left[x(t) - x(t-\varepsilon)\right] \to \partial_t x(t) \end{cases}} \quad (1.2.195)$$

$\hookrightarrow$ as $\varepsilon \to 0$

$$\partial^{(\varepsilon)} x_j = \frac{1}{\varepsilon}\left[x_{j+1} - x_j\right] \quad , 0 < j \leq N$$

$$\bar{\partial}^{(\varepsilon)} x_j = \frac{1}{\varepsilon}\left[x_j - x_{j-1}\right] \quad , \quad 1 \leq j \leq N+1$$

"Integration by parts",

$$\boxed{\varepsilon \sum_{j=1}^{N+1} p_j \bar{\partial}^{(\varepsilon)} x_j = p_j x_j \Big|_0^{N+1} - \varepsilon \sum_{j=0}^{N} \left(\partial^{(\varepsilon)} p_j\right) x_j} \quad (1.2.196)$$

Indeed:

$$LHS = \varepsilon \sum_{j=1}^{N+1} p_j \frac{1}{\varepsilon}\left[x_{j} - x_{j-1}\right] = \sum_{j=1}^{N+1}\left(p_j x_j - p_j x_{j-1}\right)$$

$$RHS = p_{N+1} x_{N+1} - p_0 x_0 - \varepsilon \sum_{j=0}^{N} \frac{1}{\varepsilon}\left(p_{j+1} - p_j\right) x_j$$

$$= p_{N+1} x_{N+1} - p_0 x_0 - \sum_{j=0}^{N}\left(p_{j+1} x_j - p_j x_j\right)$$

$$= \sum_{j=1}^{N} p_j x_j - \underbrace{\sum_{j=0}^{N} p_{j+1} x_j}_{j \to j+1} + \left(p_{N+1} x_{N+1} - \cancel{p_0 x_0}\right) + \cancel{p_0 x_0} =$$

$$= \sum_{j=1}^{N} p_j x_j - \sum_{j=1}^{N+1} p_j x_{j-1} + p_{N+1} x_{N+1} = \sum_{j=1}^{N+1} p_j x_j - \sum_{j=1}^{N+1} p_j x_{j-1} \equiv RHS$$

If $X_{N+1}=X_0=0$

$$\sum_{j=1}^{N+1} p_j\, \bar\partial^{(\varepsilon)} x_j = -\sum_{j=0}^{N} \left(\partial^{(\varepsilon)} p_j\right) x_j \qquad (1.2.198)$$

If $p_0=p_{N+1}$, $x_0=x_{N+1}$, then

$$\sum_{j=1}^{N+1} p_j\, \bar\partial^{(\varepsilon)} x_j = -\sum_{j=1}^{N+1}\left(\partial^{(\varepsilon)} p_j\right) x_j \cancel{-\left(\partial^{(\varepsilon)} p_0\right) x_0} \qquad (1.2.199)$$

since $\left(\partial^{(\varepsilon)} p_0\right) X_0 = \frac{1}{\varepsilon}\left[p_1-p_0\right] X_0 = \frac{1}{\varepsilon}\left[p_1-p_{N+1}\right] X_{N+1} = \frac{1}{\varepsilon}\left[p_{N+2}-p_{N+1}\right] X_{N+1}$
$= \left(\partial^{(\varepsilon)} p_{N+1}\right) X_{N+1}$

simply rearrangement of terms

• For functions with vanishing endpoints $X_{N+1}=X_0=0$:

$$\sum_{j=1}^{N+1}\left(\bar\partial^{(\varepsilon)} x_j\right)^2 = \sum_{j=1}^{N+1} \frac{1}{\varepsilon^2}\left(x_j-x_{j-1}\right)^2 = \sum_{j=1}^{N+1}\frac{1}{\varepsilon^2}\left(x_j^2+x_{j-1}^2-2x_j x_{j-1}\right)$$

$$= -\sum_{k,j=0}^{N} x_j \left[\partial^{(\varepsilon)}\bar\partial^{(\varepsilon)}\right]_{jk} x_k \qquad \boxed{\equiv \sum_{j=0}^{N} x_j\, \partial^{(\varepsilon)}\bar\partial^{(\varepsilon)} x_j}$$

$$\left[\partial^{(\varepsilon)}\bar\partial^{(\varepsilon)}\right]_{jk} = \frac{1}{\varepsilon^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \qquad (1.2.202)$$

$(N+1)\times(N+1)$ matrix

• $$i\partial^{(\varepsilon)} e^{-i\omega t} = i\frac{1}{\varepsilon}\left[e^{-i\omega(t+\varepsilon)} - e^{-i\omega t}\right] = \frac{i}{\varepsilon}\left(e^{-i\omega\varepsilon}-1\right) e^{-i\omega t} \equiv \Omega e^{-i\omega t} \qquad (1.2.206)$$

$$i\bar\partial^{(\varepsilon)} e^{-i\omega t} = \bar\Omega e^{-i\omega t} \text{ with } \bar\Omega = \frac{i}{\varepsilon}\left(1-e^{i\omega\varepsilon}\right) \equiv \Omega^* \qquad (1.2.207)$$

In the continuum limit $\varepsilon\to 0$:

$$\Omega\, \cancel{\bar\Omega} \to \frac{i}{1}\left(e^{-i\omega\varepsilon}(-i\omega)\right) \Rightarrow \omega, \quad \bar\Omega \to \omega$$
as well

The Wiener measure in terms of Fourier coefficients

$X(\tau) \Rightarrow X(\tau_j)$, $\quad j = 0, \dots, N+1$

Then the sum (1.2.197) will be carried out only up to N:

$$\boxed{\begin{aligned} X(\tau_j) &= \sum_{n=1}^{N} \sqrt{\frac{2}{(N+1)\varepsilon}}\, a_n \sin(\nu_n \tau_j) \\ \nu_n &= \frac{\pi n}{t} \to \frac{\pi n}{(N+1)\varepsilon} \end{aligned}} \equiv X_j \qquad (1.2.209)$$

$\tau_j = \varepsilon j$

$\boxed{\psi_n(\tau_0) = 0}$

$\boxed{\psi_n(\tau_{N+1}) = 0}$

The expansion functions are orthogonal:

$$\psi_n(\tau_j) = \sqrt{\frac{2}{N+1}} \sin(\nu_n \tau_j)$$

$$\sum_{j=1}^{N} \psi_n(\tau_j)\psi_m(\tau_j) = \frac{2}{N+1} \sum_{j=1}^{N} \sin(\nu_n \tau_j) \sin(\nu_m \tau_j)$$

$$= \frac{2}{N+1} \sum_{j=1}^{N} \frac{1}{2i}\left[e^{i\nu_n\tau_j} - e^{-i\nu_n\tau_j}\right] \frac{1}{2i}\left[e^{i\nu_m\tau_j} - e^{-i\nu_m\tau_j}\right]$$

$$= \frac{2}{N+1}\left(-\frac{1}{4}\right) \sum_{j=1}^{N} \left\{ e^{i(\nu_n+\nu_m)\tau_j} - e^{-i(\nu_n-\nu_m)\tau_j} - e^{i(\nu_n-\nu_m)\tau_j} + e^{-i(\nu_n+\nu_m)\tau_j} \right\}$$

$$\cancel{= -\frac{1}{2(N+1)} \cdot 2\,\mathrm{Re} \cdot \sum_{j=1}^{N} \left\{ e^{i\nu_{n+m}\tau_j} - e^{-i\nu_{n-m}\tau_j} \right\}}$$

$$\cancel{\tau_j = \varepsilon j, \quad \sum_{j=1}^{N} e^{i\nu\tau_j} = \sum_{j=1}^{N} \left(e^{i\nu\varepsilon}\right)^j = \frac{e^{i\nu\varepsilon} - e^{i\nu\varepsilon(N+1)}}{1 - e^{i\nu\varepsilon}}}$$

$$\cancel{\curvearrowright -\frac{1}{2(N+1)}\,\mathrm{Re}\left[\frac{e^{i\nu_{n+m}\varepsilon} - e^{i\nu_{n+m}\varepsilon(N+1)}}{1 - e^{i\nu_{n+m}\varepsilon}} - \frac{e^{i\nu_{n-m}\varepsilon} - e^{i\nu_{n-m}\varepsilon(N+1)}}{1 - e^{i\nu_{n-m}\varepsilon}}\right]}$$

$$\cancel{= -\frac{1}{N+1}\left\{ \frac{\left(1 - e^{-i\nu_{n+m}\varepsilon}\right)\left(e^{i\nu_{n+m}\varepsilon}\right.}{2(1 - \cos\nu_{n+m}\varepsilon)} \right.}$$

$\cancel{\frac{1}{2(N+1)}}$ But $\sum_{j=1}^{N} e^{i\nu_k \tau_j} = \sum_{j=1}^{N} e^{i\nu_k \varepsilon j} = \sum_{j=1}^{N} \left(e^{i\nu_k \varepsilon}\right)^j$

$$= \sum_{j=0}^{N} \ldots - 1 = \frac{1 - e^{i\nu_k \varepsilon (N+1)}}{1 - e^{i\nu_k \varepsilon}} - 1 \quad ;$$

Here

$$\nu_k \varepsilon = \frac{\pi k}{(N+1)\varepsilon} \cdot \varepsilon = \frac{\pi k}{(N+1)\varepsilon}$$

$$e^{i\nu_k \varepsilon (N+1)} = e^{i\pi k} = (-1)^k$$

and $$\sum_{j=1}^{N} = \begin{cases} 0 - 1 = -1, & \text{if } k = \text{even} \\ \dfrac{2}{1 - e^{i\nu_k \varepsilon}} - 1, & \text{if } k = \text{odd} \end{cases}$$

~~When $n \neq m$, then~~. We have:

$$\sum_{j=1}^{N} \psi_n(\tau_j)\psi_m(\tau_j) = -\frac{1}{2(N+1)} \left\{ \left[\frac{1 - (-1)^{n+m}}{1 - e^{i\nu_{n+m}\varepsilon}} + \frac{1 - (-1)^{n+m}}{1 - e^{-i\nu_{n+m}\varepsilon}} \right] - \right.$$

$$\left. - \left[\frac{1 - (-1)^{n-m}}{1 - e^{-i\nu_{n-m}\varepsilon}} + \frac{1 - (-1)^{n-m}}{1 - e^{i\nu_{n-m}\varepsilon}} \right] \right\}$$

If $n-m$ = even, then, $n+m = (n-m) + 2m$ = even as well. $\to 0$ if $n \neq m$. If $n-m$ = odd, then $n+m = (n-m) + 2m$ = odd as well.

$$[\ldots]_{(k=n+m)} = 2\left[\frac{1}{1 - e^{i\nu_k \varepsilon}} + \frac{1}{1 - e^{-i\nu_k \varepsilon}}\right] = 2\,\frac{1 - e^{-i\nu_k \varepsilon} + 1 - e^{i\nu_k \varepsilon}}{2 - 2\cos \nu_k \varepsilon}$$

$$= 2\,\frac{2(1 - \cos \nu_k \varepsilon)}{1 - \cos \nu_k \varepsilon} = 2$$

The same for $k = n-m$ term $\to 0$ again.

If, however, $n = m$, then

$$\sum_{j=1}^{N} \psi_n(\tau_j)^2 = -\frac{1}{2(N+1)} \sum_{j=1}^{N} \left\{ \left(e^{i\nu_{2n}\tau_j} - e^{-i\nu_{2n}\tau_j}\right) - 2 \right\}$$

~~$\to 0$ after summation~~

$$= -\frac{1}{2(N+1)}\left\{\cancel{2N} -2N + \left[\underbrace{\frac{1-e^{i\nu_{2n}\varepsilon(N+1)}}{1-e^{i\nu_{2n}\varepsilon}}}_{=0} - 1 + \underbrace{\frac{1-e^{-i\nu_{2n}\varepsilon(N+1)}}{1-e^{-i\nu_{2n}\varepsilon}}}_{=0} - 1\right]\right\}$$

$$= -\frac{1}{2(N+1)}\left\{-2N-2\right\} = 1$$

$$\boxed{\sum_{j=1}^{N} \psi_n(\tau_j)\psi_m(\tau_j) = \delta_{nm} \equiv \frac{2}{N+1}\sum_{j=1}^{N} \sin(\nu_n\tau_j)\sin(\nu_m\tau_j)} \quad (1.2.2)$$

So, they are orthogonal. Completeness:

$$\sum_{n=1}^{N} \psi_n(\tau_j)\psi_n(\tau_{j'}) = \frac{2}{N+1}\sum_{n=1}^{N} \sin(\nu_n\tau_j)\sin(\nu_n\tau_{j'})$$

$$= \frac{2}{N+1}\left(-\frac{1}{4}\right)\sum_{n=1}^{N}\left\{e^{i\nu_n\varepsilon(j'+j)} + e^{-i\nu_n\varepsilon(j'+j)} - e^{-i\nu_n\varepsilon(j'-j)} - e^{i\nu_n\varepsilon(j'-j)}\right\}$$

If $j=j'$, then

$$\hookrightarrow -\frac{1}{2(N+1)}\sum_{n=1}^{N}\left\{e^{i\nu_n\varepsilon 2j} + e^{-i\nu_n\varepsilon 2j} - 2\right\},$$

$$\sum_{n=1}^{N} e^{i\nu_n\varepsilon k} = \sum_{n=1}^{N}\left(e^{i\frac{\pi k}{N+1}}\right)^n = \sum_{n=0}^{N}\left(e^{\frac{i\pi k}{N+1}}\right)^n - 1$$

$$= \frac{1-e^{\frac{i\pi k}{N+1}(N+1)}}{1-e^{\frac{i\pi k}{N+1}}} - 1 = \frac{1-e^{i\pi k}}{1-e^{i\pi k/(N+1)}} - 1 = \frac{1-(-1)^k}{1-e^{\frac{i\pi k}{N+1}}} - 1$$

So, if $k=2j \to$ we obtain -1 from each sum,

$$\hookrightarrow -\frac{1}{2(N+1)}\left\{-2-2N\right\} = 1$$

If $j\neq j'$, then

$$\hookrightarrow -\frac{1}{2(N+1)}\left\{\left[\frac{1-(-1)^{j+j'}}{1-e^{i\pi(j+j')/(N+1)}} + \frac{1-(-1)^{j+j'}}{1-e^{-\frac{i\pi(j+j')}{N+1}}}\right] - \right.$$

$$- \left[\frac{1-(-1)^{j-j'}}{1-e^{\frac{i\pi(j'-j)}{N+1}}} + \frac{1-(-1)^{j+j'}}{1-e^{\frac{i\pi(j'-j)}{N+1}}} \right] \Bigg\}$$

It is $= 0$ if $j-j' =$ even. If $j-j' =$ odd, then

$$[\ldots]_k = 2\, \frac{2}{1-e^{\frac{i\pi k}{N+1}}} + \frac{2}{1-e^{-\frac{i\pi k}{N+1}}}$$

$$= 2\, \frac{2\left(1-\cos\frac{\pi k}{N+1}\right)}{2\left(1-\cos\frac{\pi k}{N+1}\right)} = 2 \quad , \quad k = j+j' \text{ or } j-j'$$

and both brackets cancell out. We hence obtain:

$$\boxed{\sum_{n=1}^{N} \psi_n(\tau_j)\,\psi_n(\tau_{j'}) = \frac{2}{N+1}\sum_{n=1}^{N} \sin(\nu_n\tau_j)\sin(\nu_n\tau_{j'}) = \delta_{jj'}} \quad (1.2.211)$$

- Jacobian of transformation $X(\tau_j) \leftrightarrow a_m$:

$$\frac{\partial X(\tau_j)}{\partial a_m} = \sqrt{\frac{2}{(N+1)\varepsilon}}\, \sin(\nu_m\tau_j) = \frac{1}{\sqrt{\varepsilon}}\, \psi_m(\tau_j) = \varepsilon^{-1/2}\, \psi_m(\tau_j)$$

and

$$J = \left| \frac{\partial X(\tau_j)}{\partial a_m} \right| = \varepsilon^{-N/2}\, \underbrace{|\psi_m(\tau_j)|}_{\substack{\shortparallel \\ 1 \text{ as } \psi_m \text{ are orthogonal}}} \equiv \varepsilon^{-N/2} \quad (1.2.212)$$

and we obtain in new variables:

$$d_W X(\tau) = \left(\prod_{j=1}^{N} \frac{dX_j}{\sqrt{4\pi D\varepsilon}} \right) \frac{1}{\sqrt{4\pi D\varepsilon}}\, \exp\left\{ -\frac{1}{4D} \underbrace{\sum_{j=1}^{N+1} \frac{(X_j - X_{j-1})^2}{\varepsilon}}_{\varepsilon\left(\bar\partial^{(\varepsilon)} X_j\right)^2} \right\}$$

- Consider the exp. term. It is diagonalised in the new variables:

$$\cancel{\sum_{j=0}^{N+1} \frac{(X_j - X_{j-1})^2}{\varepsilon} = \varepsilon \sum_{j=0}^{N} \left(\bar\partial^{(\varepsilon)} X_j\right)^2 = \varepsilon \sum_{j=0}^{N} \left(\bar\partial^{(\varepsilon)} \sum_{n=1}^{N} \frac{1}{\sqrt{\varepsilon}}\, \psi_n(\tau_j)\, a_n \right)^2 =}$$

~~NOT FINISHED~~

$$\sum_{j=0}^{N} \frac{1}{\varepsilon}(x_{j+1}-x_j)^2 = \frac{1}{\varepsilon}\sum_{j=0}^{N}(x_{j+1}-x_j)^2,$$

$$x_{j+1}-x_j = \sum_n a_n\left[\varphi_n(\tau_{j+1}) - \varphi_n(\tau_j)\right]$$

$$= \sum_{n=1}^{N} a_n \sqrt{\frac{2}{N+1}}\left[\sin(\underbrace{\nu_n\tau_{j+1}}_{\nu_n\tau_j+\nu_n\varepsilon}) - \sin(\nu_n\tau_j)\right]$$

$$= \sqrt{\frac{2}{N+1}}\sum_{n=1}^{N} a_n\, 2\cos\left(\nu_n\tau_j + \frac{\nu_n\varepsilon}{2}\right)\sin\left(\tfrac{1}{2}\nu_n\varepsilon\right)$$

$$= \sqrt{\frac{2}{N+1}}\sum_{n=1}^{N} a_n\, 2\frac{1}{2}\left[e^{i\nu_n\tau_j}e^{i\nu_n\varepsilon/2} + e^{-i\nu_n\tau_j}e^{-i\nu_n\varepsilon/2}\right]\sin\left(\tfrac{1}{2}\nu_n\varepsilon\right)$$

Denote $\lambda_n = \exp(i\nu_n\varepsilon) = \exp\left(\frac{i\pi n}{N+1}\right)$, then

$$x_{j+1}-x_j = \sqrt{\frac{2}{N+1}}\sum_{n=1}^{N} a_n \sin\left(\tfrac{1}{2}\nu_n\varepsilon\right)\left[\sqrt{\lambda_n}\, e^{i\nu_n\tau_j} + \frac{1}{\sqrt{\lambda_n}}e^{-i\nu_n\tau_j}\right]$$

and hence

$$\sum_{j=0}^{N}\frac{1}{\varepsilon}(x_{j+1}-x_j)^2 = \left(\frac{2}{N+1}\right)\frac{1}{\varepsilon}\sum_{n,m=1}^{N} a_n a_m \sin\left(\tfrac{1}{2}\nu_n\varepsilon\right)\sin\left(\tfrac{1}{2}\nu_m\varepsilon\right)\times$$

$$\times\left\{\sum_{j=0}^{N}\left[\sqrt{\lambda_n}\, e^{i\nu_n\tau_j} + \frac{1}{\sqrt{\lambda_n}}e^{-i\nu_n\tau_j}\right]\left[\sqrt{\lambda_m}\, e^{i\nu_m\tau_j} + \frac{1}{\sqrt{\lambda_m}}e^{-i\nu_m\tau_j}\right]\right\} \quad (*)$$

Consider $\{\ldots\}$ for $\underline{n \neq m}$:

$$\{\ldots\} = \sqrt{\lambda_n\lambda_m}\left(\sum_j e^{i\nu_{n+m}\tau_j}\right) + \sqrt{\frac{\lambda_m}{\lambda_n}}\left(\sum_j e^{-i\nu_{n-m}\tau_j}\right) + \sqrt{\frac{\lambda_n}{\lambda_m}}\left(\sum_j e^{i\nu_{n-m}\tau_j}\right),$$

$$+ \frac{1}{\sqrt{\lambda_n\lambda_m}}\left(\sum_j e^{-i\nu_{n+m}\tau_j}\right)$$

Here

$$\sum_{j=0}^{N} e^{i\nu_k\tau_j} = \sum_{j=0}^{N} e^{\frac{i\pi k}{(N+1)\varepsilon}\varepsilon j} = \sum_{j=0}^{N}\left(e^{\frac{i\pi k}{N+1}}\right)^j = \frac{1-e^{\frac{i\pi k}{N+1}(N+1)}}{1-e^{\frac{i\pi k}{N+1}}}$$

$$= \frac{1 - e^{i\pi k}}{1-\lambda_k} = \frac{1-(-1)^k}{1-\lambda_k}$$

Therefore, it is $=0$ if $k =$ even.

$$\{\dots\} = \sqrt{\lambda_n\lambda_m}\,\frac{1-(-1)^{n+m}}{1-\lambda_n\lambda_m} + \frac{1}{\sqrt{\lambda_n\lambda_m}}\,\frac{1-(-1)^{n+m}}{1-\frac{1}{\lambda_n\lambda_m}} +$$

$$+ \sqrt{\frac{\lambda_m}{\lambda_n}}\,\frac{1-(-1)^{n+m}}{1-\frac{\lambda_m}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_m}}\,\frac{1-(-1)^{n+m}}{1-\frac{\lambda_n}{\lambda_m}}$$

$$= \cancel{\lambda_n}\left[1-(-1)^{n+m}\right]\left\{ \cancel{\sqrt{\lambda_n\lambda_m}\,\frac{1}{1-\lambda_n\lambda_m} + \sqrt{\lambda_n\lambda_m}\,\frac{1}{\lambda_n\lambda_m-1}}\right.$$

$$\left. + \cancel{\sqrt{\frac{\lambda_m}{\lambda_n}}\,\frac{\lambda_n}{\lambda_n-\lambda_m} + \sqrt{\frac{\lambda_n}{\lambda_m}}\,\frac{\lambda_m}{\lambda_m-\lambda_n}} \right\} = 0$$

Therefore, terms in $(*)$ with $n\neq m$ vanish.

Consider now $\underline{n=m}$; the $\{\dots\}$ brackets from $(*)$:

$$\{\dots\} = \lambda_n \sum_{j=0}^{N} e^{i\nu_{2n}\tau_j} + 2(N+1) + \frac{1}{\lambda_n}\sum_{j=0}^{N} e^{-i\nu_{2n}\tau_j}$$

$$= 2(N+1) + \lambda_n \underbrace{\frac{1-(-1)^{2n}}{\dots\dots}}_{=0} + \frac{1}{\lambda_n}\underbrace{\frac{1-(-1)^{2n}}{\dots\dots}}_{=0} = 2(N+1)$$

Therefore,

$$\sum_{j=0}^{N} \frac{1}{\varepsilon}(x_{j+1}-x_j)^2 = \frac{2}{\varepsilon(N+1)}\cdot\sum_{n=1}^{N} a_n^2 \sin^2\!\left(\tfrac{1}{2}\nu_n\varepsilon\right) 2(N+1)$$

$$= \frac{4}{\varepsilon}\sum_{n=1}^{N}\sin^2\!\left(\tfrac{1}{2}\nu_n\varepsilon\right) a_n^2 \equiv \frac{4}{\varepsilon}\sum_{n=1}^{N}\frac{1}{2}\left(1-\cos(\nu_n\varepsilon)\right) a_n^2$$

$$= \frac{1}{\varepsilon}\sum_{n=1}^{N}\Big[\underbrace{2\left(1-\cos(\nu_n\varepsilon)\right)}_{\varepsilon^2\Omega_n\bar{\Omega}_n}\Big]a_n^2 = \varepsilon\sum_{n=1}^{N}\underbrace{\Omega_n\bar{\Omega}_n}_{|\Omega_n^*|^2} a_n^2$$

• Now

$$d_w X(\tau) = \left(\prod_{j=1}^{N} \frac{dX_j}{\sqrt{4\pi \mathcal{D} \varepsilon}} \right) \frac{1}{\sqrt{4\pi \mathcal{D}\varepsilon}} \cdot \exp\left\{ -\frac{1}{4\mathcal{D}} \sum_{n=1}^{N} \Omega_n \bar{\Omega}_n a_n^2 \right\} \quad (1.2.213)$$

Show the diagonalisation via the discrete differential calculus (as in the book).

$$\sum_{j=0}^{N} \frac{1}{\varepsilon}(X_{j+1}-X_j)^2 = \frac{1}{\varepsilon} \sum_{j=0}^{N} (X_{j+1}^2 + X_j^2 - 2X_j X_{j+1})$$

$$= \frac{1}{\varepsilon}\Big[X_1^2 + (X_2^2 + X_1^2 - 2X_1 X_2) + (X_3^2 + X_2^2 - 2X_2 X_3) + \dots$$

$$+ (X_N^2 + X_{N-1}^2 - 2X_{N-1}X_N) + X_N^2 \Big]$$

if $X_0 = X_{N+1} \equiv 0$.

This can also be written as:

$$-\varepsilon \sum_{j=0}^{N} X_j\, \partial^{(\varepsilon)} \bar{\partial}^{(\varepsilon)} X_j \equiv -\varepsilon \sum_{j=0}^{N} X_j\, \partial^{(\varepsilon)} \frac{1}{\varepsilon}[X_j - X_{j-1}]$$

$$= -\sum_{j=0}^{N} X_j \left(\partial^{(\varepsilon)} X_j - \partial^{(\varepsilon)} X_{j-1} \right) = -\frac{1}{\varepsilon} \sum_{j=0}^{N} X_j \left[X_{j+1} - X_j - (X_j - X_{j-1}) \right]$$

$$= -\frac{1}{\varepsilon} \sum_{j=0}^{N} X_j \left(X_{j+1} - 2X_j + X_{j-1} \right) = -\frac{1}{\varepsilon} \sum_{j=0}^{N} \left(-2X_j^2 + X_j X_{j+1} + X_{j-1} X_j \right)$$

$$= \frac{1}{\varepsilon} \sum_{j=1}^{N} \left(2X_j^2 - X_j X_{j+1} - X_{j-1} X_j \right)$$

$$\equiv \frac{1}{\varepsilon}\Big[(2X_1^2 - X_1 X_2) + (2X_2^2 - X_2 X_3 - X_1 X_2) + (2X_3^2 - X_3 X_4 - X_2 X_3) + \dots$$

$$+ (2X_{N-1}^2 - X_{N-1} X_N - X_{N-2} X_{N-1}) + (2X_N^2 - X_{N-1} X_N) \Big]$$

which is the same thing. Hence, the term in the exponential

$$\boxed{\sum_{j=0}^{N} \frac{1}{\varepsilon}(X_{j+1} - X_j)^2 = -\varepsilon \sum_{j=0}^{N} X_j\, \partial^{(\varepsilon)} \bar{\partial}^{(\varepsilon)} X_j} \quad (1.2.200)$$

Now,

$$X_j = \sum_{n=1}^{N} a_n \sqrt{\frac{2}{\varepsilon(N+1)}}\, \sin(\nu_n \tau_j)$$

so that

$$\sum_{j=0}^{N} \frac{1}{\varepsilon}(X_{j+1}-X_j)^2 = -\varepsilon \frac{2}{\varepsilon(N+1)} \sum_{n,m=1}^{N} a_n a_m \sum_{j=0}^{N} \sin(\nu_n\tau_j)\left[\partial^{(\varepsilon)}\bar\partial^{(\varepsilon)} \sin(\nu_m\tau_j)\right]$$

Consider

$$\partial^{(\varepsilon)}\bar\partial^{(\varepsilon)} \sin \nu_m\tau_j = \frac{1}{2i}\, \partial^{(\varepsilon)}\bar\partial^{(\varepsilon)}\left[e^{i\nu_m\tau_j} - e^{-i\nu_m\tau_j}\right]$$

$$= -\frac{1}{2i}\, \partial^{(\varepsilon)}\left\{ \bar\Omega_m(-\nu_m)\, e^{i\nu_m\tau_j} - \bar\Omega_m(+\nu_m)\, e^{-i\nu_m\tau_j}\right\}$$

$$= -\frac{1}{2i}\left\{\bar\Omega(-\nu_m)\,\Omega(-\nu_m)\, e^{i\nu_m\tau_j} - \bar\Omega(\nu_m)\,\Omega(\nu_m)\, e^{-i\nu_m\tau_j}\right\}$$

where

$$\bar\Omega(-\nu_m)\,\Omega(-\nu_m) = \frac{i}{\varepsilon}(1-e^{-i\nu_m\varepsilon})\,\frac{i}{\varepsilon}(e^{i\nu_m\varepsilon}-1)$$

$$= \frac{-1}{\varepsilon^2}(e^{i\nu_m\varepsilon}-1-1+e^{-i\nu_m\varepsilon}) = \frac{-1}{\varepsilon^2}(-2+2\cos\nu_m\varepsilon)$$

$$= \frac{2}{\varepsilon^2}(1-\cos\nu_m\varepsilon)$$

and also

$$\bar\Omega(\nu_m)\,\Omega(\nu_m) = \frac{i}{\varepsilon}(1-e^{+i\nu_m\varepsilon})\,\frac{i}{\varepsilon}(e^{-i\nu_m\varepsilon}-1)$$

$$= -\frac{1}{\varepsilon^2}(-1-1+e^{-i\nu_m\varepsilon}+e^{i\nu_m\varepsilon}) = \frac{2}{\varepsilon^2}(1-\cos\nu_m\varepsilon) \equiv \Omega_m\bar\Omega_m,$$

i.e. the same. Therefore,

$$\partial^{(\varepsilon)}\bar\partial^{(\varepsilon)} \sin\nu_m\tau_j = -\frac{1}{2i}\,\bar\Omega_m\Omega_m\left(e^{i\nu_m\tau_j}-e^{-i\nu_m\tau_j}\right)$$

$$= -\bar\Omega_m\Omega_m \sin(\nu_m\tau_j)$$

which gives

$$\sum_{j=0}^{N}\frac{1}{\varepsilon}(X_{j+1}-X_j)^2 = -\frac{2}{N+1}\sum_{n,m=1}^{N} a_n a_m \sum_{j=0}^{N}\left(-\bar\Omega_m\Omega_m\right)\sin\nu_n\tau_j\,\sin\nu_m\tau_j$$

$$= -\frac{2}{N+1}\sum_{n,m}\left(-\Omega_m\bar{\Omega}_m\right) a_n a_m \left(\underbrace{\sum_{j=0}^{N} \sin\nu_n\tau_j \sin\nu_m\tau_j}_{\delta_{nm}\frac{N+1}{2}}\right)$$

$$= + \sum_{n,m} \Omega_m\bar{\Omega}_m a_n^2 = \sum_{n,m} |\Omega_m|^2 a_n^2 ,$$

ie the same result.