Quadratic approximation

$$W(xt|x_0 0) = \int_{C[x_0 0;\, xt]} \prod \frac{dx(\tau)}{\sqrt{4\pi \mathcal{D} d\tau}} \exp\left\{-\frac{1}{4\mathcal{D}} \int_0^t ds \left[\dot{x}^2(s) + 4\mathcal{D} V(x(s))\right]\right\}$$

This is the Feynman-Kac formula for the solution of the Bloch equation with $V(x(s))$.

$$x = x_c + X(\tau), \qquad X(0) = X(t) = 0$$

We expand the exponent of the functional in X:

$$\frac{1}{4\mathcal{D}} \int_0^t ds \left[(\dot{x}_c + \dot{X})^2 + 4\mathcal{D} V(x_c + X) \right]$$

$$= \frac{1}{4\mathcal{D}} \int_0^t \left[\dot{x}_c^2 + 4\mathcal{D} V(x_c)\right] ds + \frac{1}{4\mathcal{D}} \int_0^t ds \left[2\dot{x}_c \dot{X} + 4\mathcal{D} V'(x_c)\cdot X \right]$$

$$+ \frac{1}{4\mathcal{D}} \int_0^t ds \left[\dot{X}^2 + \frac{1}{2} 4\mathcal{D} V''(x_c)\cdot X^2 \right] + \left(\begin{matrix}\text{higher order}\\ \text{terms}\end{matrix}\right)$$

The 1st order term $= 0$ due to definition of x_c:

$$\delta \int_0^t L(x(s), \dot{x}(s), s)\, ds = 0 \quad , \quad \cancel{\delta x}\ \delta X(0) = \delta X(s) = 0$$

means

$$\int_0^t ds \left\{ \frac{\delta L}{\delta X(s)} \delta X(s) + \frac{\delta L}{\delta \dot{X}(s)} \delta \dot{X}(s) \right\} = 0$$

where
$$\int_0^t ds \frac{\delta L}{\delta \dot{x}(s)} \cdot \delta \dot{X}(s) = \cancel{\int ds \frac{\delta L}{\delta \dot{X}(s)}} \text{ (by parts)}$$

$$= \delta X(s) \frac{\delta L}{\delta \dot{x}(s)}\Big|_0^t - \int_0^t ds \frac{\partial}{\partial s}\left(\frac{\partial L}{\delta \dot{x}(s)}\right) \delta x(s)$$

giving

$$\delta S = \int_0^t ds\, \delta x(s) \Big[\underbrace{\frac{\partial L}{\partial x(s)} - \frac{\partial}{\partial s}\Big(\frac{\partial L}{\partial \dot x(s)}\Big)}_{=0 \text{ (Euler eq.)}} \Big] = 0$$

Here

$$L = \dot x^2 + 4D V(x)$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial s}\frac{\partial L}{\partial \dot x} \to 2\ddot x_c = 4D V'(x_c) \to \boxed{\ddot x_c = 2D V'(x_c)}$$

The 1st order term:

$$\int_0^t ds \left[2\dot x_c \dot X + 4D V'(x_c) X\right] = \left(\text{by parts the 1st term only}\right)$$

$$= \underbrace{2\dot x_c X\Big|_0^t}_{0} - \int_0^t ds \Big(\underbrace{2\ddot x_c X - 4D V'(x_c) X}_{0}\Big) = 0$$

Therefore, the exponential up to the 2nd order:

$$-\frac{1}{4D}\int_0^t ds\left[\dot x_c^2 + 4D V''(x_c)\right] \quad (1.2.148) \qquad \cancel{+}\,\frac{1}{4D}\int_0^t ds\left[\dot X^2 + 2D V''(x_c) X^2\right] \quad (1.2.149)$$

and hence

$$\boxed{\begin{aligned} W(xt|x_0 0) &= \exp\Big[-\frac{1}{4D}\int_0^t ds \left(\dot x_c^2 + 4D V''(x_c)\right)\Big] \times F(xt; x_0 0) \quad (1.2.150) \\ F(xt; x_0 0) &= \int_{C[00;0t]} \prod \frac{dX(\tau)}{\sqrt{4\pi D\, d\tau}} \exp\Big[-\frac{1}{4D}\int_0^t d\tau \left[\dot X^2 + 2D V''(x_c) X^2\right]\Big] \quad (1.2.151) \end{aligned}}$$

The F-part we take by discretisation; $\varepsilon = t/(N+1)$

Recall that $X(0) = X(t) = 0 \to X_0 = X_{N+1} = 0$

(this is conditional)

$$F = \int \frac{dX_1 \ldots}{\sqrt{4\pi \mathcal{D}\varepsilon}} \ldots \int \frac{dX_N}{\sqrt{4\pi\mathcal{D}\varepsilon}} \frac{1}{\sqrt{4\pi\mathcal{D}\varepsilon}} \cdot \exp\left\{ -\frac{1}{\varepsilon 4 \mathcal{D}} \sum_{j=0}^{N} (X_{j+1}-X_j)^2 - \frac{\varepsilon}{2} \sum_{j=0}^{N} V_j'' X_j^2 \right\} \qquad (1.2.151)$$

In $\{\ldots\} \to -\frac{1}{4\mathcal{D}\varepsilon} \sum_{ij=1}^{N} X_i B_{ij} X_j$ with

$$B_N = \begin{pmatrix} b_1 & -1 & & & \\ -1 & b_2 & -1 & & \\ & -1 & b_3 & & \\ & & & \ddots & -1 \\ & & & -1 & b_N \end{pmatrix} \qquad B_{ii} = b_i = 2 + 2\mathcal{D}\varepsilon^2 V_i''$$

Integration over X_i gives:

$$\frac{1}{\sqrt{4\pi\mathcal{D}\varepsilon}} \cdot \frac{(\sqrt{\pi})^N}{(\sqrt{4\pi\mathcal{D}\varepsilon})^N} \cdot \frac{(\sqrt{4\mathcal{D}\varepsilon})^N}{\sqrt{\det B_N}} = \frac{1}{\sqrt{4\pi\mathcal{D}\varepsilon \det B_N}},$$

Hence

$$F = \lim_{\varepsilon \to 0, N \to \infty} (4\pi\mathcal{D}\varepsilon \det B_N)^{-1/2} \qquad (1.2.152)$$

Recurrence relation for $\det B_N \equiv G_N$:

$$G_N = (2 + 2\mathcal{D}\varepsilon^2 V_N'') G_{N-1} - G_{N-2}$$

Introduce $C_N = \frac{G_N}{N+1} \to G_N = C_N(N+1)$

$$\hookrightarrow C_N(N+1) = 2C_{N-1} \cdot N + 2N\mathcal{D}\varepsilon^2 V_N'' C_{N-1} - C_{N-2}(N-1)$$

$$\cancel{N[C_N - 2C_{N-1} + C_{N-2}] = -C_N + C_{N-2} + 2N\mathcal{D}\varepsilon^2 V_N'' C_{N-1}}$$

$$\cancel{\frac{C_N - 2C_{N-1} - C_{N-2}}{\varepsilon^2} = C_N}$$

$$(N+1)C_N + (N+1)C_{N-2} - 2(N+1)C_{N-1} =$$
$$= 2N\mathcal{D}\varepsilon^2 V_N'' C_{N-1} - 2C_{N-1} + 2C_{N-2}$$

$$C_N - 2C_{N-1} + C_{N-2} = -\frac{2}{N+1}(C_{N-1} - C_{N-2}) + 2\frac{N\varepsilon^2}{N+1}\mathcal{D}V_N'' C_{N-1} \quad (1.2.155)$$

$$\frac{C_N - 2C_{N-1} + C_{N-2}}{\varepsilon^2} = -\frac{2}{\varepsilon(N+1)}\frac{(C_{N-1} - C_{N-2})}{\varepsilon} + 2\frac{N}{N+1}\mathcal{D}V_N'' C_{N-1}$$

In the $\varepsilon \to 0$ limit $\varepsilon(N+1) = t - t_0$,

$$\boxed{\frac{\partial^2 C(t)}{\partial t^2} = -\frac{2}{t-t_0}\frac{\partial C}{\partial t} + 2\mathcal{D}V''(x_c(t))\, C} \quad (1.2.156)$$

The boundary conditions:

$$C(x, t_0) \Rightarrow N = 1 \to \frac{B_1}{1+1} = \frac{2 + 2\mathcal{D}\varepsilon^2 V_1''}{2} \to 1$$

$$\left.\frac{\partial C}{\partial t}\right|_{t=t_0} \Rightarrow \left(\frac{G_2}{3} - \frac{G_1}{2}\right)\frac{1}{\varepsilon} = \frac{1}{3\varepsilon}\begin{vmatrix} B_1 & -1 \\ -1 & B_2 \end{vmatrix} - \frac{1}{2\varepsilon}\begin{vmatrix} B_1 \end{vmatrix}$$

$$= \frac{1}{3\varepsilon}\left[\left(2 + 2\mathcal{D}\varepsilon^2 V_1''\right)\left(2 + 2\mathcal{D}\varepsilon^2 V_2''\right) - 1\right] - \frac{1}{2\varepsilon}\left(2 + 2\mathcal{D}\varepsilon^2 V_1''\right)$$

$$= \frac{1}{3\varepsilon}\left[3 + O(\varepsilon^2)\right] - \frac{1}{2\varepsilon}\left(2 + O(\varepsilon^2)\right) = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} + O(\varepsilon^2) = 0$$

$$\boxed{C(x, t_0) = 1, \quad \left.\frac{\partial C}{\partial t}\right|_{t=t_0} = 0} \quad (1.2.157)$$

Therefore,

$$F(x,t) = \frac{1}{\sqrt{4\pi\varepsilon\mathcal{D}\underbrace{\det B_N}_{G_N}}} = \frac{1}{\sqrt{4\pi\mathcal{D}\varepsilon(N+1)\cdot G_N}} \to \frac{1}{\sqrt{4\pi\mathcal{D}(t-t_0)C(t)}} \quad (1.2.158)$$

- Next we derive an equation for $C(x,t)$ and solve it.

$$H(x,t) = (t-t_0)\,C(x,t)$$

$$H_t' = C + (t-t_0)\,C_t' \,,\quad H_t'' = C_t' + C_t' + (t-t_0)\,C_t'' = 2C_t' + (t-t_0)\,C_t''$$

↳ into (1.2.156):

$$C_t'' = \frac{H_t''}{t-t_0} - \frac{2}{t-t_0}\,C_t'$$

and from (1.2.156)

$$\left(\frac{H_t''}{t-t_0} - \cancel{\frac{2}{t-t_0}C_t'}\right) + \cancel{\frac{2}{t-t_0}C_t'} = 2DV''(x_c(t))\,\frac{H}{t-t_0}$$

$$\hookrightarrow \quad \boxed{H_t'' = 2DV''(x_c(t))\,H} \qquad (1.2.160)$$

with the boundary conditions:

$$\begin{cases} H(x,t_0) = \cancel{\left.\dfrac{C(x,t)}{t-t_0}\right|_{t\to 0}} = (t-t_0)\,C(x,t)\Big|_{t\to t_0} = 0 \\ \left.\dfrac{\partial H}{\partial t}\right|_{t=t_0} = \underbrace{C(x,t_0)}_{=1} + \underbrace{\lim_{t\to t_0}\left[(t-t_0)\dfrac{\partial C}{\partial t}\right]}_{=0} = 1 \end{cases} \qquad (1.2.161)$$

To solve (1.2.160), we try to change to an independent variable $x_c = x_c(t)$ as V'' depends on t only via x_c.

$$L_f = \dot{x}_c^2(\tau) + 4DV(x_c(\tau))$$

$$\frac{\partial L_f}{\partial x_c} = \frac{\partial}{\partial \tau}\frac{\partial L_f}{\partial \dot{x}_c} \to 2\ddot{x}_c = 4DV' \to \boxed{\ddot{x}_c = 2DV'(x_c)}$$

The conserved "energy" $E = \dot{x}_c^2 - 4DV \to \boxed{\dot{x}_c = \sqrt{E+4DV}}$ (1.2.163)

and $\cancel{\dfrac{\partial^2 H}{\partial t^2} = \dfrac{\partial}{\partial x_c}\left(\dfrac{\partial H}{\partial t}\right)\dfrac{\partial x_c}{\partial t} = \dot{x}_c\dfrac{\partial}{\partial x_c}\left[\dfrac{\partial H}{\partial x_c}\dot{x}_c\right]} =$

$$= \cancel{\dot{x}_c \left[\frac{\partial^2 H}{\partial x_c^2} \dot{x}_c + \frac{\partial H}{\partial x_c} \ddot{x}_c \right.}$$

$$\frac{\partial H}{\partial t} = \frac{dH}{dx_c} \dot{x}_c \,, \quad \frac{\partial^2 H}{\partial t^2} = \ddot{x}_c \frac{dH}{dx_c} + \dot{x}_c^{\,2} \frac{d^2 H}{dx_c^2}$$

↳ (1.2.160) now reads:

$$\underbrace{\ddot{x}_c}_{2DV'(x_c)} \frac{dH}{dx_c} + \underbrace{\dot{x}_c^{\,2}}_{E+4DV(x_c)} \frac{d^2 H}{dx_c^2} = 2DV''(x_c)\, H$$

$$\boxed{2DV' \frac{dH}{dx} + (E+4DV) \frac{d^2 H}{dx^2} = 2DV'' H} \qquad (1.2.164)$$

with

$$\boxed{H\Big|_{x_c = x_0} = 0 \,, \quad \frac{\partial H}{\partial x_c}\Big|_{x_c = x_0} = \frac{(\partial H/\partial t)_{t=t_0}}{\underbrace{(\dot{x}_c)_{t=t_0}}} = \frac{1}{\sqrt{E+4DV}}} \qquad (1.2.165)$$

(1.2.164) can also ~~be~~ written differently:

$$LHS = \frac{d^2}{dx^2}\Big[H(E+4DV) \Big] = H''(E+4DV) + 2H' \cdot 4DV' + H\, 4DV''$$

$$RHS = \frac{d}{dx}\Big[6DV'H\Big] = 6DV''H + 6DV'H'$$

↳ $LHS = RHS \rightarrow H''(E+4DV) + 8H'DV' + 4DHV'' = 6DV''H + 6DV'H'$

which is exactly (1.2.164)! Hence,

$$\boxed{\Big[(E+4DV)H\Big]'' = \Big[6DV'H\Big]'} \qquad (1.2.166)$$

which can be integrated:

$$C_1 + 6DV'H = \Big[(E+4DV)H\Big]' \equiv (E+4DV)H' + 4DV' \cdot H$$

$\cancel{(E+4DV)H' = 2DV'H}$ From the initial conditions ($x_c = x_0$):

$$C_1 + 6DV' \cdot 0 = (E+4DV) \cdot \frac{1}{\sqrt{E+4DV}} = 4DV' \cdot 0$$

$$C_1 = \sqrt{E+4\Delta V(x_0)}$$

$$\hookrightarrow (E+4\Delta V)H' - 2\Delta V' H = \sqrt{E+4\Delta V(x_0)}$$

$$H' - \frac{2\Delta V'}{E+4\Delta V}\cdot H = \frac{\sqrt{E+4\Delta V_0}}{E+4\Delta V}$$

Solving: $H' - \alpha H = 0 \rightarrow \frac{dH}{H} = \int \alpha \, dx$ ~~$\rightarrow H = e\exp(\alpha x)$~~

$\hookrightarrow$ ~~$H = C(x) e^{\alpha x} \rightarrow H' - \alpha H = \beta$~~

~~$C' e^{\alpha x} + \alpha C e^{\alpha x} - \alpha C e^{\alpha x} = \beta \Rightarrow C' = \beta e^{-\alpha x}$~~

$$\alpha(x) = \frac{2\Delta V'(x)}{E+4\Delta V}$$

$$\ln \frac{H}{C} = \int_{x_0}^{x} \alpha(x)\,dx \rightarrow H = C \exp\left[\int_{x_0}^{x} \alpha(x)\,dx\right]$$

$$H' - \alpha H = \beta(x) \rightarrow C \equiv C(x)$$

$$C' e^{\int \alpha dx} + C e^{\int \alpha dx} \cancel{\alpha(x)} - \cancel{\alpha H} \equiv \beta(x) \equiv \frac{(E+4\Delta V_0)^{1/2}}{E+4\Delta V(x)}$$

$$C' = \beta(x)\, e^{-\int_{x_0}^{x} \alpha dx}$$

$$C(x) = C_1 + \int_{x_0}^{x} \beta(x')\, e^{-\int_{x_0}^{x'} \alpha(x'')dx''} dx'$$

and the full solution

$$H(x) = \left[C_1 + \int_{x_0}^{x} \beta(x')\, e^{-\int_{x_0}^{x'} \alpha(x'')dx''} dx'\right] e^{\int_{x_0}^{x} \alpha(x')dx'}$$

Initial condition of $H(x_0) = 0 \rightarrow C_1 \equiv 0$.

Next, consider

$$\int_{x_0}^{x'} \alpha(x')\,dx' = \int_{x_0}^{x} \frac{2\Delta V'(x')\,dx'}{E+4\Delta V} = \frac{1}{2}\int \frac{d(E+4\Delta V)}{E+4\Delta V}$$

$$= \frac{1}{2} \ln |E+4\Delta V| \Big|_{x_0}^{x} = \frac{1}{2} \ln \frac{E+4\Delta V(x)}{E+4\Delta V(x_0)} = \ln \sqrt{\frac{E+4\Delta V}{E+4\Delta V_0}},$$

and

$$\exp\left[\int_{x_0}^{x} \alpha(x')dx\right] = \sqrt{\frac{E+4\Delta V}{E+4\Delta V_0}}$$

so that

$$H(x) = \left(\frac{E+4DV(x)}{\cancel{E+4DV_0}}\right)^{1/2} \int_{x_0}^{x} \frac{\cancel{(E+4DV_0)^{1/2}}}{E+4DV(x')} \cdot \sqrt{\frac{E+4DV_0}{E+4DV(x')}}\, dx',$$

$$\boxed{H(x) = \left(E+4DV_0\right)^{1/2}\left(E+4DV\right)^{1/2} \int_{x_0}^{x} dx' \left(E+4DV(x')\right)^{-3/2}} \qquad (1.2.167)$$

This $H(x)$ can now be used in $F(x,t)$:

$$F(x,t) = \Big(4\pi D \underbrace{(t-t_0)\,C(x,t)}_{H(x,t)}\Big)^{-1/2} = \Big(4\pi D\, H(x_c(t))\Big)^{-1/2}$$

The final result from (1.2.150):

$$\boxed{\begin{aligned} W(xt|x_0 0) = &\left\{4\pi D\left(E+4DV(x_0)\right)^{1/2}\left(E+4DV(x)\right)^{1/2}\int_{x_0}^{x}\left(E+4DV(y)\right)^{-3/2}dy\right\}^{-1/2} \times \\ &\times \exp\left\{-\frac{1}{4D}\int_0^t ds\left(\dot{x}_c^2 + 4DV''(x_c)\right)\right\} \end{aligned}} \qquad (1.2.168)$$

This is exact if $V(x)$ is of the up to 2nd order in x.

<u>Example</u> $F(x) = -\gamma k x$ (a harmonic oscillator force)

First, we need to work out the "potential" $V(x)$:

$$V(x) = \frac{3\pi R}{2\gamma k_B T} F^2 + \frac{\operatorname{div}\vec{F}}{2\gamma} \xrightarrow{1D} \cancel{\frac{3\pi R}{2\gamma k_B T}\gamma^2 k^2 x^2 + \frac{1}{2\gamma}\frac{\partial F}{\partial x}}$$

$$\cancel{\frac{3\pi R}{}} \rightarrow \frac{1}{4\gamma^2 D}(-\gamma k x)^2 + \frac{1}{2\gamma}\frac{\partial}{\partial x}(-\gamma k x)$$

$$= \frac{k^2}{4D}x^2 - \frac{1}{2\gamma}k\gamma = \frac{k^2}{4D}x^2 - \frac{k}{2} \qquad (1.2.170)$$

Then, find the classical trajectory:

$$L_f = \dot{x}_c^2 + 4DV(x) = \dot{x}^2 + k^2x^2 - 2Dk$$

$$\frac{\partial L_f}{\partial x} = \frac{\partial}{\partial t}\frac{\partial L_f}{\partial \dot{x}} \rightarrow 2k^2 x = 2\ddot{x}, \quad \ddot{x} = k^2 x$$

with the solution:

$$x = c_1 e^{k\tau} + c_2 e^{-k\tau}, \quad x(0) = x_0, \; x(t) = x$$

$$\hookrightarrow \quad x_c(\tau) = x_0 e^{-k\tau} + A\left(e^{k\tau} - e^{-k\tau}\right)$$

$$A = \frac{x - x_0 \exp(-kt)}{e^{kt} - e^{-kt}}$$

Then

$$W(xt|x_0 0) = e^{\frac{1}{2\gamma D}\int_{x_0}^{x} F\,dx'} \underbrace{\int_{C[x_0 0;\, xt]} d_W x(\tau) \exp\left(-\int_0^t V(x(\tau))\,d\tau\right)}_{\text{this is what we just calculated for the quadratic potential.}}$$

So, the prefactor to the path integral:

$$\frac{1}{2\gamma D}\int_{x_0}^{x} F\,dx' = \frac{1}{2\gamma D}\int_{x_0}^{x}(-\gamma k x')\,dx' = -\frac{k}{2D}\left(\frac{x^2}{2} - \frac{x_0^2}{2}\right) = \frac{k}{4D}\left(x_0^2 - x^2\right)$$

Then, we need to work out the exponent with x_c:

$$\left(4DV \equiv k^2 x^2 - 2kD\right)$$

$$\int_0^t d\tau\left[\dot{x}_c^2 + 4DV(x_c)\right] = \int_0^t d\tau \left\{\left[-kx_0 e^{-k\tau} + Ak\left(e^{k\tau} + e^{-k\tau}\right)\right]^2 + k^2\left[x_0 e^{-k\tau} + A\left(e^{k\tau} - e^{-k\tau}\right)\right]^2 - 2kD\right\}$$

$$= -2kDt + \int_0^t d\tau \left\{\left[k^2 x_0^2 e^{-2k\tau} - 2k^2 A x_0 e^{-k\tau}\left(e^{k\tau} + e^{-k\tau}\right) + A^2k^2\left(e^{2k\tau} + e^{-2k\tau} + 2\right)\right] + k^2\left[x_0^2 e^{-2k\tau} + 2Ax_0 e^{-k\tau}\left(e^{k\tau} - e^{-k\tau}\right) + A^2\left(e^{2k\tau} + e^{-2k\tau} - 2\right)\right]\right\}$$

$$= -2kDt + \int_0^t d\tau \left\{e^{-2k\tau}\left[k^2x_0^2 - 2k^2x_0A + A^2k^2 + k^2x_0^2 - 2Ax_0k^2 + A^2k^2\right] + e^{+2k\tau}\left[A^2k^2 + A^2k^2\right] - \cancel{2k^2Ax_0} + \cancel{2A^2k^2} + \cancel{2Ax_0k^2} - \cancel{2k^2A^2}\right\}$$

$$= -2k\Delta t + \frac{1}{2k}\left(1-e^{-2kt}\right)\left[2k^2x_0^2 + 2A^2k^2 - 4Ax_0k^2\right]$$

$$+ \frac{1}{2k}\left(e^{2kt}-1\right) 2A^2k^2 \cancel{[A^2k^2]t} = -2k\Delta t +$$

$$+ \frac{1-e^{-2kt}}{2k}\, 2k^2 \big(\underbrace{x_0^2 - 2Ax_0 + A^2}_{(x_0-A)^2}\big) + \frac{e^{2kt}-1}{2k}\, 2A^2k^2$$

$$= -2k\Delta t + k(x_0-A)^2\left(1-e^{-2kt}\right) + A^2k\left(e^{2kt}-1\right)$$

$$= -2k\Delta t + k\left(1-e^{-2kt}\right)\left[x_0 - \frac{x-x_0e^{-kt}}{e^{kt}-e^{-kt}}\right]^2 +$$

$$+ k\left(\frac{x-x_0e^{-kt}}{e^{kt}-e^{-kt}}\right)^2\left(e^{2kt}-1\right) = -2k\Delta t + B_1 + B_2$$

where

$$B_1 = k\cancel{\left(1-e^{-2kt}\right)}\frac{\left(x_0e^{kt} - x_0\cancel{e^{-kt}} - x + x_0\cancel{e^{-kt}}\right)^2}{e^{2kt}\left(1-e^{-2kt}\right)^{\cancel{2}}} =$$

$$= \frac{ke^{-2kt}}{1-e^{-2kt}}\left(x-x_0e^{kt}\right)^2,$$

$$B_2 = k\,\frac{\left(x-x_0e^{-kt}\right)^2}{e^{-2kt}\left(e^{2kt}-1\right)^{\cancel{2}}}\cancel{\left(e^{2kt}-1\right)} = \frac{ke^{2kt}\left(x-x_0e^{-kt}\right)^2}{e^{2kt}-1}$$

$$= \frac{k\left(x-x_0e^{-kt}\right)^2}{1-e^{-2kt}},$$

$$B_1+B_2 = \frac{k}{1-e^{-2kt}}\left\{e^{-2kt}\left(x-x_0e^{kt}\right)^2 + \left(x-x_0e^{-kt}\right)^2\right\}$$

(need to check algebra here)

The factor $F(x,t) = \left(4\pi D H(x,t)\right)^{-1/2}$ is calculated from (1.2.160), (1.2.161):

$$V'' = \frac{d^2}{dx^2}\left(\frac{k^2}{4D}x^2 - \frac{k}{2}\right) = \frac{k^2}{4D}2 = \frac{k^2}{2D},$$

$$H''_t = 2DV''H \equiv 2D\frac{k^2}{2D}H = k^2 H,$$

$$H(t) = C_1 e^{-kt} + C_2 e^{kt}, \text{ with } H_{t_0=0} = 0,\ H'_t\big|_{t_0} = 1$$

$$\cancel{H(t_0)=0,\ H_{x_0}=0,\ H'_{x_0}} \text{ giving } C_1 + C_2 = 0,$$

$$-kC_1 + C_2 k = 1 \rightarrow C_2 = 1/2k,\ C_1 = -1/2k,$$

$$H(t) = \frac{1}{2k}\left(e^{kt} - e^{-kt}\right) \qquad (1.2.176)$$

So, combining all contributions, we have:

$$W = \underbrace{\exp\left[\frac{k}{4D}(x_0^2 - x^2)\right]}_{\text{prefactor}} \cdot \left(4\pi D H(x,t)\right)^{-1/2} \times \exp\left[\frac{1}{4D}\int_0^t d\tau \left[\dot{x}_c^2 + 4DV''\right]\right]$$

$$\equiv \sqrt{\frac{k}{2\pi D(1-e^{-2kt})}}\ \exp\left\{-\frac{k}{2D}\frac{(x - x_0 e^{-kt})^2}{1 - e^{-2kt}}\right\}$$