This finally gives:

$$\boxed{W(x_t v_t t \mid x_0 v_0 0) = \int\limits_{C[v_0 0;\, v_t t]} \underbrace{d_W v(\tau)}_{\prod\limits_{\tau=0}^{t} \frac{d v(\tau)}{\sqrt{\pi\, d\tau}} \exp\left(-\int_0^t \dot{v}(\tau)^2 d\tau\right)} \; \delta\left(x(t) - x(0) - \int_0^t v(\tau)\, d\tau\right)}$$

(1.2.75)

1.2.5. Bloch eq. and Feynman-Kac formula

- <u>Bloch eq.</u> Particles can be removed (annihilated) with probability $V(x,t)$ per unit time. Then

$$\underbrace{j = -D\frac{\partial \rho}{\partial x}}_{\text{current}} \quad \text{and} \quad \underbrace{\frac{\partial \rho}{\partial t}}_{\text{change of } \rho} = \underbrace{-\frac{\partial j}{\partial x}}_{\text{reduced due to current}} \underbrace{- V\rho}_{\text{reduced due to annihilation}} \Rightarrow \boxed{\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} - V\rho} \quad (1.2.76)$$

~~For~~ Via the path integral:

$$\boxed{W_B(x_t t \mid x_0 t_0) = \int\limits_{C[x_0, t_0;\, x_t, t]} d_W x(\tau)\, \underbrace{\exp\left[-\int_0^t V(x_t, s)\, ds\right]}_{\text{probability to remain on the trajectory up to time } t}} \quad (1.2.79)$$

- $t > t_0$ condition can be imposed by putting $\Theta(t-t_0)$ in front:

$$W_D'(xt \mid x_0 t_0) = \Theta(t-t_0)\, W_D(xt \mid x_0 t_0)$$

where W_D satisfies usual diffusion eq.:

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) W_D' = \Theta(t-t_0) \underbrace{\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) W_D}_{= 0} + \delta(t-t_0) \underbrace{W_D(xt \mid x_0 t_0)}_{\substack{\to \delta(x-x_0) \\ \text{when } t \to t_0}}$$

Therefore, W_D' corresponds to the solution of the diff. eq. with the δ-function. We shall use W_D instead of W_D':

$$\boxed{\left[\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right] W_D(x,t\,|\,x_0 t_0) = \delta(x-x_0)\,\delta(t-t_0)} \qquad (1.2.80)$$

which gives explicitly a solution which $= 0$ when $t<t_0$.

Similarly, we seek solution of the block eq. with the δ-f on the right:

$$\boxed{\left[\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2} + V(x,t)\right] W_B(x,t\,|\,x_0 t_0) = \delta(x-x_0)\,\delta(t-t_0)} \qquad (1.2.81)$$

which gives zero solution automatically for $t<t_0$.

L_D'

$$\boxed{W_B(xt\,|\,x_0 t_0) = W_D(xt\,|\,x_0 t_0) - \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt'\, W_D(xt\,|\,x't')\, V(x't')\, W_B(x't'\,|\,x_0 t_0)} \qquad (1.2.82)$$

Indeed, apply $\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)$ on both sides:

$$LHS \Rightarrow \left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) W_B = \delta(x-x_0)\delta(t-t_0) - V(xt)\, W_B(xt\,|\,x_0 t_0)$$

$$RHS \Rightarrow \underbrace{\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) W_D}_{\delta(x-x_0)\delta(t-t_0)} - \int dx' \int dt' \underbrace{\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) W_D(xt\,|\,x't')}_{\delta(x-x')\,\delta(t-t')} V(x't')\, W_B(x't'\,|\,x_0 t_0)$$

$$= \delta(x-x_0)\,\delta(t-t_0) - V(xt)\,W_B(xt\,|\,x_0 t_0) \equiv LHS.$$

- Consider (1.2.79) in the discrete approximation: $\left(\varepsilon = (t-t_0)/(N+1)\right)$

$$W_B(xt\,|\,x_0 t_0) = \frac{1}{(4\pi D\varepsilon)^{(N+1)/2}} \int_{-\infty}^{\infty} dx_1 \ldots \int dx_N \exp\left\{-\frac{1}{4D\varepsilon}\sum_{j=0}^{N}(x_{j+1}-x_j)^2 - \varepsilon\sum_{j=1}^{N} V(x_j, t_j)\right\}$$

$$= \frac{1}{(4\pi D\varepsilon)^{(N+1)/2}} \int dx_1 \ldots \int dx_N \exp\left(-\frac{1}{4D\varepsilon}\sum_{j=0}^{N}(x_{j+1}-x_j)^2\right) \exp\left(-\varepsilon\sum_{j=1}^{N} V(x_j, t_j)\right)$$

The exponent with V we expand:

$$\exp\left[-\varepsilon\sum_j V_j\right] = 1 - \varepsilon\sum_{j_1} V_{j_1} + \frac{\varepsilon^2}{2!}\sum_{j_1 j_2} V_{j_1} V_{j_2} - \ldots$$

Which gives:

$$W_B(xt|x_0t_0) = \underbrace{\int \frac{dx_1}{\sqrt{4\pi\mathcal{D}\varepsilon}} \cdots \int \frac{dx_N}{\sqrt{4\pi\mathcal{D}\varepsilon}} \frac{1}{\sqrt{4\pi\mathcal{D}\varepsilon}} \exp\left[-\frac{1}{4\mathcal{D}\varepsilon}\sum_j (x_{j+1}-x_j)^2\right]}_{W_\Delta(xt|x_0t_0) \,\equiv\, \int_{C[x_0t_0;xt]} d_W x(\tau)}$$

$$- \varepsilon \sum_{j_1} \underbrace{\int \frac{dx_1}{\sqrt{4\pi\mathcal{D}\varepsilon}} \cdots \int \frac{dx_N}{\sqrt{\cdots}} \frac{1}{\sqrt{\cdots}} \exp\left[-\frac{1}{4\mathcal{D}\varepsilon}\sum_j (x_{j+1}-x_j)^2\right]}_{\text{will be split into } t_0 \to t_{j_1} \,\&\, t_{j_1} \to t} \cdot V(x_{j_1}t_{j_1})$$

$$+ \frac{\varepsilon^2}{2!} \sum_{j_1 j_2} \underbrace{\int \frac{dx_1}{\sqrt{\cdots}} \cdots \int \frac{dx_N}{\sqrt{\cdots}} \frac{1}{\sqrt{\cdots}} \exp[\ \cdots\ ]}_{\text{to split between } t_0, t_{j_1}, t_{j_2}, t} V(x_{j_1},t_{j_1})\, V(x_{j_2},t_{j_2}) + \ldots$$

Consider the 2nd term:

$$(2nd) = -\varepsilon \sum_{j_1} \left[\int_{C[x_0t_0;\,x_{j_1}t_{j_1}]} d_W x(\tau)\right] V(x_{j_1}, t_{j_1}) \left[\int_{C[x_{j_1},t_{j_1}|xt]} d_W x(\tau)\right]$$

$\int dx_{j_1} \to \int dt'$

$$= -\int dt' \int dx'\, W_\Delta(x't'|x_0t_0)\, V(x't')\, W_\Delta(xt|x't')$$

When considering the 3rd term, we must consider two cases; $t_{j_1} < t_{j_2}$ and $t_{j_1} > t_{j_2}$ in the sum $\sum_{j_1 j_2}$.

(a) $\underline{t_{j_1} < t_{j_2}}$

$\int dx_{j_1} \int dx_{j_2}$

$$(3rd) = \frac{\varepsilon^2}{2} \sum_{j_1<j_2} \left[\int_{C[x_0t_0;\,x_{j_1}t_{j_1}]} d_W x(\tau)\right] V(x_{j_1}t_{j_1}) \left[\int_{C[x_{j_1}t_{j_1};\,x_{j_2}t_{j_2}]} d_W x(\tau)\right]$$

$$\times V(x_{j_2}t_{j_2}) \left[\int_{C[x_{j_2}t_{j_2};\,xt]} d_W x(\tau)\right]$$

$\int dx_j\, dx_{j'}$

$$= \frac{\varepsilon^2}{2} \sum_{j<j'} W_\Delta(xt|x_{j'}t_{j'})\, V(x_{j'}t_{j'})\, W_\Delta(x_{j'}t_{j'}|x_jt_j)\, V(x_jt_j)\, W_\Delta(x_jt_j|x_0t_0)$$

~~(a) $t_{j_1} > t_{j_2}$ → the same is obtained, but $\sum_{j<j'} \to \sum_{j>j'}$~~

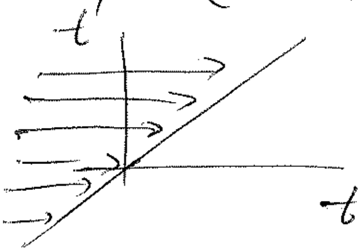
$$\equiv \frac{1}{2} \int_{-\infty}^{\infty} dt' \int_{t'}^{\infty} dt'' \int dx' dx'' \, W_\Delta(xt|x''t'') V(x''t'') W_\Delta(x''t''|x't') V(x't') W_\Delta(x't'|x_0 t_0)$$

$(t'' > t')$

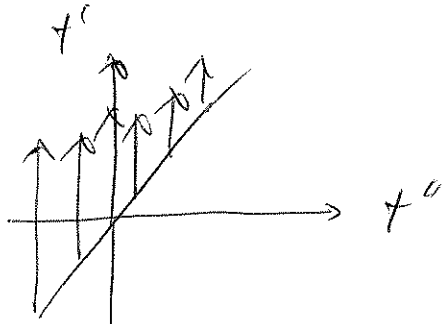
(b) $t_{j_1} > t_{j_2}$

$$\hookrightarrow \frac{1}{2} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{t'} dt'' \int dx' dx'' \, W_\Delta(xt|x't') V(x't') W_\Delta(x't'|x''t'') V(x''t'') W_\Delta(x''t''|x_0 t_0)$$

$(t'' < t')$



here we change into



$$\hookrightarrow \frac{1}{2} \int_{-\infty}^{\infty} dt'' \int_{t''}^{\infty} dt' \; (\text{the same expression}) = \Big| \, t' \leftrightarrow t'' \Big|$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dt' \int_{t'}^{\infty} dt'' \, W_\Delta(xt|x''t'') V(x''t'') W_\Delta(x''t''|x't') V(x't') W_\Delta(x't'|x_0 t_0)$$

which is exactly the same as for the 1st case. We combine both and $1/2$ disappears. Then we can extend dt'' integration to $-\infty$ as for $t'' < t'$ $W_\Delta(x''t''|x't') = 0$ anyway.

$$\hookrightarrow (3rd) = -\int_{-\infty}^{\infty} dt' \int dt'' \, W_\Delta V W_\Delta V W_\Delta$$

This is the same as iterate (1.2.82), i.e. (1.2.79) is indeed the representation of the solution via the path integral.

Brownian particle in an external field

In 3D the Fokker-Planck eq. looks like this:

$$\frac{\partial W}{\partial t} = \mathcal{D}\,\Delta W - \frac{1}{\eta}\,\mathrm{div}\,(\vec{F}\cdot W) \qquad (1.2.87)$$

We would like to derive W via the path integral. We write: for the conservative force (only function of $\vec{r}$, <u>not</u> time):

$$\vec{F} = -\nabla\phi\ :$$

$$W(\vec{r}t|\vec{r}_0 t_0) = e^{\alpha[\phi(\vec{r})-\phi(\vec{r}_0)]}\,\tilde{W}(rt|r_0 t_0) \equiv \psi\tilde{W}$$

with some α. To substitute this into (1.2.87), we need:

$$\nabla\psi = \nabla e^{\alpha(\phi-\phi_0)} = e^{\alpha(\phi-\phi_0)}\alpha\nabla\phi = \alpha\psi\nabla\phi = -\alpha\psi\vec{F},$$

$$\Delta\psi = \nabla\cdot(-\alpha\psi\vec{F}) = -\alpha\nabla\cdot(\psi\vec{F}) = -\alpha[\nabla\psi\cdot\vec{F} + \psi\nabla\cdot\vec{F}]$$

$$= -\alpha[-\alpha\psi\vec{F}\cdot\vec{F} + \psi\nabla\cdot\vec{F}] = \alpha^2\psi F^2 - \alpha\psi\nabla\vec{F};$$

$$\Delta W = \Delta(\psi\tilde{W}) = \Delta\psi\cdot\tilde{W} + 2\nabla\psi\cdot\nabla\tilde{W} + \psi\Delta\tilde{W}$$

$$= (\alpha^2 F^2 - \alpha\nabla\vec{F})\psi\tilde{W} - 2\alpha\psi\vec{F}\cdot\nabla\tilde{W} + \psi\Delta\tilde{W};$$

$$\mathrm{div}(W\vec{F}) = \nabla W\cdot\vec{F} + W\nabla\cdot\vec{F} = \nabla(\psi\tilde{W})\cdot\vec{F} + \psi\tilde{W}\nabla\cdot\vec{F}$$

$$= \nabla\psi\cdot\vec{F}\cdot\tilde{W} + \nabla\tilde{W}\cdot\vec{F}\cdot\psi + \psi\tilde{W}\nabla\cdot\vec{F}$$

$$= -\alpha\psi\vec{F}\cdot\vec{F}\cdot\tilde{W} + \psi\vec{F}\cdot\nabla\tilde{W} + \psi\tilde{W}\nabla\cdot\vec{F}$$

$$= -\alpha\psi F^2\tilde{W} + \psi\vec{F}\cdot\nabla\tilde{W} + \psi\tilde{W}\nabla\cdot\vec{F}$$

Substitute into the equation (1.2.87):

$$\cancel{\psi}\frac{\partial\tilde{W}}{\partial t} = \mathcal{D}[\alpha^2F^2 - \alpha\nabla\vec{F}]\cancel{\psi}\tilde{W} \underbrace{- 2\alpha\mathcal{D}\cancel{\psi}\vec{F}\cdot\nabla\tilde{W}} + \mathcal{D}\cancel{\psi}\Delta\tilde{W}$$

$$- \frac{1}{\eta}[-\alpha\cancel{\psi}F^2\tilde{W} + \underbrace{\cancel{\psi}\vec{F}\cdot\nabla\tilde{W}} + \cancel{\psi}\tilde{W}\nabla\cdot\vec{F}]$$

Choose α so to cancell on $\nabla\tilde{w}$ terms:

$$-2\alpha\mathcal{D} - \frac{1}{\gamma} = 0 \rightarrow \boxed{\alpha = -\frac{1}{2\gamma\mathcal{D}}}$$

Then:

$$\frac{\partial\tilde{w}}{\partial t} = \mathcal{D}\Delta\tilde{w} + \tilde{w}\left\{\mathcal{D}\alpha^2F^2 - \alpha\mathcal{D}\nabla\cdot\vec{F} + \frac{\alpha F^2}{\gamma} - \frac{1}{\gamma}\nabla\cdot\vec{F}\right\}$$

where

$$\{\ldots\} = \mathcal{D}\frac{1}{4\gamma^2\mathcal{D}^2}F^2 + \frac{\cancel{\mathcal{D}}}{2\gamma\cancel{\mathcal{D}}}\nabla\cdot\vec{F} - \frac{1}{2\gamma^2\mathcal{D}}F^2 - \frac{1}{\gamma}\nabla\cdot\vec{F}$$

$$= \left(\frac{1}{4\gamma^2\mathcal{D}} - \frac{1}{2\gamma^2\mathcal{D}}\right)F^2 + \left(\frac{1}{2\gamma} - \frac{1}{\gamma}\right)\nabla\vec{F} = -\frac{F^2}{4\gamma^2\mathcal{D}} - \frac{\nabla\cdot\vec{F}}{2\gamma} \equiv -V(\vec{r})$$

$$\boxed{\frac{\partial\tilde{w}}{\partial t} = \mathcal{D}\Delta\tilde{w} - V(\vec{r})\,\tilde{w}} \quad V\notin r \qquad (1.2.91)$$

with the

$$\boxed{V(\vec{r}) = \frac{F^2}{4\gamma^2\mathcal{D}} + \frac{\nabla\cdot\vec{F}}{2\gamma}} \qquad (1.2.92)$$

serves as an annihilation probability in the Bloch eq. (1.2.91)
Hence $\tilde{w}$ can be written as a path integral, and so W:

$$\boxed{W(\vec{r}\,t|\vec{r}_0\,t_0) = \exp\left[\frac{1}{2\gamma\mathcal{D}}\int_{\vec{r}_0}^{\vec{r}}\vec{F}\cdot d\vec{r}\right]\int_{C[\vec{r}_0,0;\vec{r}\,t]} d_Wx(\tau)\,d_Wy(\tau)\,d_Wz(\tau)\,e^{-\int_0^t V(\vec{r}(\tau))\,d\tau}} \qquad (1.2.93)$$

Feynman-Kac Theorem

- $$W_B(xt|x_0 0) = \int_{C[x_0 0;\, xt]} d_W x(\tau)\, \exp\left[-\int_0^t V(x(s))\,ds\right] \qquad (1.2.94)$$

is the fundamental solution of the Bloch eq.

$$\frac{\partial W_B}{\partial t} = \mathcal{D}\frac{\partial^2 W_B}{\partial x^2} - V(x)\,W_B \qquad (1.2.95)$$

We shall assume $4\mathcal{D} = 1$.

- W_B above satisfies the ESKC relation:

$$\int dx'\, W_B(xt|x't')\, W_B(x't'|x_0 0)$$

$$= \int dx' \int_{C[x_0 0;\, x't']} d_W x(\tau)\, e^{-\int_0^{t'} V ds} \int_{C[x't';\, xt]} d_W x(\tau)\, e^{-\int_{t'}^{t} V ds}$$

$$= \underbrace{\int dx' \int_{C[x_0 0;\, x't']} d_W x(\tau) \int_{C[x't';\, xt]} d_W x(\tau)}_{= \int_{C[x_0 0;\, xt]}} \exp\left[-\int_0^t V ds\right] \equiv W_B(xt|x_0 0)$$

as required.

- The initial condition

$$W_B(xt|x_0 0)\Big|_{t\to 0} = \lim_{t\to 0} \int_{C[x_0 0;\, xt]} d_W x(\tau) \equiv \delta(x - x_0) \qquad (1.2.96)$$

- We should an eq. for W_B. We start from the identity:

$$\exp\left[-\int_0^t ds\, V(x(s))\right] = 1 - \int_0^t ds'\, V(x(s'))\, \exp\left[-\int_0^{s'} V(x(s))\,ds\right] \qquad (1.2.97)$$

Indeed, it is satisfied at $t = 0$. Then, differentiate:

$$e^{-\int_0^t V ds}\left[-V(x(t))\right] = -V(x(t))\exp\left[-\int_0^t V ds\right]$$

which is the same. [~~This is simply expansion of the exponent into the Taylor series~~ Integrate (1.2.97) with the Wiener measure:

$$\int_{C[x_0 0; xt]} d_W x(\tau)\, e^{-\int_0^t V ds} = \underbrace{\left[\int_{C[x_0 0; xt]} d_W x(\tau)\right]}_{\to W_D(xt|x_0 0)} -$$

$$-\int_{C[x_0 0; xt]} d_W x(\tau) \int_0^t ds\, V(x(s)) \exp\left[-\int_0^s ds' V(x(s'))\right]$$

On the left we have $W_B(xt|x_0 0)$, the 1st term in the RHS is W_D; we need to consider the 2nd one:

$$\int_{C[x_0 0; xt]} d_W x(\tau) \int_0^t ds\, V(x(s)) \exp\left[-\int_0^s ds' V(x(s'))\right]$$

$$= \int_0^t ds \underbrace{\int_{C[x_0 0; xt]} d_W x(\tau) \left\{ V(x(s)) \exp\left[-\int_0^s ds' V(x(s'))\right] \right\}}_{\text{split into } 0\to s, \text{ and } s\to t}$$

$$= \int_0^t ds \int dx_s \int_{C[x_0 0; x_s s]} d_W x(\tau)\, V(x(s)) \exp\left[-\int_0^s ds' V(x(s'))\right] \times$$

$$\times \underbrace{\left[\int_{C[x_s s; xt]} d_W x(\tau)\right]}_{W_D} = \int_0^t ds \int dx_s\, W_B(x_s s|x_0 0)\, V(x(s)) \times$$

$$\times W_D(xt|x_s s)$$

Finally, we obtain: (1.2.100)

$$\boxed{W_B(xt|x_0 0) = W_\Delta(xt|x_0 0) - \int_0^t d\tau \int_{-\infty}^{\infty} dx_\tau \, W_\Delta(xt|x_\tau \tau) V(x(\tau)) W_B(x_\tau \tau|x_0 0)}$$

Here $\tau < t$ and $\tau > 0$. If we extend $\int d\tau$ to $\int_{-\infty}^{\infty} d\tau$, then the extra bits $= 0$ because W_Δ and W_B equal to zero at wrong times. Then (1.2.100) becomes (1.2.82).

- From (1.2.100) we can show that W_B satisfies (1.2.95).

$$\hat{L} = \frac{\partial}{\partial t} - \frac{1}{4}\frac{\partial^2}{\partial x^2}$$

$$\hat{L} W_\Delta = 0$$

Act with $\hat{L}$ on both sides of (1.2.100):

$$L W_B = \underbrace{L W_\Delta}_{=0} - \left(\frac{\partial}{\partial t} - \frac{1}{4}\frac{\partial^2}{\partial x^2}\right) \int_0^t d\tau \int dx_\tau \, W_\Delta(xt|x_\tau \tau) V(\tau) W_B(x_\tau \tau|x_0 0)$$

$$\text{RHS} = -\frac{\partial}{\partial t}\left[\int_0^t d\tau \int dx_\tau \, W_\Delta(xt|x_\tau \tau) V_\tau W_B(x_\tau \tau|x_0 0)\right] +$$

$$+ \int_0^t d\tau \int dx_\tau \left(\frac{1}{4}\frac{\partial^2}{\partial x^2} W_\Delta(xt|x_\tau \tau)\right) V_\tau W_B$$

$$= -\int dx_\tau \underbrace{W_\Delta(xt|x_\tau t)}_{\delta(x-x_\tau)} V_\tau W_B(x_\tau t|x_0 0)$$

$$\cancel{- \int_0^t d\tau \int dx_\tau \underbrace{\left(\frac{\partial}{\partial t} W_\Delta(xt|x_\tau \tau)\right)}_{\frac{1}{4}\frac{\partial^2}{\partial x^2} W_\Delta} V_\tau W_B} + \cancel{\int_0^t d\tau \int dx_\tau \left(\frac{1}{4}\frac{\partial^2}{\partial x^2} W_\Delta\right) V_\tau W_B}$$

$$= - V(x)\, W_B(xt|x_0 0)\,,$$

i.e. $\hat{L} W_B = -V(x) W_B$, i.e. it is (1.2.95) as required.