(1.2) Wiener path integrals & stochastic processes

The Fokker-Planck eq.

$W(x_t, t \mid x_\tau, \tau)$ – transition probability $\tau \to t$

Assumptions:

$$\lim_{t\to\tau} \left\langle \frac{x_t - x_\tau}{t-\tau} \right\rangle = A(x_\tau, \tau) \quad (\text{"mean velocity"}) \qquad (1.2.15)$$

$$\lim_{t\to\tau} \frac{\langle (x_t - x_\tau)^2 \rangle}{t-\tau} = 2B(x_\tau, \tau) \qquad (1.2.16)$$

$$\lim_{t\to\tau} \left\langle \frac{(x_t - x_\tau)^3}{t-\tau} \right\rangle = 0 \qquad (1.2.17)$$

$$\langle f(x_t - x_\tau) \rangle = \int dx_t\, f(x_t - x_\tau)\, W(x_t, t \mid x_\tau, \tau)$$

We would like to derive an eq. for W. It satisfies the ESKC:

$$W(x_t, t \mid x_0, 0) = \int dx_\tau\, W(x_\tau, \tau \mid x_0, 0)\, W(x_t, t \mid x_\tau, \tau)$$

Multiply it by an arbitrary $g(x)$ which satisfies:

$$g(x) \xrightarrow[x\to\pm\infty]{} 0, \quad g'(x) \xrightarrow[x\to\pm\infty]{} 0 \qquad (1.2.18)$$

$$\hookrightarrow \int_{-\infty}^{\infty} g(x_t)\, W(x_t, t \mid x_0, 0)\, dx_t =$$

$$= \int dx_\tau\, dx_t\; g(x_t)\, W(x_t, t \mid x_\tau, \tau)\, W(x_\tau, \tau \mid x_0, 0) \qquad (1.2.19)$$

Expand $g(x)$:

$$g(x_t) = g(x_\tau) = (x_t - x_\tau)\, g'(x_\tau) + \frac{1}{2}(x_t - x_\tau)^2 g''(x_\tau) +$$

$$+ \frac{1}{6}(x_t - x_\tau)^3 g'''(\xi),$$

where $\xi \in [x_\tau, x_t]$. Substitute this in (1.2.19):

$$\text{LHS} = \cancel{g(x_\xi) \int W\, dx_t} \int_{-\infty}^{\infty} g(x_t)\, W(x_t, t \mid x_0, 0)\, dx_t$$

$$\mathrm{RHS} = \int dx_\tau \underbrace{dx_t\, g(x_\tau)\, W(x_t,t|x_\tau,\tau)}\, W(x_\tau,\tau|x_0,0)$$

$$= \int dx_\tau \Big[\underbrace{\int dx_t\, W(x_t,t|x_\tau,\tau)}_{=1}\Big]\, g(x_\tau)\, W(x_\tau,\tau|x_0,0)$$

$$+ \underbrace{\int dx_\tau dx_t\, (x_t-x_\tau)\, W(x_\tau,\tau|x_0,0)\, W(x_t,t|x_\tau,\tau)}_{\int dx_\tau \langle x_t - x_\tau\rangle W(x_\tau,\tau|x_0,0)}\, g'(x_\tau)$$

$$+ \frac{1}{2}\int dx_\tau \underbrace{dx_t\, (x_t-x_\tau)^2\, W(x_\tau,\tau|x_0,0)\, W(x_t,t|x_\tau,\tau)}_{\frac{1}{2}\int dx_\tau \langle (x_t-x_\tau)^2\rangle W(x_\tau,\tau|x_0,0)}\, g''(x_\tau)$$

$$+ \frac{1}{6}\int dx_\tau \underbrace{dx_t\, (x_t-x_\tau)^3\, W(x_t,t|x_\tau,\tau)\, W(x_\tau,\tau|x_0,0)}_{\frac{1}{6}\int dx_\tau \langle (x_t-x_\tau)^3\rangle W(x_\tau,\tau|x_0,0) \;=\; 0}\, g'''(\xi)$$

$$= \int dx_\tau\, g(x_\tau)\, W(x_\tau,\tau|x_0,0) + \int dx_\tau\, g'(x_\tau) \langle x_t - x_\tau\rangle W(x_\tau,\tau|x_0,0)$$

$$+ \frac{1}{2}\int dx_\tau\, g''(x_\tau)\, \langle (x_t-x_\tau)^2\rangle\, W(x_\tau,\tau|x_0,0)$$

$$+ \frac{1}{6}\int dx_\tau\, g'''(\xi)\, W(x_\tau,\tau|x_0,0)\, \langle (x_t-x_\tau)^3\rangle$$

Take the 1st term in the LHS and divide by $t-\tau$:

$$\cancel{\int dx_t\, g(x_t)\, \underbrace{\frac{W(x_t,t|x_0,0) - W(x_\tau,\tau|x_0,0)}{t-\tau}} \to \frac{\partial W(x_t,t}{\partial t}}$$

$$LHS = \left[\frac{\int dx_t\, g(x_t)\, W(x_t t | x_0, 0) - \int dx_\tau\, g(x_\tau)\, W(x_\tau \tau | x_0, 0)}{t-\tau} \right]_{t \to \tau}$$

$$= \int dx\, g(x)\, \frac{W(x t | x_0 0) - W(x \tau | x_0, 0)}{t-\tau} \to \int dx_t\, g(x_t)\, \frac{\partial W(x_t t | x_0 0)}{\partial t}$$

$$RHS = \int dx_\tau\, g'(x_\tau) \underbrace{\left[\frac{\langle x_t - x_\tau \rangle}{t-\tau} \right]}_{\to A(x_\tau; \tau)} W(x_\tau, \tau | x_0, 0)$$

$$+ \frac{1}{2} \int dx_\tau\, g''(x_\tau) \underbrace{\left[\frac{\langle (x_t - x_\tau)^2 \rangle}{t-\tau} \right]}_{\to 2B(x_\tau, \tau)} W(x_\tau, \tau | x_0, 0) + \left(\begin{matrix} \text{3rd term} \\ \to 0 \end{matrix} \right)$$

$$\overset{t}{\to} = \int dx_t\, g'(x_t)\, A(x_t, t)\, W(x_t, t | x_0 0)$$

$$+ \int dx_t\, g''(x_t)\, B(x_t, t)\, W(x_t t | x_0, 0) = \left| \text{by parts} \right|$$

$$= \left\{ \underbrace{g(x_t)}_{\overset{\shortparallel}{0}} \frac{\partial}{\partial x_t} \left[A(x_t, t) W(x_t, t | x_0, 0) \right] \Big|_{-\infty}^{\infty} - \int dx_t\, g(x_t) \frac{\partial}{\partial x_t} \left[A(x_t, t) W(x_t t | x_0 \right. \right.$$

$$+ \left\{ \underbrace{g'(x_t)}_{\overset{\shortparallel}{0}} \frac{\partial}{\partial x_t} [BW] \Big|_{-\infty}^{\infty} - \int g'(x_t) \frac{\partial}{\partial x_t} [BW]\, dx_t \right\} = \left(\begin{matrix} \text{once} \\ \text{again for} \\ \text{the 2nd term} \end{matrix} \right)$$

$$= - \int dx_t\, g(x_t) \frac{\partial}{\partial x_t} \left[A(x_t, t) W(x_t t | x_0 0) \right] \cancel{+} \left\{ \underbrace{g(x_t)}_{\overset{\shortparallel}{0}} \frac{\partial^2}{\partial x_t^2} [BW]\, dx_t \Big|_{-\infty}^{\infty} \right.$$

$$\left. \cancel{+} \int g(x_t) \frac{\partial^2}{\partial x_t^2} [BW]\, dx_t \right\} =$$

$$= -\int dx_t\, g(x_t)\left[\frac{\partial}{\partial x_t}[AW] - \frac{\partial^2}{\partial x_t^2}[BW]\right]$$

Since LHS = RHS for any $g(x)$, we obtain:

$$\boxed{\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_t}(AW) - \frac{\partial^2}{\partial x_t^2}(BW) = 0} \qquad (1.2.21)$$

which is the Fokker-Plank (2nd Kolmogorov) equation.

If $A = 0$, $B = D =$ Constant, we obtain

$$\frac{\partial W}{\partial t} = D\frac{\partial^2 W}{\partial x^2} \quad \text{- diffusion eq.}$$

1.2.2. Brownian particle under external force

- $$m\ddot{x} + \gamma\dot{x} = \underbrace{F}_{\text{external force}} + \underbrace{\phi}_{\text{random}}$$

Over long times $t \gg m/\gamma$ the $m\ddot{x}$ term can be dropped.

$$\rightarrow \quad \dot{x} = f + \phi \qquad (1.2.25)$$

- Here we consider the harmonic force:

$$\dot{x}(t) + kx(t) = \phi(t) \qquad (1.2.31)$$

(Ornstein-Uhlenbeck process)

Compare this with $\dot{y} = \phi(t) \rightarrow \dot{y} = \dot{x} + kx$

$$\boxed{y(t) = x(t) + k\int_0^t d\tau\, x(\tau)}, \quad \boxed{y(0) = x(0)} \qquad (1.2.34)$$

Reversely, $\dot{x} + kx = 0 \rightarrow x = c\,e^{-kt}$

$$\dot{x} + kx = \dot{y}(t) \rightarrow x(t) = c(t)\,e^{-kt},$$

$$\left[\dot{c}\,e^{-kt} - \cancel{k c e^{-kt}}\right] + \cancel{k c e^{-kt}} = \dot{y}$$

$$\dot{c} = \dot{y}\,e^{kt} \rightarrow c(t) = \int_0^t d\tau\, \dot{y}(\tau)\, e^{+k\tau} + C'$$

$$= \cancel{y(t) e^{+kt}\Big|_0^t - (k)\int_0^t d\tau\, y(\tau) e^{+k\tau}} = \cancel{y(t)e^{\pm kt} + k\int y(\tau) e^{+k\tau} d\tau}$$

~~so that~~ $X(0) = C(0) \to C(0) = y(0)$

$$= C' + y(\tau) e^{k\tau}\Big|_0^t - \int_0^t d\tau\, y(\tau)\, k\, e^{k\tau} =$$

$$= C' + y(t) e^{kt} - y(0) - k\int_0^t y(\tau) e^{k\tau} d\tau$$

$$\hookrightarrow X(t) = C' e^{-kt} + y(t) - y(0) e^{-kt} - k\int_0^t y(\tau)\, e^{k(\tau-t)} d\tau$$

C' is obtained from $X(0) = y(0)$:

$$X(0) = C' + y(0) - y(0) = C' \to$$

$$\boxed{X(t) = y(t) - k\int_0^t e^{-k(t-\tau)} y(\tau)\, d\tau} \qquad (1.2.35)$$

- Integration is to be understood in terms of the stochastic calculus. For linear functions integration by parts is legitimate.
- If we know probability density for the process $y(t)$,

$$d_W y(\tau) = \exp\left[-\int_0^t d\tau\, \dot{y}(\tau)^2\right] \prod_{\tau=0}^{t} \frac{dy(\tau)}{\sqrt{\pi\, d\tau}} \qquad (4D=1)$$

then probability density for $X(t)$ related to $y(t)$ via (1.2.35) is simply given by the change of coordinates:

$$d_W y(\tau) = J \exp\left\{-\int_0^t d\tau\, (\dot{x} + kx)^2\right\} \prod \frac{dx(\tau)}{\sqrt{\pi\, d\tau}}$$

the Jacobian $|\partial y / \partial x|$

The Jacobian was calculated earlier for a general change of variables

$$y(t) = x(t) + \lambda\int_0^t ds\, K(s)\, x(s) \qquad (1.1.122)$$

$$J = \exp\left[\frac{\lambda}{2}\int_0^t ds\, K(s)\right] \qquad (1.1.127)$$

So in our case

$$J = \exp\left[-\frac{k}{2}\int_0^t ds \cdot 1\right] = e^{kt/2}$$

The initial points of $x(t)$, $y(t)$ are fixed to $x(0)=y(0)$; we put $x_t = y_t$ for the final points as well.

$$\hookrightarrow W(x_t t|x_0,0) = e^{kt/2}\int_{C[x_0,0;x_t,t]} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{\pi d\tau}} \exp\left[-\int_0^t d\tau\,(\dot{x}+kx)^2\right]$$

Here

$$\int_0^t (\dot{x}+kx)^2 d\tau = \int d\tau\,(\dot{x}^2 + 2kx\dot{x} + k^2x^2) = \int_0^t \dot{x}^2 d\tau + \underbrace{2k\int_0^t d\tau \cdot x \cdot \frac{dx}{d\tau}}_{k\int_0^t d(x^2)}$$

$$+ k^2\int_0^t x^2 d\tau = k(x_t^2 - x_0^2) + \int_0^t \dot{x}^2 d\tau + k^2\int_0^t x^2 d\tau$$

and we obtain:

$$\boxed{W(x_t t|x_0 0) = e^{kt/2 - k(x_t^2 - x_0^2)} \underbrace{\int_{C[x_0,0;x_t t]} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{\pi d\tau}} \exp\left[-\int_0^t \dot{x}^2 d\tau - \int_0^t d\tau\, k^2 x^2\right]}_{\int_{C[x_0,0;x_t t]} d_W x(\tau)\, \exp\left[-\int_0^t d\tau\, k^2 x(\tau)^2\right]}} \quad (1.2.38)$$

[this is the integral from Example 1.5 for $p(\tau) \equiv k^2$]

- General non-stationary and non-linear external force

$$\dot{x}(t) + f(x(t),t) = \phi(t) \quad (1.2.39)$$

$$\hookrightarrow \dot{y} = \dot{x} + f \rightarrow y(t) = x(t) + \int_0^t f(x(\tau),\tau)\,d\tau \quad (1.2.40)$$

The Jacobian in this case after discretisation:

$$\cancel{y_1 = x_1 + \tfrac{1}{2}\varepsilon} \quad y_i = x_i + \left[f(x_1,\varepsilon) + f(x_2,2\varepsilon) + \dots + \tfrac{1}{2} f(x_N, N\varepsilon)\right]\varepsilon$$

or

$$y_1 = x_1 + \tfrac{1}{2} f(x_1,\varepsilon)\,\varepsilon$$

$$y_2 = x_2 + \left[f(x_1,\varepsilon) + \tfrac{1}{2} f(x_2,2\varepsilon)\right]\varepsilon$$

$$\boxed{\varepsilon = t/N}$$

$$\left|\frac{\partial y}{\partial x}\right| = \begin{vmatrix} 1 + \frac{\varepsilon}{2} f'(x_1,\varepsilon) & 0 & 0 & \dots \\ f'(x_1,\varepsilon)\varepsilon & 1 + \frac{\varepsilon}{2} f'(x_2,2\varepsilon) & 0 & \dots \\ f'(x_1,\varepsilon)\varepsilon & f'(x_2,2\varepsilon)\varepsilon & 1 + \frac{\varepsilon}{2} f'(x_3,3\varepsilon) & 0 \dots \\ \vdots & \vdots & & \vdots \\ f'(x_1,\varepsilon)\varepsilon & f'(x_2,2\varepsilon)\varepsilon & \dots & 1 + \frac{\varepsilon}{2} f'(x_N,N\varepsilon) \end{vmatrix}$$

$$= \prod_{i=1}^{N}\left[1 + \frac{\varepsilon}{2} f'(x_i,\varepsilon)\right] \Rightarrow \prod_{i=1}^{N} \exp\left[\frac{\varepsilon}{2} f'(x_i,\varepsilon) + O(\varepsilon^2)\right]$$

$$\hookrightarrow \exp\left[\frac{1}{2}\sum_{i=1}^{N} \varepsilon f'(x_i,\varepsilon)\right] \to \exp\left[\frac{1}{2}\int_0^t f'(x(\tau),\tau)\,d\tau\right] \qquad (1.2.42)$$

where $f' = \dfrac{\partial f}{\partial x}$.

Therefore,

$$\boxed{W(x_t t \mid x_0 0) = \int_{C[x_0 0;\, x_t t]} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{\pi\, d\tau}} \exp\Big[-\underbrace{\int_0^t d\tau\, \dot{y}(\tau)^2}_{\int_0^t d\tau\,[\dot{x} + f(x(\tau),\tau)]^2}\Big] \cdot J} \qquad (1.2.43)$$

- Brownian particles with interactions

Two particles, no interaction:

$$\dot{y}_1 = \Phi_1 \; ; \quad \dot{y}_2 = \Phi_2 \qquad (1.2.44)$$

$$W(y_{1t} t; y_{2t} t \mid y_{10} 0; y_{20} 0) = \int_{C[y_{10} 0;\, y_{1t} t]} d_W y(\tau) \int_{C[y_{20} 0;\, y_{2t} t]} d_W y(\tau)$$

$$= W(y_{1t}\, t | y_{10}\, 0)\, W(y_{2t}\, t | y_{20}\, 0) \qquad (1.2.45)$$

a product (since they are entirely independent).

Let now 2 particles (X-particles) do interact:

$$\begin{cases} \mathring{X}_1 + k_{11} X_1 + k_{12} X_2 = \phi_1 \\ \mathring{X}_2 + k_{21} X_1 + k_{22} X_2 = \phi_2 \end{cases} \qquad (1.2.46)$$

and

$$\mathring{X} + k X = \phi \;,\; k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \;,\; X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \;,\; \phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\mathring{y} = \phi \;,\; y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\hookrightarrow \mathring{y} = \mathring{x} + kX \;\rightarrow\; y(t) = x(t) + k \int_0^t x(s)\, ds \qquad (1.2.52)$$

all via vectors and matrices.

The Jacobian: – the same as above but now with 2x2 matrices:

$$y_i = x_i + k\varepsilon\left[x_1 + x_2 + \dots + \tfrac{1}{2} x_N\right]$$

$$J = \left|\frac{\partial y_i}{\partial x_j}\right| = \begin{vmatrix} 1\!\!1 + \frac{1}{2}k\varepsilon & 0 & \dots & & \\ k\varepsilon & 1\!\!1 + \frac{1}{2}k\varepsilon & 0 & \dots & \\ k\varepsilon & k\varepsilon & 1\!\!1 + \frac{1}{2}k\varepsilon & 0 & \dots \\ \dots & & & & \end{vmatrix} = \prod_{i=1}^{N} \left|1\!\!1 + \tfrac{1}{2}k\varepsilon\right|$$

$$\cancel{= \prod_{i=1}^{N} \exp\left(\tfrac{1}{2}k\varepsilon\right)} = \prod_{i=1}^{N} \begin{vmatrix} 1 + \frac{1}{2}k_{11}\varepsilon & \frac{1}{2}k_{12}\varepsilon \\ \frac{1}{2}k_{21}\varepsilon & 1 + \frac{1}{2}k_{22}\varepsilon \end{vmatrix}$$

$$= \prod_{i=1}^{N} \Big\{ \underbrace{\left(1 + \tfrac{1}{2}k_{11}\varepsilon\right)\left(1 + \tfrac{1}{2}k_{22}\varepsilon\right) - \tfrac{1}{4}k_{12}k_{21}\varepsilon^2}_{1 + \frac{1}{2}(k_{11}+k_{22})\varepsilon + O(\varepsilon^2)} \Big\} \rightarrow \exp\left[\tfrac{1}{2}\varepsilon \sum_{i=1}^{N} (k_{11}+k_{22})\right]$$

$$= \exp\left[\tfrac{1}{2}\varepsilon N\, \mathrm{Tr}\, k\right] = e^{t\, \mathrm{Tr}\, k/2} \qquad (1.2.54)$$

Therefore,

$$\boxed{W(\vec{x}_t t|\vec{x}_0 t) = e^{\frac{t}{2}\operatorname{Tr}(k)} \int_{C[\vec{x}_0 0;\vec{x}_t t)} \prod \frac{dx_1(\tau)}{\sqrt{\pi\, d\tau}} \prod \frac{dx_2(\tau)}{\sqrt{\pi\, d\tau}} \times \exp\left[-\int_0^t \underbrace{(\dot{x}(s)+k\,x(s))^2}_{\sum_{i=1}^{2}\left[\dot{x}_i(s)+\sum_{j=1}^{2}k_{ij}x_j(s)\right]^2} ds\right]} \qquad (1.2.58)$$

- Interacting k particles subject to general forces

$$\dot{x}_i(t) + f_i(x(t),t) = \varphi_i(t) \quad , \quad i = 1,\dots,k$$

In ~~vector~~/matrix form:

$$\dot{x}(t) + f(x(t),t) = \varphi(t) \equiv \dot{y}(t) \Rightarrow \{y_i^j, x_i^j\} \; (j = \text{time})$$

$$\cancel{\mathcal{J} = \left(\frac{\partial y}{\partial x}\right) = \begin{vmatrix} \partial y_1^1/\partial x_1^1 & \partial y_1^1/\partial x_2^1 & \dots & \partial y_1^1/\partial x_k^1 \\ \partial y_2^1/\partial x_1^1 & \partial y_2^1/\partial x_2^1 & \dots & \partial y_2^1/\partial x_k^1 \\ \vdots & & & \\ \partial y_k^1/\partial x_1^1 & & & \end{vmatrix}} \qquad \boxed{\vec{y}(t) = \vec{x}(t) + \int_0^t \vec{f}(\vec{x},s)\,ds}$$

$$y_i^j = x_i^j + \left[f_i(\vec{x}^1,\varepsilon) + f_i(\vec{x}^2, 2\varepsilon) + \dots + \frac{1}{2} f_i(\vec{x}^N, N\varepsilon)\right]\varepsilon$$

$$\mathcal{J} = \begin{vmatrix} (\partial y^1/\partial x^1) & 0 & \dots & \\ (\partial y^2/\partial x^1) & (\partial y^2/\partial x^2) & 0 & \dots \\ \vdots & \vdots & \vdots & \\ (\partial y^{(N)}/\partial x^1) & (\partial y^{(N)}/\partial x^2) & \dots & (\partial y^{(N)}/\partial x^{(N)}) \end{vmatrix}$$

where

$$\left(\partial y^{(j)}/\partial x^{(j)}\right) = \begin{vmatrix} \partial y_1^{(j)}/\partial x_1^{(j)} & \partial y_1^{(j)}/\partial x_2^{(j)} & \dots & \partial y_1^{(j)}/\partial x_k^{(j)} \\ \partial y_2^{(j)}/\partial x_1^{(j)} & \dots & & \\ \vdots & & & \vdots \\ \partial y_k^{(j)}/\partial x_1^{(j)} & \dots & & \partial y_k^{(j)}/\partial x_k^{(j)} \end{vmatrix}$$

$$= \begin{pmatrix} 1+\frac{1}{2}\frac{\partial f_1}{\partial x_1}\varepsilon & \frac{1}{2}\frac{\partial f_1}{\partial x_2} & \frac{1}{2}\frac{\partial f_1}{\partial x_3} & \dots & \frac{1}{2}\frac{\partial f_1}{\partial x_k} \\ \frac{1}{2}\frac{\partial f_i}{\partial x_1}\varepsilon & 1+\frac{1}{2}\frac{\partial f_i}{\partial x_2}\varepsilon & & \dots & \frac{1}{2}\frac{\partial f_i}{\partial x_k}\varepsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2}\frac{\partial f_i}{\partial x_k}\varepsilon & \frac{1}{2}\frac{\partial f_i}{\partial x} & & & \end{pmatrix}$$

We differentiate only wrt to the j-th time: (diagonal matrices):

$$y_i^1 = x_i^1 + \frac{1}{2} f_i(x^1, \varepsilon)\,\varepsilon$$

$$y_i^2 = x_i^2 + \left[f_i(x^1, \varepsilon) + \frac{1}{2} f_i(x^2, 2\varepsilon)\right]\varepsilon$$

$$y_i^3 = x_i^3 + \left[f_i(x^1, \varepsilon) + f_i(x^2, 2\varepsilon) + \frac{1}{2} f_i(x^3, 3\varepsilon)\right]\varepsilon$$

$$\dots$$

So, the $j=1$ block is ($f_i^j \leftarrow j = \text{time}$)

$$A^1(\varepsilon) = \begin{vmatrix} 1+\frac{1}{2}\frac{\partial f_1^1}{\partial x_1}\varepsilon & \frac{1}{2}\frac{\partial f_1^1}{\partial x_2}\varepsilon & \frac{1}{2}\frac{\partial f_1^1}{\partial x_3}\varepsilon & \dots & \frac{1}{2}\frac{\partial f_1^1}{\partial x_k}\varepsilon \\ \frac{1}{2}\frac{\partial f_2^1}{\partial x_1}\varepsilon & 1+\frac{1}{2}\frac{\partial f_2^1}{\partial x_2}\varepsilon & \frac{1}{2}\frac{\partial f_2^1}{\partial x_3}\varepsilon & \dots & \frac{1}{2}\frac{\partial f_2^1}{\partial x_k}\varepsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1+\frac{1}{2}\frac{\partial f_k^1}{\partial x_k}\varepsilon \end{vmatrix}$$

$$\simeq \left(\text{up to the linear in } \varepsilon \text{ terms}\right) \simeq 1 + \frac{1}{2}\sum_{i=1}^{k} \frac{\partial f_i^1}{\partial x_k}\varepsilon$$

At time $j=2$ (the 2nd diagonal element in the matrix J we would have

$$A^2(\varepsilon) = 1 + \frac{1}{2}\sum_{i=1}^{k}\frac{\partial f_i^2}{\partial x_i}\varepsilon \quad , \text{etc.}$$

So

$$J = \prod_{j=1}^{N} A^j(\varepsilon) = \prod_{j=1}^{N}\left(1 + \frac{1}{2}\sum_{i=1}^{K}\frac{\partial f_i^j}{\partial x_i}\varepsilon\right) \simeq$$

$$\simeq \prod_{j=1}^{N} \exp\left[\frac{1}{2}\sum_{i=1}^{k}\frac{\partial f_i^j}{\partial x_i}\varepsilon\right] = \exp\left[\frac{1}{2}\sum_{i=1}^{k}\left(\sum_{j=1}^{N}\frac{\partial f_i^j}{\partial x_i}\varepsilon\right)\right]$$

$$\hookrightarrow \boxed{\exp\left[\frac{1}{2}\sum_{i=1}^{k}\int_0^t \frac{\partial f_i(\vec{x}(s),s)}{\partial x_i}\,ds\right]}$$

and then

$$\boxed{\begin{aligned} W(\vec{x}_t t|\vec{x}_0 0) &= \int_{C[\vec{x}_0,0;\vec{x}_t t]} \prod \frac{dx_1(\tau)}{\sqrt{\pi d\tau}} \cdots \prod \frac{dx_k(\tau)}{\sqrt{\pi d\tau}} \\ &\times \exp\left[\frac{1}{2}\sum_{i=1}^{k}\int_0^t \frac{\partial f_i}{\partial x_i}ds\right] \exp\left[-\sum_{i=1}^{k}\int_0^t ds \left(\dot{x}_i(s) + f_i(x(s),s)\right)^2\right] \end{aligned}}$$

(1.2.58)

1.2.4. Brownian particles with inertia

- 2nd order DE

$$m\ddot{x} + \gamma\dot{x} = F + \phi$$

is first changed into two 1st order eqs:

$$\begin{cases} \dot{x} - v = 0 \\ \dot{v} + \frac{\gamma}{m}v - \frac{F}{m} = \frac{1}{m}\phi \end{cases} \qquad (1.2.59)$$

We introduce "temperatures":

$$\begin{cases} \dot{v} + \frac{\gamma}{m}v - \frac{1}{m}f = T_1 \frac{1}{m}\phi_1 \\ \dot{x} - v = T_2\phi_2 \end{cases} \qquad (1.2.60)$$

[$T_2 \to 0$ and $T_1 \to 1$ later on]. "Temperatures" are introduced via a substitution: $\mathcal{D} \to \mathcal{D}T$.

• Consider the process

$$\ddot{x} = \phi \quad \longrightarrow \quad \begin{cases} \dot{v} = T_1 \phi_1 \\ \dot{x} - v = T_2 \phi_2 \end{cases} \qquad (1.2.64)$$

Then:

$$W^{T_1 T_2}(x_t v_t t \mid x_0 v_0 0) = \int_{C[x_0 v_0 0;\, x_t v_t t]} \prod \frac{dv(\tau)}{\sqrt{\pi T_1 d\tau}} \prod \frac{dx(\tau)}{\sqrt{\pi T_2 d\tau}} \times$$

$$\times \exp\left\{ -(?) \right\} \qquad (1.2.65)$$

This is the case of "interacting" two particles considered above:

$$\begin{cases} \dot{x}_1 + k_{11} x_1 + k_{12} x_2 = \phi_1 \\ \dot{x}_2 + k_{21} x_1 + k_{22} x_2 = \phi_2 \end{cases} \longrightarrow \quad k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

with $\mathrm{Tr}\, k = 0 \rightarrow$ Jacobian $= \underline{1}$. Then, we only have

$$(?) = \frac{1}{T_1} \int_0^t \dot{v}^2 ds + \frac{1}{T_2} \int_0^t (\dot{x} - v)^2 ds$$

in the exponential.

• Use the discrete time approximation for a part of the integral:

$$\prod \frac{1}{\sqrt{\pi\, \delta\tau \cdot T_2}} \exp\left(-\frac{1}{T_2} \int_0^t (\dot{x} - v)^2 ds \right)$$

$$\longrightarrow \prod_{i=0}^{N} \frac{1}{\sqrt{\pi \varepsilon T_2}} \cdot \exp\left[-\frac{1}{T_2} \sum_{j=0}^{N} \left(\frac{x_{j+1} - x_j}{\varepsilon} - v_j \right)^2 \varepsilon \right]$$

$$= \prod_{i=0}^{N} \left(\frac{1}{\sqrt{\pi \varepsilon T_2}} \exp\left[-\frac{1}{T_2} \left(\frac{x_{j+1} - x_j}{\varepsilon} - v_j \right)^2 \varepsilon \right] \right)$$

We know:

$$\delta(x) = \lim_{\alpha \to \infty} \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}$$

or, if $T = 1/\alpha^2 \longrightarrow$ $\boxed{\delta(x) = \lim_{T \to 0} \frac{1}{\sqrt{\pi T}} e^{-x^2/T}}$

so that in the $T_2 \to 0$ limit we get

$$\prod_{j=0}^{N} \frac{1}{\sqrt{\varepsilon}}\, \delta\left[\left(\frac{x_{j+1}-x_j}{\varepsilon} - v_j\right)\sqrt{\varepsilon}\right] = \prod_{j=0}^{N} \frac{1}{\varepsilon}\, \delta\left(\frac{x_{j+1}-x_j}{\varepsilon} - v_j\right)$$

$$\equiv \prod_{\tau=0}^{t} \frac{1}{\varepsilon}\, \delta(\dot{x}(\tau) - v(\tau)) = \cancel{\frac{1}{\varepsilon^N}} \underbrace{\delta[\dot{x}(\tau) - v(\tau)]}_{\substack{\text{product of } \delta\text{-f.} \\ \text{at each slice of time}}} \prod_{\tau=0}^{t} \frac{1}{d\tau}$$

Insert into (1,2.65) [$T_1 \to 1$ here], while considering the $dx(\tau)$

$$\cancel{W(x_1 v_1 t \mid x_0 v_0 0) = \int_{C[\ldots]} \prod \frac{dv(\tau)}{\sqrt{\pi d\tau}} \cdot \prod \frac{dx(\tau)}{\sqrt{\pi d\tau}} \exp\left[-\int_0^t d\tau\, \dot{v}^2\right] \times}$$

$$\cancel{\times\, \delta[\dot{x}(\tau) - v(\tau)]}$$

integral in the discretised way:

$$\int \frac{dx(\tau)}{\sqrt{\pi d\tau}} \ldots \to \int \frac{dx_1}{\varepsilon} \int \frac{dx_2}{\varepsilon} \ldots \int \frac{dx_N}{\varepsilon} \prod_{j=0}^{N} \delta\left(\frac{x_{j+1}-x_j}{\varepsilon} - v_j\right)$$

$$= \int dx_1 \int dx_2 \ldots \int dx_N \prod_{j=0}^{N} \delta(x_{j+1} - x_j - v_j \varepsilon)$$

$$= \int dx_1 \ldots \int dx_N\, \delta(x_1 - x_0 - v_0 \varepsilon)\, \delta(x_2 - x_1 - v_1 \varepsilon) \ldots \delta(x_N - x_{N-1} - v_{N-1}\varepsilon) \times$$

$$\times\, \delta(\underbrace{x_{N+1}}_{x_t} - x_N - v_N \varepsilon) = \int dx_1 \ldots \int dx_{N-1}\, \delta(x_1 - x_0 - v_0 \varepsilon) \times \ldots \times \delta(x_{N-1} - x_{N-2} - v_{N-2}\varepsilon)$$

$$\times\, \delta(x_t - v_N \varepsilon - x_{N-1} - v_{N-1}\varepsilon) =$$

$$= \int dx_1 \ldots \int dx_{N-2} \ldots \delta(x_{N-3} - x_{N-4} - v_{N-4}\varepsilon)\, \delta(x_{N-2} - x_{N-3} - v_{N-3}\varepsilon) \times$$

$$\times\, \delta(x_t - x_{N-2} - (v_{N-2} + v_{N-1} + v_N)\varepsilon) = \ldots =$$

$$= \cancel{\int dx_1}\, \delta[x_t - x_0 - (v_0 + v_1 + \ldots + v_N)\varepsilon] = \delta\left(x_t - x_0 - \int_0^t v(\tau)\, d\tau\right)$$

This finally gives:

$$\boxed{W(x_t v_t t \mid x_0 v_0 0) = \int_{C[v_0 0;\, v_t t]} \underbrace{d_w v(\tau)}_{\prod_{\tau=0}^{t} \frac{dv(\tau)}{\sqrt{\pi\, d\tau}} \exp\left(-\int_0^t \dot v(\tau)^2 d\tau\right)} \delta\left(x(t) - x(0) - \int_0^t v(\tau)\, d\tau\right)} \tag{1.2.70}$$

1.2.5. Bloch eq. and Feynman-Kac formula

- Bloch eq. Particles can be removed (annihilated) with probability $V(x,t)$ per unit time. Then

$$\underbrace{j = -D\frac{\partial \rho}{\partial x}}_{\text{current}} \quad \text{and} \quad \underbrace{\frac{\partial \rho}{\partial t}}_{\text{change of } \rho} = -\underbrace{\frac{\partial j}{\partial x}}_{\text{reduced due to current}} - \underbrace{V\rho}_{\text{reduced due to annihilation}} \Rightarrow \boxed{\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} - V\rho} \tag{1.2.76}$$

~~For~~ Via the path integral:

$$\boxed{W_B(x_t\, t \mid x_0 t_0) = \int_{C[x_0, t_0;\, x_t, t]} d_w x(\tau)\, \underbrace{\exp\left[-\int_0^t V(x, s)\, ds\right]}_{\text{probability to remain on the trajectory up to time } t}} \tag{1.2.79}$$

- $t > t_0$ condition can be imposed by putting $\theta(t-t_0)$ in front:

$$W_D'(xt \mid x_0 t_0) = \theta(t-t_0)\, W_D(xt \mid x_0 t_0)$$

where W_D satisfies usual diffusion eq:

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) W_D' = \theta(t-t_0) \underbrace{\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) W_D}_{=0} + \delta(t-t_0) \underbrace{W_D(xt \mid x_0 t_0)}_{\to\, \delta(x-x_0) \text{ when } t \to t_0}$$