1.1.5 Substitutions

- $$\int \cdots \int f(x_1, \dots, x_N)\, dx_1 \dots dx_N = \int \cdots \int \underbrace{\tilde{f}(y_1, \dots, y_N)}_{f(x_1(y), \dots, x_N(y))} J\, dy_1 \dots dy_N$$

$$J = \frac{\partial(x_1, \dots, x_N)}{\partial(y_1, \dots, y_N)}$$ where $x_1 = x_1(y_1, \dots, y_N) = x_1(y)$, $x_2 = x_2(y), \dots, x_N = x_N(y)$

- Integral eq.

$$y(t) = x(t) + \lambda \int_a^b K(t,s)\, x(s)\, ds$$

If $K(t,s) = 0$ for $s > t \Rightarrow y(t) = x(t) + \lambda \int_a^t K(t,s)\, x(s)\, ds$

- If $K(t,s)$ does not depend on ~~t~~, t, then $y(t) = x(t) + \lambda \int_a^t K(s)\, x(s)\, ds$

$K(t,s)$ must be discontinuous at $s = t$:

$$\dot{y} = \dot{x} + \lambda K(t,t) X(t) + \lambda \int_a^t \left(\partial_t K(t,s)\right) X(s)\, ds$$

If $K(t,s)$ does not depend on t, then $\partial_t K = 0$ and $\dot{y} = \dot{x} + \lambda K(t,t)\, x(t)$

If $K(t,t) = 0$, then $y(t) = x(t) + C$, i.e. this substitution is not good.

Normally, via averaging:

$$K(t) = \frac{1}{2}\Big[\underset{\overset{\shortparallel}{0}}{K(t+0,s)} + K(t-0,s)\Big]_{t \to s} = \frac{1}{2} K(\cancel{s}-0,s)\Big|_{\cancel{s\to t}}$$

$$= \frac{1}{2} K(t,s)\Big|_{t \to s-0} \qquad (1.1.129)$$

Let us ~~obtain~~ the Jacobian in this case:

$$y(t) = x(t) + \lambda \int_a^t K(\cancel{s}) X(s)\, ds, \quad \cancel{K(t,t) = \tfrac{1}{2}}$$

$$\hookrightarrow y_i = x_i + \lambda \left(K_1 x_1 + \dots + \frac{1}{2} K_N x_N\right) \varepsilon$$

which is:

$$y_1 = x_1 + \lambda \frac{1}{2} K_1 x_1 \varepsilon$$

$$y_2 = x_2 + \lambda \left(K_1 x_1 + \frac{1}{2} K_2 x_2 \varepsilon\right)$$

$$y_3 = x_3 + \lambda \left(K_1 x_1 + K_2 x_2 + \frac{1}{2} K_3 x_3 \varepsilon\right) \dots$$

$$\left|\frac{\partial y}{\partial x}\right| = \begin{vmatrix} 1+\frac{1}{2}\lambda k_1\varepsilon & 0 & 0 & 0 & 0 \\ \lambda k_1 & 1+\frac{1}{2}\lambda k_2\varepsilon & 0 & 0 & 0 \\ \lambda k_1 & \lambda k_2 & 1+\frac{1}{2}\lambda k_3\varepsilon & 0 & 0 \\ \lambda k_1 & \lambda k_2 & \lambda k_3 & 1+\frac{1}{2}\lambda k_4\varepsilon & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \prod_{i=1}^{N}\left(1+\frac{1}{2}\lambda k_i\varepsilon\right) = \prod_{i=1}^{N}\exp\left[\frac{1}{2}\lambda k_i\varepsilon + O(\varepsilon^2)\right] \to \exp\left[\int_a^t \frac{1}{2}\lambda k(s)\,ds\right]$$

$$\boxed{J(\lambda) = \exp\left[\frac{\lambda}{2}\int_a^t k(s)\,ds\right]} \tag{1.1.127}$$

Example

$$\int_{C[0,0;t]} d_w y(\tau) = 1 \tag{1.1.128}$$

Change of function: $$y(t) = x(t) - \int_0^t ds\, \frac{\dot{\mathcal{D}}(s)}{\mathcal{D}(s)}\, x(s) \tag{1.1.129}$$

where $$\ddot{\mathcal{D}}(s) + p(s)\mathcal{D}(s) = 0,\quad \mathcal{D}(t) = 1,\quad \dot{\mathcal{D}}(t) = 0 \tag{1.1.130}$$

We have after the substitution: (4D = 1 here again)

$$\int_{C[0,0;t]} d_w y(\tau) = \int_{C[0,0;t]} \prod_{\tau=0}^{t} \frac{dy(\tau)}{\sqrt{\pi d\tau}} \exp\left[-\int_0^t \dot{y}^2(\tau)\,d\tau\right]$$

$$= \int \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{\pi d\tau}} \exp\left[-\int_0^t d\tau\left(\dot{x} - \frac{\dot{\mathcal{D}}}{\mathcal{D}}x\right)^2\right] \underbrace{\exp\left[-\frac{1}{2}\int_0^t \frac{\dot{\mathcal{D}}(s)}{\mathcal{D}(s)}\,ds\right]}_{\text{Jacobian}}$$

Here

$$\int_0^t \frac{\dot{\mathcal{D}}(s)}{\mathcal{D}(s)}\,ds = \int \frac{d\mathcal{D}(s)}{\mathcal{D}(s)} = \ln|\mathcal{D}(s)|\Big|_0^t = \underbrace{\ln \mathcal{D}(t)}_{=0} - \ln \mathcal{D}(0) =$$

$= -\ln \mathcal{D}(0)$, since $\mathcal{D}(t) = 1$, and hence

$$\exp\left[-\frac{1}{2}\int_0^t \frac{\dot{\mathcal{D}}(s)}{\mathcal{D}(s)}\,ds\right] = \exp\left[+\frac{1}{2}\ln \mathcal{D}(0)\right] = \sqrt{\mathcal{D}(0)}$$

The other exponent contains the integral:

$$\int_0^t d\tau \left(\dot{x} - \frac{\dot{\mathcal{D}}}{\mathcal{D}} x\right)^2 = \int_0^t \dot{x}^2 d\tau + \int_0^t d\tau \left(\frac{\dot{\mathcal{D}}}{\mathcal{D}}\right)^2 x^2 - 2\int_0^t d\tau\, \dot{x} \frac{\dot{\mathcal{D}}}{\mathcal{D}} x$$

Here

$$2\int_0^t \underbrace{d\tau\ x \cdot \dot{x}}_{dx^2} \frac{\dot{\mathcal{D}}}{\mathcal{D}} \xrightarrow{\text{by parts}} x^2 \frac{\dot{\mathcal{D}}}{\mathcal{D}}\Big|_0^t - \int_0^t x^2 \left(\frac{\dot{\mathcal{D}}}{\mathcal{D}}\right)' d\tau$$

$$= \underbrace{x^2(t)}_{=0} \frac{\dot{\mathcal{D}}(t)}{\mathcal{D}(t)} - \underbrace{x^2(0)}_{=0} \frac{\dot{\mathcal{D}}(0)}{\mathcal{D}(0)} - \int_0^t d\tau \frac{d}{d\tau}\left(\frac{\dot{\mathcal{D}}}{\mathcal{D}}\right) x^2(\tau)$$

$(C[0,0;t]\,!)$

$$= - \int_0^t d\tau \frac{\ddot{\mathcal{D}}\mathcal{D} - (\dot{\mathcal{D}})^2}{\mathcal{D}^2} x^2 = - \int_0^t \frac{\ddot{\mathcal{D}}}{\mathcal{D}} x^2 d\tau + \int_0^t d\tau \left(\frac{\dot{\mathcal{D}}}{\mathcal{D}}\right)^2 x^2$$

so that

$$\int_0^t d\tau \left(\dot{x} - \frac{\dot{\mathcal{D}}}{\mathcal{D}} x\right)^2 = \int_0^t \dot{x}^2 d\tau + \cancel{\int_0^t d\tau \left(\frac{\dot{\mathcal{D}}}{\mathcal{D}}\right)^2 x^2} + \int_0^t \frac{\ddot{\mathcal{D}}}{\mathcal{D}} x^2 d\tau - \cancel{\int_0^t d\tau \left(\frac{\dot{\mathcal{D}}}{\mathcal{D}}\right)^2 x^2}$$

$$= \int_0^t \dot{x}^2 d\tau + \int_0^t \frac{\ddot{\mathcal{D}}}{\mathcal{D}} x^2 d\tau$$

But $\ddot{\mathcal{D}} = -p(\tau)\mathcal{D}(\tau) \rightarrow \int_0^t \dot{x}^2 d\tau + \int_0^t (-p(\tau))\, x^2(\tau)\, d\tau$

Therefore,

$$\int_{C[0,0;t]} d_W y(\tau) = \int \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{\pi d\tau}} \exp\left[-\int_0^t \dot{x}^2 d\tau\right] \exp\left[+\int_0^t p(\tau) x^2(\tau) d\tau\right] \times$$

$$\times \sqrt{\mathcal{D}(0)} \equiv \sqrt{\mathcal{D}(0)}\, I_4,$$

where I_4 we considered above (with $\alpha p(\tau) \equiv -p(\tau)$).
The result was $I_4 = 1/\sqrt{\mathcal{D}(0)}$. We get the same as our

$$\int_{C[0,0;t]} = 1 \rightarrow I_4 = 1/\sqrt{\mathcal{D}(0)}$$

• Example: correlation function

$$F[x(\tau)] = x(s)x(\rho), \quad s, \rho \text{ - two fixed times}$$
$$t_0 < s < \rho < t$$

$$\int_{C[0,t_0;t]} d_W x(\tau) \cdot x(s)x(\rho) \overset{\text{def}}{\equiv} \langle x(s)x(\rho) \rangle_W$$

We have: (still $4D = 1$):

$$\int_{C[0,t_0;t]} = \int_{-\infty}^{\infty} dx_s dx_\rho \cdot x_s x_\rho \cdot \underbrace{\int_{C[0,t_0;x_s,s]} d_W x(\tau)}_{\frac{1}{\sqrt{\pi(s-t_0)}} \exp\left[-\frac{x_s^2}{s-t_0}\right]} \times \underbrace{\int_{C[x_s,s;x_\rho,\rho]} d_W x(\tau)}_{\frac{1}{\sqrt{\pi(\rho-s)}} \exp\left[-\frac{(x_\rho - x_s)^2}{\rho-s}\right]} \times$$

$$\times \underbrace{\int_{C[x_\rho,\rho;t]} d_W x(\tau)}_{=1}$$

$$= \int_{-\infty}^{\infty} dx_s dx_\rho \frac{1}{\sqrt{\pi(s-t_0)}\sqrt{\pi(\rho-s)}} e^{-x_s^2/(s-t_0)} e^{-\frac{(x_\rho-x_s)^2}{\rho-s}} \cdot x_s x_\rho$$

Here

$$\int_{-\infty}^{\infty} dx_\rho \, x_\rho \, e^{-\frac{(x_\rho-x_s)^2}{\rho-s}} = \underbrace{\int dx_\rho (x_\rho - x_s) e^{\cdots}}_{=0} + x_s \underbrace{\int e^{-\cdots} dx_\rho}_{\sqrt{\pi(\rho-s)}} =$$

$$= \sqrt{\pi(\rho-s)} \cdot x_s$$

Then,

$$\int_{-\infty}^{\infty} dx_s \, x_s^2 \frac{1}{\sqrt{\pi(s-t_0)}} \cdot e^{-x_s^2/(s-t_0)} = \frac{1}{\sqrt{\pi(s-t_0)}} \cdot \frac{\sqrt{\pi}}{2\left(\frac{1}{s-t_0}\right)^{3/2}}$$

$$= \frac{1}{2}(s-t_0), \quad (s < \rho):$$

$$\boxed{\int_{C[0,t_0;t]} d_W x(\tau) \, x(s) x(\rho) = \frac{1}{2} \min(s-t_0, \rho-t_0)} \quad (1.1.13)$$

Example $\quad$ s – fixed time (< t): $\quad \overbrace{\qquad\qquad}^{\frac{1}{\sqrt{\pi s}} e^{-x_s^2/s}}$

$$I_2^{cond} = \int_{C[0,0;x,t]} d_w x(\tau)\, x(s) = \int_{-\infty}^{\infty} dx_s \cdot x_s \underbrace{\int_{C[0,0;x_s,s]} d_w x(\tau)}_{} \times$$

$$\times \underbrace{\int_{C[x_s,s;x,t]} d_w x(\tau)}_{\frac{1}{\sqrt{\pi(t-s)}} \exp\left(-\frac{(x-x_s)^2}{t-s}\right)} = \int_{-\infty}^{\infty} dx_s \cdot x_s \frac{1}{\sqrt{\pi s}} \frac{1}{\sqrt{\pi(t-s)}} e^{-x_s^2/s} e^{-\frac{(x-x_s)^2}{t-s}}$$

Here $\displaystyle -\frac{x_s^2}{s} - \frac{(x-x_s)^2}{t-s} \equiv -\frac{x^2}{t} - \frac{t}{s(t-s)}\left(x_s - \frac{s}{t}x\right)^2$

$$\hookrightarrow \int_{-\infty}^{\infty} x_s \exp\left[-\frac{t}{s(t-s)}\left(x_s - \frac{s}{t}x\right)^2\right] dx_s = \underbrace{\int \left(x_s - \frac{s}{t}x\right) \exp[\ldots]\, dx_s}_{= 0}$$

$$+ \frac{s}{t} x \underbrace{\int \exp[\ldots]\, dx_s}_{\sqrt{\pi\frac{(t-s)s}{t}}} = \frac{s}{t} x \cdot \cancel{\sqrt{\frac{t}{\pi(t-s)s}}} \sqrt{\frac{\pi(t-s)\cdot s}{t}}$$

so that

$$I_2^{cond} = \frac{1}{\sqrt{\pi s}\sqrt{\pi(t-s)}} \frac{s}{t} \cdot x \cdot \frac{\sqrt{\pi(t-s)s}}{\sqrt{t}} e^{-x^2/t}$$

$$= \frac{x\, s}{\sqrt{\pi}\, t^{3/2}} e^{-x^2/t}$$