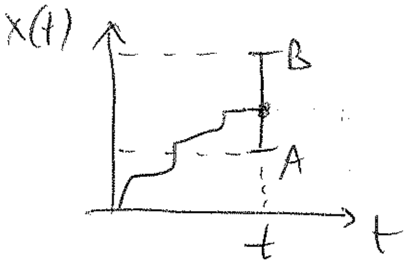
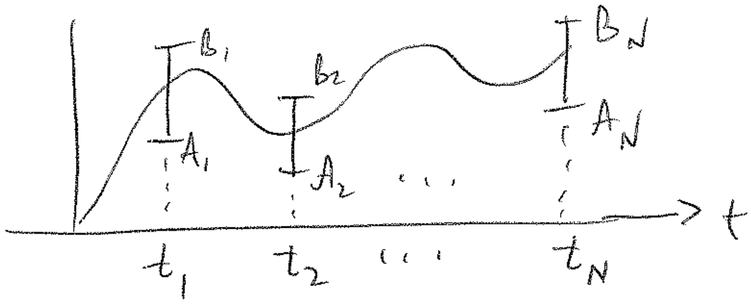
1.1.2. Wiener's path integrals



$$\mathbb{P}\left[x(t) \in [A,B]\right] = \int_A^B dx\, w(x,t) \qquad (1.1.48)$$

↳ to be anywhere within $x \in [A,B]$ at t.

Probability to go inside a tube of "gates":



$\boxed{\text{a product of individual probabilities}}$

↑

$\boxed{\text{Markov chain}}$

$$\mathbb{P}\left[x(t_1) \in [A_1, b_1]; \ldots ; x(t_N) \in [A_N, B_N]\right] =$$

$$= \int_{A_1}^{B_1} dx_1\, w(x_1 - x_0, t_1 - t_0) \times \int_{A_2}^{b_2} dx_2\, w(x_2 - x_1, t_2 - t_1) \times \ldots \times$$

$$\times \int_{A_N}^{B_N} dx_N\, w(x_N - x_{N-1}, t_N - t_{N-1}) = \left(\begin{array}{l} \text{assuming } t_0 = 0 \\ \text{and } x_0 = 0 \end{array} \right) =$$

$$= \int_{A_1}^{B_1} dx_1\, \frac{e^{-x_1^2/4\mathcal{D}t_1}}{\sqrt{4\pi\mathcal{D}t_1}} \times \int_{A_2}^{b_2} dx_2\, \frac{e^{-(x_2 - x_1)^2/4\mathcal{D}(t_2 - t_1)}}{\sqrt{4\pi\mathcal{D}(t_2 - t_1)}} \times \ldots$$

$$\ldots \times \int_{A_N}^{B_N} dx_N\, \frac{e^{-(x_N - x_{N-1})^2/4\mathcal{D}(t_N - t_{N-1})}}{\sqrt{4\pi\mathcal{D}(t_N - t_{N-1})}} \qquad (1.1.50)$$

- Stochastic process is obtained via a continuous limit of time increments

$$\Delta t = t_i - t_{i-1} \to 0, \quad 1 \le i \le N \ (N \to \infty)$$

For a general Markov process

$$\boxed{W(x,t \mid x_0,t_0)\, dx \overset{Def}{=} \mathbb{P}\left[x(t) \in [x, x+dx], \text{ given } x(t_0)=x_0 \ \boxed{t>t_0}\right]}$$

Initially, $W(x,t) = \delta(x-x_0)$ at $t \to t_0$, and the transition ~~pb~~ probabilities become Gaussian → Wiener process.

- Probability for the brownian particle moves along $x(t)$ via an ∞ number of 'gates' (combining all exponents):

$$\lim_{\substack{\Delta t_i \to 0 \\ N\to\infty}} \exp\left\{-\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{4D(t_i - t_{i-1})}\right\} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{4\pi D (t_i - t_{i-1})}}$$

Here $\exp\{\dots\}$ can be also rewritten as:

$$\exp\{\dots\} = \exp\left\{-\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{4D(t_i - t_i)^2}\Delta t_i\right\} =$$

$$= \exp\left\{-\sum_{i=1}^{N} \frac{1}{4D}\dot{x}_i^2 \Delta t_i\right\} = \exp\left\{-\frac{1}{4D}\int_0^t \dot{x}^2 d\tau\right\}$$

and we arrive at the probability for a single trajectory:

$$\lim \boxed{\exp\left\{-\frac{1}{4D}\int_0^t d\tau\, \dot{x}(\tau)^2\right\} \prod_{t=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D\, d\tau}}} \quad (1.1.51)$$

probability inside thin tube around $x(\tau)$

- Notations:

$C(x_1 t_1; x_2, t_2)$ – all trajectories starting at $x_1 t_1$, ending at $x_2 t_2$.

$C(x_1 t_1; [A,B], t_2)$ – ending at the gate $[A,B]$;

$C(x_1, t_1; t_2)$ – for $[A, B]$ (ending gate) to be from the whole $\mathbb{R}$.

For instance, $C(0,0;t)$ – all trajectories starting at the origin and ending anywhere.

- Probability to find the particle at the gate $[A,B]$:

$$\mathbb{P}\left[x(t)\in[A,B]\right] = \int_{C[0,0;[A,B],t]} \underbrace{\exp\left\{-\frac{1}{4D}\int_0^t d\tau\, \dot{x}(\tau)^2\right\}\prod_{\tau=0}^{t}\frac{dx(\tau)}{\sqrt{4\pi D\, d\tau}}} \qquad (1.1.52)$$

sum over a continuous set of all trajectories

At the same time, this probability is

$$\int_A^B dx\, w(x,t) = \int_A^B dx\, \frac{1}{\sqrt{4\pi D t}}\, e^{-x^2/4Dt}$$

If $A = B = X_t$, then

$$\mathbb{P}\left[x(t) = X_t\right] = \int_{C[0,0;X_t,t]} \underbrace{d_W x(\tau)}_{\text{Wiener measure}} \qquad (1.1.53)$$

$$\equiv \frac{1}{\sqrt{4\pi D t}}\exp\left(-\frac{X_t^2}{4Dt}\right)$$

Here the measure is conditional since X_t is fixed. The measure in

$$\int_{C[0,0;t]} d_W x(\tau)$$

is unconditional.

Also,

$$\boxed{\int_{C[0,0;t]} d_W x(\tau) = 1} \qquad (1.1.55)$$

Obviously:

$$\underbrace{\int_{C[0,0;t]} d_W x(\tau)}_{\text{all possible endpoints}} = \int_{-\infty}^{\infty} dx_t \underbrace{\int_{C[0,0;x_t,t]} d_W x(\tau)}_{\text{fixed endpoints at } x_t} \qquad (1.1.56)$$

• <u>ESKC condition</u>.

$$W(x-x_0,t-t_0) = \int_{-\infty}^{\infty} dx' \, W(x-x',t-t') \, W(x'-x_0,t'-t_0)$$

in terms of Wiener's integrals:

$$\boxed{\int_{C[x_0,t_0;x,t]} d_W x(\tau) = \int_{-\infty}^{\infty} dx' \int_{C[x',t';x,t]} d_W x(\tau) \int_{C[x_0 t_0;x' t']} d_W x(\tau)} \qquad (1.1.58)$$

which means that the integral over paths can always be split into a product of integrals over smaller paths.

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1.1.3 Wiener's theorem

- The Wiener path integral $= 0$ for any <u>discontinuous</u> or <u>differentiable</u> trajectory.

(A) Discontinuous trajectories

$0 < t < 1$ is split into 2^m intervals to

$$t_j = \frac{j}{2^m}\ ;\ j = 0, \dots, 2^m - 1 \qquad t_0 = 0 \quad t_1 = \frac{1}{2^m} \quad t_2 = \frac{2}{2^m} \quad \dots \quad t_m = 1 \quad t$$

Consider all $x(t)$ which have a discontinuity:

$$|x(t_{j+1}) - x(t_j)| > \frac{h}{A^m} \qquad (1.1.64)$$

$$h > 0, \quad A \in [1, \sqrt{2}\,].$$

We shall also use $D = 1/4 \to W(x, t|00) = \frac{1}{\sqrt{\pi t}} e^{-x^2/t}$ for the transition probability.

$\boxed{\text{Step 1}}$ Estimate the measure for all trajectories which are discontinuous as in (1.1.64)

$$\int_{C(0,0;t)} d_W x(\tau)\, \underbrace{\chi[x(\tau)]}_{} =$$

$$\to \begin{cases} 1, & \text{if } x(\tau) \text{ satisfies (1.1.64)} \\ 0, & \text{if } x(\tau) \text{ does not} \end{cases}$$

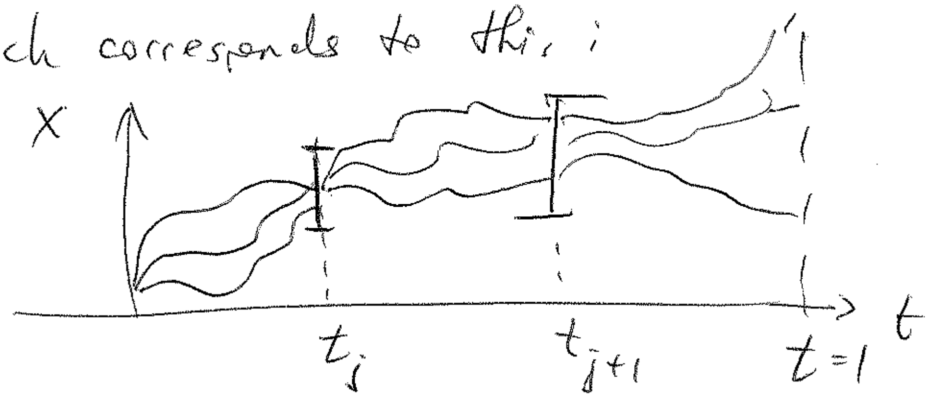
$$\cancel{= \int_{|x_1 - x_2| > \frac{h}{A^m}} dx_1 dx_2 \int_{C[0,0;x_1,t_j]}}$$

Let $x_{j+1} = x(t_{j+1})$, $x_j = x(t_j)$, then

$$\to \int_{|x_{j+1} - x_j| > h/A^m} dx_{j+1}\, dx_j \cdot \int_{C[0,0;x_j,t_j]} d_W x(\tau)\ \times$$

$$\times \int_{C[x_j,t_j;x_{j+1},t_{j+1}]} d_w x(\tau) \times \int_{C[x_{j+1},t_{j+1};t=1]} d_w x(\tau)$$

which corresponds to this:



The last integral $= 1$ (normalisation, it should arrive somewhere). Also,

$$\int_{C[x,t|x',t']} d_w x(\tau) = \frac{1}{\sqrt{\pi(t'-t)}} e^{-(x'-x)^2/(t'-t)},$$

so that

$$\int_{C[0,0;t]} d_w x(\tau) \overbrace{\mathcal{X}[x(\tau)]}^{\text{functional}} = \left| \begin{matrix} \xi_1 = x_{j+1} \\ \xi_2 = x_j \end{matrix} \right|$$

$$= \int_{|\xi_1-\xi_2|>\frac{h}{A^m}} d\xi_1 d\xi_2 \; \frac{1}{\sqrt{\pi \frac{j}{2^m}}} e^{-\xi_1^2/(j/2^m)} \cdot \frac{1}{\sqrt{\pi\underbrace{\left(\frac{(j+1)}{2^m} - \frac{j}{2^m}\right)}_{1/2^m}}} e^{-(\xi_1-\xi_2)^2/\frac{1}{2^m}}$$

$$= \sqrt{\frac{2^m}{\pi j}} \sqrt{\frac{2^m}{\pi}} \int_{|\xi_1-\xi_2|>\frac{h}{A^m}} d\xi_1 d\xi_2 \; e^{-2^m\xi_1^2/j} e^{-2^m(\xi_1-\xi_2)^2}$$

Making change of variables:

$$(\xi_1-\xi_2)2^{m/2} = \eta, \quad \xi_1 = \xi \to \begin{cases} \xi_1 = \xi \\ \xi_2 = \xi - \eta 2^{-m/2} \end{cases}$$

Jacobian:

$$\frac{\partial(\xi_1,\xi_2)}{\partial(\eta,\zeta)} = \begin{vmatrix} \partial\xi_1/\partial\eta & \partial\xi_2/\partial\eta \\ \partial\xi_1/\partial\zeta & \partial\xi_2/\partial\zeta \end{vmatrix} = \begin{vmatrix} 1 & -2^{-m/2} \\ 0 & 1 \end{vmatrix} = 2^{-m/2}$$

$$\hookrightarrow 2^{-m/2}\frac{2^m}{\pi\sqrt{j}} \int\limits_{|\eta|>\frac{h}{A^m}2^{m/2}} d\eta \int d\zeta\, e^{-\left(\frac{2^m}{j}\right)\cdot\zeta^2}\, e^{-\cancel{2^m}\eta^2/\cancel{2^m}} =$$

$$= \frac{2^{m/2}}{\sqrt{j}\,\pi}\int\limits_{|\eta|>h(\sqrt{2}/A)^m} d\eta\, e^{-\eta^2}\underbrace{\left[\int_{-\infty}^{\infty} d\zeta\, e^{-\left(\frac{2^m}{j}\right)\zeta^2}\right]}_{\sqrt{\frac{\pi}{2^m/j}} = \sqrt{\frac{\pi j}{2^m}} = \sqrt{\pi j}\,2^{-m/2}}$$

$$= \frac{\cancel{\sqrt{2^m}}\,1}{\sqrt{\pi}}\int\limits_{|\eta|>h(\sqrt{2}/A)^m} d\eta\, e^{-\eta^2} = \frac{2}{\sqrt{\pi}}\int_{h(\sqrt{2}/A)^m}^{\infty} e^{-\eta^2}\,d\eta$$

We use inequality (Abramovitz & Stegun):

$$\int_x^\infty e^{-t^2}dt \le \frac{e^{-x^2}}{x+\sqrt{x^2+\frac{4}{\pi}}} < \frac{e^{-x^2}}{2x} \quad \text{as} \quad x+\sqrt{x^2+\tfrac{4}{\pi}} > 2x$$

Then,

$$\int\limits_{C[0,0;t]} d_w x(\tau)\,\chi[x(\tau)] = \frac{2}{\sqrt{\pi}}\int_{h\theta^m}^{\infty} e^{-\eta^2}\,d\eta <$$

$$< \frac{2}{\sqrt{\pi}}\,\frac{e^{-h^2\theta^{2m}}}{2h\theta^m} = \frac{e^{-h^2\theta^{2m}}}{\sqrt{\pi}\,h\,\theta^m}, \quad \theta = \sqrt{2}/A.$$

$1 < \theta < \sqrt{2}$ as $1 < A < \sqrt{2}$,

$\boxed{\text{Step 2}}$ Estimate the measure for all trajectories from (1.1.64) for all possible m and j:

$$\ldots \leq \sum_{m=1}^{\infty} \sum_{j=0}^{2^m-1} \frac{e^{-h^2\theta^{2m}}}{\sqrt{\pi}\, h\, \theta^m} = \sum_{m=1}^{\infty} 2^m \frac{e^{-h^2\theta^{2m}}}{\sqrt{\pi}\, h\, \theta^m} =$$

$$= \frac{1}{h\sqrt{\pi}} \sum_{m=1}^{\infty} e^{-h^2\theta^{2m}} \cdot \left(\frac{2}{\theta}\right)^m \cancel{\leq}$$

Since $1 < \theta < \sqrt{2}$, $\theta^2 \in [1,2]$ $\to$ ~~$e^{-h^2(\theta^2)^m} < e^{-h^2}$~~

For large $h \to h^2 > m/(\theta^{2m}-1)$, then

$$e^{-h^2\theta^{2m}} \cancel{\leq} e^{-h^2(\theta^{2m}-1)} e^{-h^2} < e^{-h^2} e^{-m}$$

$$\to \frac{1}{h\sqrt{\pi}} \sum_{m=1}^{\infty} e^{-h^2} \left(\frac{2}{e\theta}\right)^m = \frac{e^{-h^2}}{h\sqrt{\pi}} \frac{1}{1-\frac{2}{e\theta}} \sim \frac{e^{-h^2}}{h}$$

$\boxed{\text{Step 3}}$ All such sets have zero measure:

$e^{-h^2}/h \to 0$ as $h \to \infty$ [But this corresponds to ∞ discontinuity only ???]

$\boxed{\text{Step 4}}$ For discont. functions

$$|x(t_{j+1}) - x(t_j)| > h\,(t_{j+1} - t_j)^{\log_2 A} = h\left(\frac{j+1}{2^m} - \frac{j}{2^m}\right)^{\log_2 A}$$

$$= h\left(1/2^m\right)^{\log_2 A} = h\, 2^{-m\log_2 A} = h\, 2^{\log_2 A^{-m}} = h\, A^{-m}$$

[the last two steps of the proof are not clear!]

- **Differentiable trajectories**

$$|x(t_{j+1}) - x(t_j)|_{t_{j+1}\to t_j} \le h\,(t_{j+1}-t_j) = \frac{h}{2^m}$$

The difference is only now in the interval for ξ_1, ξ_2:

$$\int_{C[0,0;t]} d_W x(\tau)\cdot \chi[x(\tau)] = \int_{|\xi_1-\xi_2|<\frac{h}{2^m}} d\xi_1\, d\xi_2\, \frac{2^m}{\pi\sqrt{j}}\, e^{-\frac{2^m}{j}\xi_1^2}\, e^{-2^m(\xi_1-\xi_2)^2}$$

$$= \cancel{2^{m/2}}\frac{1}{\sqrt{\pi}} \int_{|y|\le \frac{h}{2^m}2^{m/2} = \frac{h}{2^{m/2}}} dy\, e^{-y^2} = \frac{2}{\sqrt{\pi}}\int_0^{h/2^{m/2}} e^{-y^2}\, dy$$

As $m\to\infty$, $h/2^{m/2} \to 0$ and the $\int \to 0$ as well.
[Why?]

$\boxed{\text{Conclusion: trajectories are continuous, but zig-zag like}}$