Wigner's operators

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n - # of degrees of freedom,

- Any operator can be written as: (1)

$$\hat{A} = \sum_{ab} |a) A_{ab} (b| \equiv \sum_{ab} A_{ab} \varphi_a(x) \varphi_b^*(x') \equiv \underbrace{\langle x|A|x'\rangle}_{\text{coordinate representation}}$$

$$A_{ab} = \langle a|\hat{A}|b\rangle \quad , \quad \langle a|b\rangle = \delta_{ab}$$

- We define the Wigner operator of A as:

$$\boxed{A_W(r,p) = \int ds\, e^{ips/\hbar} \langle r - \frac{s}{2} | \hat{A} | r + \frac{s}{2} \rangle} \quad (2)$$

$ps \to p\cdot s$ (n-dimensional)

Here p - n-dim vector ("momentum")
r - n-dim vector ("position")
s - n-dim vector

- Some properties:

(i) $$\int dp\, A_W(r,p) = \int dp \int ds\, e^{ips/\hbar} \sum_{ab} A_{ab} \varphi_a^*(r+\frac{s}{2}) \varphi_b(r-\frac{s}{2})$$

$$= \int ds \underbrace{\left(\int dp\, e^{ips/\hbar}\right)}_{(2\pi\hbar)^n \delta(s)} \sum_{ab} \dots = (2\pi\hbar)^n \sum_{ab} A_{ab} \varphi_a^*(r) \varphi_b(r)$$

$$\equiv (2\pi\hbar)^n \langle r|\hat{A}|r\rangle \quad (3)$$

(ii) $$\int dr\, A_W(r,p) = \int dr \int ds\, e^{ips/\hbar} \sum_{ab} A_{ab} \varphi_a^*(r+\frac{s}{2}) \varphi_b(r-\frac{s}{2})$$

Change variables:

$$\begin{cases} x_1 = r + \frac{s}{2} \\ x_2 = r - \frac{s}{2} \end{cases} \to \begin{cases} r = \frac{1}{2}(x_1+x_2) \\ s = x_1 - x_2 \end{cases} \qquad dr\,ds = \left|\frac{\partial(r,s)}{\partial(x_1,x_2)}\right| dx_1\, dx_2$$

$$\frac{\partial(r,s)}{\partial(x_1,x_2)} = \begin{vmatrix} 1/2 & 1/2 \\ 1 & -1 \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1 \Rightarrow dr\,ds = dx_1\, dx_2$$

$$\int dr\, A_w(r,p) = \int dx_1 dx_2\, e^{ip(x_1-x_2)/\hbar} \sum_{ab} A_{ab}\, \varphi_a^*(x_1)\, \varphi_b(x_2)$$

$$= \sum_{ab} A_{ab} \left(\int dx_1\, \varphi_a(x_1) e^{-ipx_1/\hbar} \right)^* \left(\int dx_2\, \varphi_b(x_2)\, e^{-ipx_2/\hbar} \right)$$

$$= \sum_{ab} A_{ab}\, \underbrace{\varphi_a^*(p)}_{\text{functions in the } p\text{-representation.}}\, \varphi_b(p) \tag{4}$$

(iii) $A_w^*(r,p) = \int ds\, e^{-ips/\hbar} \sum_{ab} A_{ab}^*\, \varphi_a(r+\frac{s}{2})\, \varphi_b^*(r-\frac{s}{2})$

Change $s \to -s = s'$:

$$\int ds = \int_{-\infty}^{\infty} ds = \int_{+\infty}^{-\infty} -ds' \equiv \int_{-\infty}^{\infty} ds'$$

$$A_w^*(r,p) = \int ds'\, e^{ips'/\hbar} \sum_{ab} A_{ab}^*\, \varphi_a(r-\frac{s'}{2})\, \varphi_b^*(r+\frac{s'}{2})$$

$$A_{ab}^* = A_{ba}$$

$\hookrightarrow$ $$A_w^*(r,p) = \int ds\, e^{ips/\hbar} \underbrace{\sum_{ab} A_{ba}\, \varphi_b^*(r+\frac{s}{2})\, \varphi_a(r-\frac{s}{2})}_{\langle r+\frac{s}{2} | \hat{A} | r-\frac{s}{2} \rangle} \equiv A_w(r,p)$$

$$\boxed{A_w^*(r,p) = A_w(r,p)} \tag{5}$$

- Define the "classical" distribution function

$$\boxed{f_w(r,p) \equiv \rho_w(r,p)(2\pi\hbar)^{-n}} \tag{6}$$

$$\boxed{\rho_w(r,p) = \int ds\, e^{ips/\hbar} \langle r - \frac{s}{2} | \hat{\rho} | r + \frac{s}{2} \rangle}$$

and $\hat{\rho}$ is the statistical operator, and consider ~~the~~ average of an operator $\hat{A}$:

$$\langle \hat{A} \rangle = Tr(A\rho) = \sum_{ab} A_{ab}\, \rho_{ba} = \sum_{ab} \langle a | \hat{A} | b \rangle \langle b | \hat{\rho} | a \rangle$$

On the other hand, introduce $A_w(r,p)$ and $f_w(r,p)$, and consider the "classical" analog:

$$\int f_w(r,p)\, A_w(r,p)\, dr\, dp = \int f_w A_w^* \, dr\, dp =$$

$$= (2\pi\hbar)^{-n} \int dr\, dp \left[\int ds\, e^{ips/\hbar} \langle r - \tfrac{s}{2} | \hat{\rho} | r + \tfrac{s}{2} \rangle \right] \times$$

$$\times \left[\int ds'\, e^{ips'/\hbar} \langle r - \tfrac{s'}{2} | \hat{A} | r + \tfrac{s'}{2} \rangle \right]^*$$

$$= (2\pi\hbar)^{-n} \int dr\, dp \int ds\, ds'\, e^{ip(s-s')/\hbar} \times$$

$$\times \langle r - \tfrac{s}{2} | \hat{\rho} | r + \tfrac{s}{2} \rangle \langle r + \tfrac{s'}{2} | \hat{A} | r - \tfrac{s'}{2} \rangle$$

Integration over p gives $(2\pi\hbar)^n \delta(s-s')$:

$$\int f_w A_w\, dr\, dp = \int dr \int ds \cdot \langle r - \tfrac{s}{2} | \hat{\rho} | r + \tfrac{s}{2} \rangle \langle r + \tfrac{s}{2} | \hat{A} | r - \tfrac{s}{2} \rangle$$

Change variables:

$$\begin{cases} x_1 = r + \frac{s}{2} \\ x_2 = r - \frac{s}{2} \end{cases}$$

$$\hookrightarrow \int dx_1\, dx_2 \, \langle x_2 | \hat{\rho} | x_1 \rangle \langle x_1 | \hat{A} | x_2 \rangle$$

$$= \sum_{\substack{ab \\ a'b'}} \rho_{ab} \underbrace{\langle \varphi_a | \varphi_{b'} \rangle}_{\delta_{ab'}} \underbrace{\langle \varphi_{a'} | \varphi_b \rangle}_{\delta_{a'b}} A_{a'b'} = \sum_{ab} \rho_{ab} A_{ba} \equiv \langle A \rangle$$

Therefore,

$$\boxed{\langle A \rangle = \int f_w(r,p)\, A_w(r,p)\, dr\, dp} \tag{7}$$

- We also introduce the FT of the Wigner's operator:

$$\boxed{\alpha(\sigma,\tau) = \int dr\, dp\, e^{-i(\sigma r + \tau p)/\hbar} A_w(r,p)} \tag{8}$$

with the inverse transform being:

$$\boxed{A_w(r,p) = (2\pi\hbar)^{-2n}\int d\sigma\, d\tau\; e^{i(\sigma r+\tau p)/\hbar}\,\alpha(\sigma,\tau)} \qquad (9)$$

A product of two operators

Consider $(AB)_W$, where

$$\begin{aligned} A_w(r,p) &= (2\pi\hbar)^{-2n}\int d\sigma\, d\tau\; e^{i(\sigma r+\tau p)/\hbar}\,\alpha(\sigma,\tau) \\ B_w(r,p) &= (2\pi\hbar)^{-2n}\int d\sigma\, d\tau\; e^{i(\sigma r+\tau p)/\hbar}\,\beta(\sigma,\tau) \end{aligned} \qquad (10)$$

Then,

$$(AB)_W(r,p) \equiv \int ds\; e^{ips/\hbar}\,\langle r-\tfrac{s}{2}|\,AB\,|r+\tfrac{s}{2}\rangle \qquad (11)$$

Insert the unity:

$$\cancel{\int ds'\,|s'\rangle\langle s'| \equiv \int ds' \underbrace{\sum_c \varphi_c(s')\,\varphi_c^*(s'')}_{\delta(s'-s'')}}$$

$$(AB)_W(r,p) = \int ds\; e^{ips/\hbar}\sum_{ab}(AB)_{ab}\,\varphi_a^*(r+\tfrac{s}{2})\,\varphi_b(r-\tfrac{s}{2})$$

$$= \int ds\; e^{ips/\hbar}\sum_{abc} A_{ac}B_{cb}\,\varphi_a^*(r+\tfrac{s}{2})\,\varphi_b(r-\tfrac{s}{2})$$

$$\equiv \int ds\; e^{ips/\hbar}\int dr' \sum_{abc} A_{ac}B_{cb}\,\varphi_a^*(r+\tfrac{s}{2})\,\overbrace{\varphi_c(r')\,\varphi_c^*(r')}^{=1}\,\varphi_b(r-\tfrac{s}{2})$$

$$= \int ds \int dr'\; e^{ips/\hbar}\,\langle r-\tfrac{s}{2}|A|r'\rangle\langle r'|B|r+\tfrac{s}{2}\rangle \qquad (12)$$

~~From the definitions of the Wigner's operator~~

$$\cancel{\langle r+\tfrac{s}{2}|A|r-\tfrac{s}{2}\rangle = (2\pi\hbar)^{-n}\int d\sigma\; e^{-i\sigma s/\hbar}\,A_w(r,\sigma)}$$

$$\cancel{= (2\pi\hbar)^{-n}\int d\sigma\, e^{-i\sigma s/\hbar}}$$

We should express the operators A, B via their FT:

$$\alpha(\sigma,\tau) = \int dr\,dp\; e^{-i(\sigma r+\tau p)/\hbar}\, A_W(r,p)$$

$$= \int dr\,dp\; e^{-i(\sigma r+\tau p)/\hbar}\left[\int ds\, e^{ips/\hbar}\,\langle r-\tfrac{s}{2}|\hat{A}|r+\tfrac{s}{2}\rangle\right]$$

The p-integration gives $(2\pi\hbar)^n\delta(s-\tau)$:

$$\alpha(\sigma,\tau) = (2\pi\hbar)^n\int dr\; e^{-i\sigma r/\hbar}\,\langle r-\tfrac{\tau}{2}|\hat{A}|r+\tfrac{\tau}{2}\rangle$$

and hence

$$\langle r-\tfrac{\tau}{2}|\hat{A}|r+\tfrac{\tau}{2}\rangle = (2\pi\hbar)^{-2n}\int d\sigma\, e^{i\sigma r/\hbar}\,\alpha(\sigma,\tau)$$

Replacing $r-\frac{\tau}{2} = x$, $r+\frac{\tau}{2} = x'$, we obtain:

$$\boxed{\begin{aligned}\langle x|\hat{A}|x'\rangle &= \sum_{ab} A_{ab}\,\varphi_a^*(x)\varphi_b^*(x')\\ &= (2\pi\hbar)^{-2n}\int d\sigma\, e^{i(x+x')\sigma/2\hbar}\,\alpha(\sigma, x'-x)\end{aligned}} \qquad (13)$$

- Use (13) in (12) now for both A and B:

$$(AB)_W(r,p) = \int ds\,dr'\, e^{ips/\hbar}\,\frac{1}{(2\pi\hbar)^{+4n}}\left[\int d\sigma\, e^{i(r-\frac{s}{2}+r')\sigma/2\hbar}\,\alpha(\sigma, -r+\tfrac{s}{2}+r')\right]\times$$

$$\times\left[\int d\sigma'\, e^{i(r'+r+\frac{s}{2})\sigma'/2\hbar}\,\beta(\sigma', -r'+r+\tfrac{s}{2})\right] \qquad (14)$$

Change variables: $(s,r')\to(x_1,x_2)$, via:

$$\begin{cases} x_1 = r'+\frac{s}{2}-r\\ x_2 = -r'+r+\frac{s}{2}\end{cases} \to \begin{cases} s = x_1+x_2\\ r' = r+\frac{1}{2}(x_1-x_2)\end{cases}$$

$$\frac{\partial(\sigma, r')}{\partial(x_1, x_2)} = \begin{vmatrix} 1 & 1 \\ +\frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -1,$$ but we need the abs. value.

$$(AB)_W = \int \frac{dx_1 dx_2}{(2\pi\hbar)^{2n}} \int \frac{d\sigma d\sigma'}{(2\pi\hbar)^{2n}} \; e^{ip(x_1+x_2)/\hbar} \; e^{i\left[r - \frac{1}{2}(x_1+x_2) + r + \frac{1}{2}(x_1-x_2)\right]\sigma/2\hbar}$$

$$\times \, \alpha(\sigma, x_1) \; e^{i\left[r + \frac{1}{2}(x_1-x_2) + r + \frac{1}{2}(x_1+x_2)\right]\sigma'/2\hbar} \; \beta(\sigma', x_2)$$

$$= \int \frac{dx_1 dx_2}{(2\pi\hbar)^{2n}} \int \frac{d\sigma d\sigma'}{(2\pi\hbar)^{2n}} \; e^{ip(x_1+x_2)/\hbar} \; e^{i(r\sigma)/\hbar} \; e^{-ix_2\sigma/2\hbar}$$

$$\times \, \alpha(\sigma, x_1) \; e^{ir\sigma'/\hbar} \; e^{+ix_1\sigma'/2\hbar} \; \beta(\sigma', x_2)$$

$$= \int \frac{dx_1 dx_2}{(2\pi\hbar)^{2n}} \int \frac{d\sigma d\sigma'}{(2\pi\hbar)^{2n}} \left[e^{i(px_1 + r\sigma)/\hbar} \alpha(\sigma, x_1) \right] e^{-i(x_2\sigma - x_1\sigma')/2\hbar} \times$$

$$\times \left[e^{i(px_2 + r\sigma')/\hbar} \beta(\sigma', x_2) \right]$$

change $x_1 \to \tau$
$x_1' \to \tau'$

$$= \cancel{\int dx_1 dx_2} \; \frac{1}{(2\pi\hbar)^{2n}} \left[\int d\sigma d\tau \; e^{i(p\tau + r\sigma)/\hbar} \alpha(\sigma, \tau) \right] \times \frac{1}{(2\pi\hbar)^{2n}} \times$$

$$\times \int d\sigma' d\tau' \; e^{-i(\tau'\sigma - \tau\sigma')/2\hbar} \left[e^{i(p\tau' + r\sigma')/\hbar} \beta(\sigma', \tau') \right] \qquad (15)$$

We nearly have $A_W(r,p)$ and $B_W(r,p)$ inside the square brackets; the problem is in the exponential

$$e^{-i(\tau'\sigma - \tau\sigma')/2\hbar} =$$

$$= 1 + \frac{i}{2\hbar}(\tau'\sigma - \tau\sigma') + \frac{1}{2}\left(\frac{i}{2\hbar}\right)^2 (\tau'\sigma - \tau\sigma')^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{2\hbar}\right)^n (\tau'\sigma - \tau\sigma')^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{2\hbar}\right)^n \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k} \underbrace{(\tau'\sigma)^k(\tau\sigma')^{n-k}} \qquad (16)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!}\left(\frac{i}{2\hbar}\right)^n \binom{n}{k}(-1)^k \times \left(\tau'^k\sigma'^{n-k}\right)\left(\cancel{\sigma}\tau^{n-k}\sigma^k\right)$$

Consider the following operator:

$$\boxed{e^{+\frac{i\hbar}{2}\Lambda}, \quad \Lambda = \overleftarrow{\partial}_r\cdot\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_r} \qquad (17)$$

where the arrows indicate in which direction the derivative works

Consider

$$e^{i(p\tau+r\sigma)/\hbar}\; e^{+\frac{i\hbar}{2}\Lambda}\; e^{i(p\tau'+r\sigma')/\hbar}$$

$$= e^{i(p\tau+r\sigma)/\hbar}\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\binom{n}{k}\left(\frac{i\hbar}{2}\right)^n(-1)^{n-k}\left(\overleftarrow{\partial}_r\overrightarrow{\partial}_p\right)^k\left(\overleftarrow{\partial}_p\overrightarrow{\partial}_r\right)^{n-k}\right]\times$$

$$\times\, e^{i(p\tau'+r\sigma')/\hbar} \qquad (18)$$

Here $[\ldots] \to \left(\overleftarrow{\partial}_r\right)^k\left(\overleftarrow{\partial}_p\right)^{n-k} \times \left(\overrightarrow{\partial}_p\right)^k\left(\overrightarrow{\partial}_r\right)^{n-k}$

When applying the $\leftarrow$ bit on the exponential on the left,

$$e^{i(p\tau+r\sigma)/\hbar}\left(\overleftarrow{\partial}_r\right)^k\left(\overleftarrow{\partial}_p\right)^{n-k} =$$

$$= \left(\partial_r\right)^k\left(\partial_p\right)^{n-k} e^{i(p\tau+r\sigma)/\hbar} =$$

$$= e^{i(p\tau-r\sigma)/\hbar} \cdot \left(\frac{i\sigma}{\hbar}\right)^k\left(\frac{i\tau}{\hbar}\right)^{n-k}$$

$$= e^{i(p\tau-r\sigma)/\hbar}\, \sigma^k\tau^{n-k}\left(\frac{i}{\hbar}\right)^n$$

On the other hand, the "$\rightarrow$" part produces:

$$\left(\overrightarrow{\partial}_p\right)^k\left(\overrightarrow{\partial}_r\right)^{n-k} e^{i(p\tau'+r\sigma')/\hbar} =$$

$$= \left(\frac{i\tau'}{\hbar}\right)^k\left(\frac{i\sigma'}{\hbar}\right)^{n-k} = \left(\frac{i}{\hbar}\right)^n (\tau')^k (\sigma')^{n-k},$$

so that, altogether,

$$e^{i(p\tau+r\sigma)/\hbar}\, e^{\frac{i\hbar}{2}\Lambda}\, e^{i(p\tau'+r\sigma')/\hbar}$$

$$= e^{i(p\tau+r\sigma)/\hbar}\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\binom{n}{k}(-1)^{k}\left(\frac{i\hbar}{2}\right)^n\left(\frac{i}{\hbar}\right)^{2n}\sigma^k\tau^{n-k}(\sigma')^{n-k}(\tau')^k\right]\cancel{e^{i(p\tau'+}}\times$$

$$\times\, e^{i(p\tau'+r\sigma')/\hbar}$$

Here $\left(\frac{i\hbar}{2}\right)^n\left(\frac{i}{\hbar}\right)^{2n} = \left(\frac{i}{\hbar}\right)^n\left(\frac{i\hbar}{2}\cdot\frac{i}{\hbar}\right)^n = \left(\frac{i}{\hbar}\right)^n\left(\frac{-1}{2}\right)^n = (-1)^n\left(\frac{i}{2\hbar}\right)^n$

giving

$$e^{i(p\tau+r\sigma)/\hbar}\left\{\sum_{n,k}\frac{(-1)^{k}}{n!}\binom{n}{k}\left(\frac{i}{2\hbar}\right)^n\left[\sigma^k\tau^{n-k}(\sigma')^{n-k}(\tau')^k\right]\right\}e^{i(p\tau'+r\sigma')/\hbar} \qquad (19a)$$

which ~~ex~~ is exactly the same as in (16):

$$\{\ldots\} = e^{-i(\tau'\sigma-\tau\sigma')/2\hbar} \qquad (19b)$$

Therefore, we can write from (15) and (17) [and (9)]:

$$\boxed{(AB)_W = A_W(r,p)\, e^{-\frac{i\hbar}{2}\Lambda}\, B_W(r,p)} \qquad (20)$$

- Expansion of the exponential operator introduces powers of $\hbar$.

$$e^{+\frac{i\hbar}{2}\Lambda} = 1 + \frac{i\hbar}{2}\Lambda + \ldots = 1 + \frac{i\hbar}{2}\left(\overleftarrow{\partial}_r\cdot\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\cdot\overrightarrow{\partial}_r\right) + \ldots$$

leading to:

$$(AB)_w = A_w(r,p)\,B_w(r,p) + \frac{i\hbar}{2}\Big[\big(\partial_r A_w(r,p)\big)\big(\partial_p B_w(r,p)\big) -$$

$$-\big(\partial_p A_w(r,p)\big)\big(\partial_r B_w(r,p)\big)\Big] + \dots \tag{21}$$

One can recognise the <u>classical Poisson</u> bracket in the expression

$$[\dots] = \{A_w(r,p),\, B_w(r,p)\} = \sum_j \left(\frac{\partial A_w}{\partial r_j}\frac{\partial B_w}{\partial p_j} - \frac{\partial A_w}{\partial p_j}\frac{\partial B_w}{\partial r_j}\right) \tag{22}$$

- Calculate the trace:

$$\mathrm{Tr}(A) = \mathrm{Tr}\Big(\sum_{ab} |a\rangle A_{ab} \langle b|\Big) = \sum_{cab} \underbrace{\langle c|a\rangle}_{\delta_{ca}} A_{ab} \underbrace{\langle b|c\rangle}_{\delta_{bc}} = \sum_c A_{cc}$$

On the other hand, consider

$$\int dr\,dp\, A_w(r,p) = \int dr\,dp \int ds\; e^{ips/\hbar} \langle r - \tfrac{s}{2}|\hat{A}|r+\tfrac{s}{2}\rangle = \left|\begin{matrix} p\text{-integration} \\ \to \delta(s) \end{matrix}\right|$$

$$= (2\pi\hbar)^n \int dr \langle r|\hat{A}|r\rangle = (2\pi\hbar)^n \int dr \sum_{ab} \varphi_a^*(r) A_{ab} \varphi_b(r)$$

$$= (2\pi\hbar)^n \sum_{ab} A_{ab}\delta_{ab} = (2\pi\hbar)^n \sum_a A_{aa} = (2\pi\hbar)^n\, \mathrm{Tr}\, A,$$

i.e.

$$\boxed{\mathrm{Tr}\, A = (2\pi\hbar)^{-n} \int dr\,dp\, A_w(r,p)} \tag{23}$$

- Consider $\mathrm{Tr}(AB)$:

$$\mathrm{Tr}(AB) = (2\pi\hbar)^{-n} \int dr\,dp\, (AB)_w(r,p)$$

$$= (2\pi\hbar)^{-n} \int dr\,dp\, A_w(r,p)\, e^{-\frac{i\hbar}{2}\Lambda}\, B_w(r,p) \tag{24}$$

This can be calculated by expanding both A_w and B_w in the Fourier integrals with $\alpha(\sigma,\tau)$ and $\beta(\sigma,\tau)$, and then using (19):

$$\mathrm{Tr}(AB) = (2\pi\hbar)^{-n} \int dr\,dp\, (2\pi\hbar)^{-2n} \int d\sigma\, d\tau\, e^{i(\sigma r + \tau p)/\hbar}\, \alpha(\sigma,\tau)\ \times$$

$$\times\, e^{-\frac{i\hbar}{2}\Lambda}\, (2\pi\hbar)^{-2n} \int d\sigma' d\tau'\, e^{i(\sigma' r + \tau' p)/\hbar}\, \beta(\sigma',\tau') \qquad (25)$$

$$= (2\pi\hbar)^{-5n} \int d\sigma\, d\sigma' d\tau\, d\tau' \int dr\, dp\, \alpha(\sigma,\tau)\, \beta(\sigma',\tau')\, \underbrace{e^{i(\sigma r+\tau p)/\hbar}\, e^{-\frac{i\hbar}{2}\Lambda}\, e^{i(\sigma' r + \tau' p)/\hbar}}_{e^{i(\sigma r + \tau p)/\hbar}\ \times\ e^{-i(\tau'\sigma - \tau\sigma')/2\hbar}\ \times\ \dots}$$

$$= (2\pi\hbar)^{-5n} \int d\sigma\, d\sigma' d\tau' d\tau\, dr\, dp\, \alpha(\sigma,\tau)\, \beta(\sigma',\tau')\, e^{i(\sigma r + \tau p)/\hbar}\, e^{i(\sigma' r + \tau' p)/\hbar}\ \times$$

$$\times\ e^{-i(\tau'\sigma - \tau\sigma')/2\hbar} \qquad (26)$$

Integrate over p and r:

$$\int dp\, e^{i\tau p/\hbar}\, e^{i\tau' p/\hbar} = (2\pi\hbar)^n \delta(\tau + \tau')$$

$$\int dr\, e^{i\sigma r/\hbar}\, e^{i\sigma' r/\hbar} = (2\pi\hbar)^n \delta(\sigma + \sigma')$$

which gives:

$$= (2\pi\hbar)^{-3n} \int d\sigma\, d\tau\ \alpha(\sigma,\tau)\, \beta(-\sigma,-\tau)\, \underbrace{e^{-i(-\tau\sigma + \tau\sigma)/2\hbar}}_{=\,1}$$

$$= (2\pi\hbar)^{-3n} \int d\sigma\, d\tau\ \alpha(\sigma,\tau)\, \beta(-\sigma,-\tau) \qquad (27)$$

This is basically:

$$\int dr\, dp\, A_W(r,p)\, B_W(r,p) = \int dr\, dp\, (2\pi\hbar)^{-2n} \int d\sigma\, d\tau\, e^{i(\sigma r + \tau p)/\hbar}\, \alpha(\sigma,\tau)$$

$$\times\, (2\pi\hbar)^{-2n} \int d\sigma' d\tau'\, e^{i(\sigma' r + \tau' p)/\hbar}\, \beta(\sigma',\tau')$$

$$= (2\pi\hbar)^{-4n} \int d\sigma\, d\sigma' d\tau\, d\tau'\, \alpha(\sigma,\tau)\, \beta(\sigma',\tau')\ \underbrace{\int dr\, e^{i(\sigma+\sigma')r/\hbar}}_{(2\pi\hbar)^n \delta(\sigma+\sigma')}\ \underbrace{\int dp\, e^{i(\tau+\tau')p/\hbar}}_{(2\pi\hbar)^n \delta(\tau+\tau')}$$

$$= (2\pi\hbar)^{-2n} \int d\sigma\, d\tau\, \alpha(\sigma,\tau)\, \beta(-\sigma,-\tau)$$

Therefore,

$$\boxed{\mathrm{Tr}(AB) = (2\pi\hbar)^{-n} \int dr\, dp\, A_W(r,p)\, B_W(r,p)} \qquad (28)$$

Another proof of the same result can be obtained directly using the operator

$$e^{+\frac{i\hbar}{2}\Lambda} = \sum_{h=0}^{\infty} \sum_{k=0}^{n} \left(+\frac{i\hbar}{2}\right)^n \frac{1}{n!} (-1)^{n-k} \binom{n}{k} \underbrace{(\overleftarrow{\partial}_r)^k (\overleftarrow{\partial}_p)^{n-k}}_{\text{acts on } A_W} \underbrace{(\overrightarrow{\partial}_p)^k (\overrightarrow{\partial}_r)^{n-k}}_{\text{act on } B_W} \qquad (29)$$

Consider

$$\int (A_W \overleftarrow{\partial}_r) \underbrace{(\overrightarrow{\partial}_r^{\,k} \overrightarrow{\partial}_p^{\,k'}) B_W}_{C_W} dr \overset{\text{by parts}}{=}$$

$$= \underbrace{A_W C_W \Big|_{-\infty}^{\infty}}_{0} - \int A_W \overrightarrow{\partial}_r C_W\, dr = - \int A_W (\partial_r C_W)\, dr$$

similarly $\int A_W \overleftarrow{\partial}_p C_W\, dp = - \int A_W (\partial_p C_W)\, dp$.

It is now should be obvious that generally (when $n \geq 1$)

$$\int A_W (\overleftarrow{\partial}_r)^k (\overleftarrow{\partial}_p)^{n-k} (\overrightarrow{\partial}_p)^k (\overrightarrow{\partial}_r)^{n-k} B_W\, dr\, dp$$

$$= (-1)^{k+n-k} \int A_W (\partial_r)^k (\partial_p)^{n-k} (\partial_p)^k (\partial_r)^{n-k} B_W\, dr\, dp$$

$$= (-1)^n \int A_W (\partial_r)^n (\partial_p)^n B_W\, dr\, dp \quad \rightarrow \text{does not depend on } k$$

at all! Therefore, the sum over k in (29) results in zero:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^n = 0,$$ apart from the case of $n=0$, when integration by parts is not performed.

This means, that $e^{-i\hbar\Lambda/2}$ is replaced with 1 giving

$$Tr(AB) = (2\pi\hbar)^{-n}\int dr dp\, (AB)_w = (2\pi\hbar)^{-n}\int dr dp\, A_w(r,p)\, e^{-\frac{i\hbar}{2}\Lambda} B_w$$

$$= (2\pi\hbar)^{-n}\int dr dp\, A_w B_w$$

which is again (23).

<u>Liouville equation</u>

- $$\dot{\rho} = \frac{1}{i\hbar}[H,\rho] = \frac{1}{i\hbar}(H\rho - \rho H) \qquad (30)$$

we shall rewrite in the Wigner way:

$\hat{\rho} \to \rho_W(r,p)$

$\hat{H} \to H_W(r,p)$

We obtain:

$$\boxed{\frac{\partial}{\partial t}\rho_W(r,p) = \frac{1}{i\hbar}\left(H_W \star \rho_W - \rho_W \star H_W\right)} \qquad (31)$$

where $\boxed{A \star B = A_W(r,p)\, e^{-\frac{i\hbar}{2}\Lambda} B_W(r,p)} \qquad (32)$

In the RHS ~~H_W~~, $H_W \star \rho_W - \rho_W \star H_W$ starts from the Poisson bracket:

$$\cancel{\frac{\partial}{\partial t}\rho_W(r,p) = \frac{1}{i\hbar}\left(\frac{-i\hbar}{2}\right)\{H_W(r,p), \rho_W(r,p)\} + O(\hbar)}$$

$$= +\frac{1}{2}\{\rho_W, H_W\} + O(\hbar)$$

classical quantum

- Let us work out the 1st <u>quantum</u> correction: this corresponds to $n=2$ in the expansion of

$$e^{-\frac{i\hbar}{2}\Lambda} \to \frac{1}{2}\left(\frac{-i\hbar}{2}\right)^2\left(\overleftarrow{\partial}_r\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_r\right)^2 =$$

$$H_w \star \rho_w = H_w(r,p)\, e^{+\frac{i\hbar}{2}\Lambda}\, \rho_w(r,w) =$$

$$= H_w\rho_w + \frac{i\hbar}{2}\underbrace{\left[(\partial_r H_w)(\partial_p \rho_w) - (\partial_p H_w)(\partial_r \rho_w)\right]}_{-\{\rho_w, H_w\} \leftarrow \text{classical Poisson bracket}} + O(\hbar^2)$$

$$= H_w\rho_w + \frac{i\hbar}{2}\{\rho_w, H_w\} + O(\hbar^2).$$

Also,

$$\rho_w \star H_w = H_w\rho_w + \frac{i\hbar}{2}\left[(\partial_r \rho_w)(\partial_p H_w) - (\partial_p \rho_w)(\partial_r H_w)\right] + O(\hbar^2)$$

$$= H_w\rho_w + \frac{i\hbar}{2}\{\rho_w, H_w\} + O(\hbar^2),$$

so that

$$\left[\text{RHS of (31)}\right] = \frac{1}{i\hbar}\left[\frac{i\hbar}{2}\{\rho_w, H_w\}\right] - \frac{1}{i\hbar}\left[+\frac{i\hbar}{2}\{\rho_w, H_w\}\right] + O(\hbar^2)\frac{1}{i\hbar} =$$

$$= -\{\rho_w, H_w\} + O(\hbar),$$

so that we obtain:

$$\boxed{\frac{\partial \rho_w}{\partial t} = -\{\rho_w, H_w\} + O(\hbar)}$$

This is classical eq. with quantum corrections coming from the following terms which start from $\sim\hbar$.

- Let us obtain the <u>1st</u> quantum correction (2nd order term):

$$(H_w \star \rho_w)^{(2)} = H_w\left(\frac{i\hbar}{2}\right)^2 \frac{1}{2}\left(\overleftarrow{\partial}_r\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_r\right)^2 \rho_w$$

$$= -\frac{\hbar^2}{8} H_w\left[\overleftarrow{\partial}_r^2\overrightarrow{\partial}_p^2 + \overleftarrow{\partial}_p^2\overrightarrow{\partial}_r^2 - 2\overleftarrow{\partial}_r\overleftarrow{\partial}_p\overrightarrow{\partial}_p\overrightarrow{\partial}_r\right]\rho_w$$

$$= -\frac{\hbar^2}{8}\left[(\partial_r^2 H_w)(\partial_p^2 \rho_w) + (\partial_p^2 H_w)(\partial_r^2 \rho_w) - 2(\partial_r\partial_p H_w)(\partial_p\partial_r \rho_w)\right]$$

$$(\rho_W \star H_W)^{(2)} = -\frac{\hbar^2}{8}\left[(\partial_r^2\rho_W)(\partial_p^2 H_W) + (\partial_p^2\rho_W)(\partial_r^2 H_W) - 2(\partial_r\partial_p\rho_W)(\partial_p\partial_r H_W)\right]$$

Their difference:

$$\frac{1}{i\hbar}(H_W\star\rho_W - \rho_W\star H_W)^{(2)} =$$

$$= +\frac{1}{i\hbar}\left(-\frac{\hbar^2}{8}\right)\Big\{ \cancel{(\partial_r^2 H_W)(\partial_p^2\rho_W)} + \cancel{(\partial_p^2 H_W)(\partial_r^2\rho_W)} - 2\cancel{(\partial_{rp}^2 H_W)(\partial_{rp}^2\rho_W)} -$$

$$- \cancel{(\partial_r^2\rho_W)(\partial_p^2 H_W)} - \cancel{(\partial_R^2\rho_W)(\partial_r^2 H_W)} + 2\cancel{(\partial_r\partial_r\rho_W)(\partial_r\partial_p H_W)}\Big\} = 0$$

Third-order does not cancell out:

$$(H_W\star\rho_W)^{(3)} = H_W\left(\frac{i\hbar}{2}\right)^3\frac{1}{3!}\left(\overleftarrow{\partial}_r\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_r\right)^3\rho_W$$

$$= \frac{-i\hbar^3}{48}H_W\left(\overleftarrow{\partial}_r^3\overrightarrow{\partial}_p^3 - 3\overleftarrow{\partial}_r^2\overrightarrow{\partial}_p^2\overleftarrow{\partial}_p\overrightarrow{\partial}_r + 3\overleftarrow{\partial}_r\overrightarrow{\partial}_p\overleftarrow{\partial}_p^2\overrightarrow{\partial}_r^2 - \overleftarrow{\partial}_p^3\overrightarrow{\partial}_r^3\right)\rho_W$$

$$= \frac{-i\hbar^3}{48}\Big[(\partial_r^3 H_W)(\partial_p^3\rho_W) - 3(\partial_r^2\partial_p H_W)(\partial_p^2\partial_r\rho_W) +$$

$$+ 3(\partial_r\partial_p^2 H_W)(\partial_p\partial_r^2\rho_W) - (\partial_p^3 H_W)(\partial_r^3\rho_W)\Big],$$

$$(\rho_W\star H_W)^{(3)} = \frac{-i\hbar^3}{48}\Big[(\partial_r^3\rho_W)(\partial_p^3 H_W) - 3(\partial_r^2\partial_p\rho_W)(\partial_p^2\partial_r H_W)$$

$$+ 3(\partial_r\partial_p^2\rho_W)(\partial_p\partial_r^2 H_W) - (\partial_p^3\rho_W)(\partial_p^3 H_W)\Big]$$

Their difference:

$$\frac{1}{i\hbar}(H_W\star\rho_W - \rho_W\star H_W)^{(3)} = \frac{-\hbar^2}{48}\Big\{\Big[(\partial_r^3 H_W)(\partial_p^3\rho_W) - 3(\partial_r^2\partial_p H_W)(\partial_p^2\partial_r\rho_W)$$

$$+ 3(\partial_r\partial_p^2 H_W)(\partial_p\partial_r^2\rho_W) - (\partial_p^3 H_W)(\partial_r^3\rho_W)\Big] - \Big[(\partial_r^3\rho_W)(\partial_p^3 H_W) - 3(\partial_r^2\partial_p\rho_W)\times$$

$$\times(\partial_p^2\partial_r H_W) + 3(\partial_r\partial_p^2\rho_W)(\partial_p\partial_r^2 H_W) - (\partial_p^3\rho_W)(\partial_r^3 H_W)\Big]\Big\}$$

$$= \frac{-\hbar^2}{48} \left\{ 2 (\partial_r^3 H_W)(\partial_p^3 \rho_W) - 6 (\partial_r^2 \partial_p H_W)(\partial_p^2 \partial_r \rho_W) + \right.$$

$$\left. + 6 (\partial_r \partial_p^2 H_W)(\partial_p \partial_r^2 \rho_W) - 2 (\partial_p^3 H_W)(\partial_r^3 \rho_W) \right\}$$

$$= \frac{-\hbar^2}{24} \left[(\partial_r^3 H_W)(\partial_p^3 \rho_W) - \dots \right]$$

is of the order of $\hbar^2$. These are mixed derivatives, e.g.

$$\partial_r^2 H_W = \sum_{ij} \partial_{r_i} \partial_{r_j} H_W, \text{ etc.}$$