24.04.08

Calculation of $\Pi_{22}(t)$

1. $\Pi_{22}(t) = \sum_\lambda \frac{e_\lambda e_\lambda^t}{\omega_\lambda^2} \cos\omega_\lambda t \Rightarrow \dot{\Pi}_{22}(t) = -\Omega_{22}(t)$

$\Omega_{22}(t) = \sum_\lambda \frac{e_\lambda e_\lambda^t}{\omega_\lambda} \sin\omega_\lambda t \Rightarrow \boxed{\Pi_{22}(t) = -\int_0^t \Omega_{22}(\tau)d\tau + D_{22}^{-1}}$

2. The $\underline{\underline{LT}}$ of $\Omega_{22}(t)$ is:

$$\Omega_{22}(s) = \sum_\lambda \frac{e_\lambda e_\lambda^t}{\omega_\lambda} \frac{\omega_\lambda}{s^2+\omega_\lambda^2} = \sum_\lambda \frac{e_\lambda e_\lambda^t}{s^2+\omega_\lambda^2} \equiv (s^2 \mathbb{1}_{22} + D_{22})^{-1}$$

$$\boxed{\Omega_{22}(s) = (s^2 \mathbb{1}_{22} + D_{22})^{-1}}$$

3. This can be related ~~...~~ to the GF

$$\boxed{G(z) = (z\mathbb{1} - D)^{-1}}$$

~~...~~ defined over the whole perfect system $1 \cup 2$:

$$\boxed{\varphi(z) = z\mathbb{1} - D} \quad , \quad \varphi G = \mathbb{1}$$

$$\begin{cases} \varphi_{11} G_{11} + \varphi_{12} G_{21} = 1_{11} \\ \varphi_{11} G_{12} + \varphi_{12} G_{22} = 0 \\ \varphi_{21} G_{11} + \varphi_{22} G_{21} = 0 \rightarrow \varphi_{21} = -\varphi_{22} G_{21} G_{11}^{-1} \\ \varphi_{21} G_{12} + \varphi_{22} G_{22} = 1_{22} \end{cases}$$

We need $\varphi_{22}^{-1}(z)$ only to calculate $\Omega_{22}(s)$, since

$$\Omega_{22}(s) = -(-s^2 \mathbb{1}_{22} - D_{22})^{-1} = -[\varphi_{22}(-s^2)]^{-1}$$

We have:

$\varphi_{21} = -\varphi_{22} G_{21} G_{11}^{-1} \rightarrow$ in the last equation

$$-\varphi_{22} G_{21} G_{11}^{-1} G_{12} + \varphi_{22} G_{22} = 1_{22}$$

$$\Omega_{21}\left(G_{22}-G_{21}G_{11}^{-1}G_{12}\right)=1_{22},$$

$$\boxed{\Omega_{22}(z)^{-1}=G_{22}(z)-G_{21_0}(z)\,G_{1_01_0}^{-1}(z)\,G_{1_02}(z)}$$

Therefore,

$$\boxed{\Omega_{22}(s)=-G_{22}(-s^2)+G_{21_0}(-s^2)\,G_{1_01_0}^{-1}(-s^2)\,G_{1_02}(-s^2)}$$

Thus, knowing the GF of the perfect reference system, allows calculation of $\Omega_{22}(z)$ in the complex plane.

4. The LT of $\Pi(t)$ is:

$$\Pi_{22}(s)=D_{22}^{-1}\frac{1}{s}-\frac{1}{s}\Omega_{22}(s)=\frac{1}{s}\left[D_{22}^{-1}-\Omega_{22}(s)\right]$$

and thus can also be calculated directly. By the way,

$$\boxed{D_{22}^{-1}=\Omega_{22}(s=0)},$$

so that

$$\boxed{\Pi_{22}(s)=\frac{1}{s}\left[\Omega_{22}(s=0)-\Omega_{22}(s)\right]}$$

5. To get $\Pi_{22}(t)$, one has to perform integration in the complex plane.

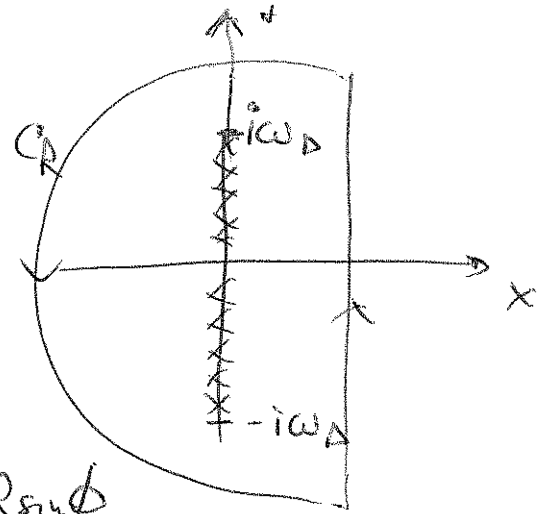
The poles of $\Pi_{22}(s)$ are:

(i) $s=0$

(ii) $s=\pm i\omega_\lambda$ (from $\Omega_{22}(s)$)

We choose the contour in the inverse LT as:

$$\Pi_{22}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Pi_{22}(s)\, e^{st}\, ds$$



with $a>0$. The contour can be closed on the left as shown →

since: $s = a + Re^{i\phi} = (a + R\cos\phi) + iR\sin\phi$

for C_R: $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$, $\cos\phi < 0$

and thus $\int_{C_R} \to 0$ as $R \to \infty$ since $\frac{1}{s^2+\omega_\lambda^2} \to 0$ as $s \to \infty$. (along any direction)

(Zhordan's lemma).

~~Therefore,~~

$$\Pi(t) = \sum_i \mathrm{Res}[\Pi(s)e^{st}, s_i] =$$

$$= \Pi(s=0) + \sum_\lambda \left[\Pi(i\omega_\lambda) e^{i\omega_\lambda t} + \Pi(-i\omega_\lambda) e^{-i\omega_\lambda t}\right]$$

$$+ \left[\Omega(s=0) - \Omega(s=0)\right] e^{0t}$$

$= 0$

$$= \sum_\lambda \frac{1}{i\omega_\lambda}\left[\Omega(s=0)\right.$$

(i) $\mathrm{Res}\left[\Pi(s)e^{st}, s=0\right] = \left[\Omega(s=0) - \Lambda(s=0)\right] e^{0t} = 0$

(ii) $\mathrm{Res}\left[\Pi(s)e^{st}, s=+i\omega_\lambda\right] = \frac{1}{i\omega_\lambda}\left[\Omega(s=0) - \sum_\lambda \frac{e_\lambda e_\lambda^\dagger}{2i\omega_\lambda}\right.$

In fact, we should only worry about $\frac{\Omega_{22}(s)}{s}$ part of $\Pi_{22}(s)$.

$$\frac{1}{2\pi i}\int_C \frac{\Omega_{22}(s)}{s} e^{st} ds = \Omega_{22}(s=0) e^{0t} +$$

$$+ \sum_\lambda e_\lambda e_\lambda^+ \left[\frac{1}{i\omega_\lambda} \cdot \frac{1}{2i\omega_\lambda} e^{i\omega_\lambda t} + \frac{1}{-i\omega_\lambda} \frac{1}{-2i\omega_\lambda} e^{-i\omega_\lambda t} \right]$$

$$= D_{22}^{-1} + \sum_\lambda \frac{e_\lambda e_\lambda^+}{\omega_\lambda^2}\left(-\frac{1}{2}\right)\left(e^{i\omega_\lambda t} + e^{-i\omega_\lambda t}\right)$$

$$= D_{22}^{-1} - \frac{1}{2}\sum_\lambda \frac{e_\lambda e_\lambda^+}{\omega_\lambda^2} 2\cos\omega_\lambda t = D_{22}^{-1} - \sum_\lambda \frac{e_\lambda e_\lambda^+}{\omega_\lambda^2}\cos\omega_\lambda t,$$

and thus

$$\Pi_{22}(t) = D_{22}^{-1} - \mathcal{L}^{-1}\left[\frac{\Omega_{22}(s)}{s}\right] = D_{22}^{-1} - \left(D_{22}^{-1} - \sum_\lambda \frac{e_\lambda e_\lambda^+}{\omega_\lambda^2}\cos\omega_\lambda t\right)$$

$$\equiv \sum_\lambda \frac{e_\lambda e_\lambda^+}{\omega_\lambda^2}\cos\omega_\lambda t,$$

the correct result!

So, the practical formula for calculating $\Pi_{22}(t)$:

$$\boxed{\Pi_{22}(t) = \frac{1}{2\pi i}\int_C \frac{1}{s}\left[\Omega_{22}(s=0) - \Omega_{22}(s)\right]e^{st} ds}$$

with the contour C as in the picture: