

# SYMMETRIC SUBMANIFOLDS OF SYMMETRIC SPACES

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ABSTRACT. This paper contains a survey about the classification of symmetric submanifolds of symmetric spaces. It is an extended version of a talk given by the author at the 7th International Workshop on Differential Geometry at Kyungpook National University in Taegu, Korea, in November 2002.

## 1. INTRODUCTION

A connected Riemannian manifold  $M$  is a *symmetric space* if for each point  $p \in M$  there exists an involutive isometry  $s_p$  of  $M$  such that  $p$  is an isolated fixed point of  $M$ . Symmetric spaces play an important role in Riemannian geometry because of their relations to algebra, analysis, topology and number theory. The symmetric spaces have been classified by E. Cartan. A thorough introduction to symmetric spaces can be found in [4].

Symmetric submanifolds play an analogous role in submanifold geometry. The precise definition of a symmetric submanifold is as follows. A submanifold  $S$  of a Riemannian manifold  $M$  is a *symmetric submanifold* of  $M$  if for each point  $p \in S$  there exists an involutive isometry  $\sigma_p$  of  $M$  such that

$$\sigma_p(p) = p, \sigma_p(S) = S \text{ and } (\sigma_p)_*(X) = \begin{cases} -X & , \text{ if } X \in T_p S, \\ X & , \text{ if } X \in \nu_p S. \end{cases}$$

Here,  $T_p S$  denotes the tangent space of  $S$  at  $p$  and  $\nu_p S$  the normal space of  $S$  at  $p$ . The isometry  $\sigma_p$  is called the *extrinsic symmetry* of  $S$  at  $p$ . It follows from the very definition that a symmetric submanifold is a symmetric space.

The aim of this paper is to present a survey about the classification problem of symmetric submanifolds of symmetric spaces. Another survey, concentrating on general theory of symmetric submanifolds and on the classification in rank one symmetric spaces, has been presented by Naitoh and Takeuchi in [18].

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## 2. TOTALLY GEODESIC SYMMETRIC SUBMANIFOLDS

Let  $S$  be a symmetric submanifold of a Riemannian manifold  $M$ . Let  $p \in S$  and  $\sigma_p$  be the extrinsic symmetry of  $S$  at  $p$ . The connected component  $S_p^\perp$  of the fixed point set of  $\sigma_p$  containing  $p$  is a totally geodesic submanifold of  $M$  such that  $T_p S_p^\perp = \nu_p S$ . The

submanifold  $S_p^\perp$  is just the image under the exponential map of  $M$  of the normal space  $\nu_p S$ . Thus a necessary condition for a submanifold to be symmetric is that tangent to each normal space there exists a totally geodesic submanifold of  $M$ . This is no restriction in a space of constant curvature, but it is quite restrictive in more general Riemannian manifolds like symmetric spaces. Since every symmetric submanifold is a symmetric space, it is geodesically complete. Let  $q_1, q_2$  be two distinct points in  $S$ . Then there exists a geodesic  $\gamma$  in  $S$  connecting  $q_1$  and  $q_2$ . The geodesic symmetry  $\sigma_p$  at the midpoint  $p$  on  $\gamma$  between  $q_1$  and  $q_2$  maps  $S_{q_1}^\perp$  to  $S_{q_2}^\perp$ , and vice versa. This shows that any two normal submanifolds  $S_{q_1}^\perp$  and  $S_{q_2}^\perp$  are congruent to each other under an isometry of  $M$ . Thus we may talk about the congruence class of the normal submanifolds associated to  $S$ . We denote by  $S^\perp$  any representative of this congruence class of normal submanifolds  $S_p^\perp$ ,  $p \in S$ .

Let  $\alpha$  be the second fundamental form of  $S$ . Since each isometry of  $M$  is an affine map with respect to the Levi Civita connection, we get

$$(\nabla_X^\perp \alpha)(Y, Z) = \sigma_{p*}(\nabla_X^\perp \alpha)(Y, Z) = (\nabla_{\sigma_{p*}X}^\perp \alpha)(\sigma_{p*}Y, \sigma_{p*}Z) = -(\nabla_X^\perp \alpha)(Y, Z)$$

for all  $p \in S$  and  $X, Y, Z \in T_p S$ . Thus the second fundamental form of a symmetric submanifold is parallel. The Codazzi equation then implies that each tangent space of  $S$  is curvature-invariant, that is,  $R(T_p S, T_p S)T_p S \subset T_p S$  for all  $p \in S$ , where  $R$  denotes the Riemannian curvature tensor of  $M$ . We summarize this in

**Proposition.** *Let  $S$  be a symmetric submanifold of a Riemannian manifold  $M$ . Then the following statements hold:*

- (i) *the second fundamental form of  $S$  is parallel;*
- (ii) *each tangent space of  $S$  is curvature-invariant, that is,  $R(T_p S, T_p S)T_p S \subset T_p S$  for all  $p \in S$ ;*
- (iii) *for each point  $p \in S$  there exists a totally geodesic submanifold  $S_p^\perp$  of  $M$  with  $p \in S_p^\perp$  and  $T_p S_p^\perp = \nu_p S$ . Any two normal submanifolds  $S_{q_1}^\perp$  and  $S_{q_2}^\perp$ ,  $q_1, q_2 \in S$ , are congruent to each other under an isometry of  $M$ .*

We assume now that  $M$  is a symmetric space. Let  $G$  be identity component of the isometry group of  $M$ ,  $o \in M$ , and  $K$  the isotropy subgroup of  $G$  at  $o$ . Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$  be the corresponding Cartan decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$ . We identify  $T_o M$  and  $\mathfrak{m}$  in the usual way. Then the curvature tensor  $R_o$  of  $M$  at  $o$  is given by

$$R_o(X, Y)Z = -[[X, Y], Z]$$

for all  $X, Y, Z \in \mathfrak{m} \cong T_o M$ . This shows that a subspace  $V$  of  $T_o M$  is curvature-invariant if and only if  $V$  is a Lie triple system, that is, if  $[[V, V], V] \subset V$ . Given a subspace  $V$  of  $T_o M$ , there exists a totally geodesic submanifold  $S$  of  $M$  with  $o \in M$  and  $T_o S = V$  if and only if  $V$  is a Lie triple system.

A connected submanifold  $F$  of a Riemannian manifold  $M$  is *reflective* if the geodesic reflection of  $M$  in  $F$  is a well-defined global isometry. As each connected component of the fixed point set of an isometry is totally geodesic, a reflective submanifold is necessarily a totally geodesic submanifold. We assume again that  $M = G/K$  is a connected Riemannian symmetric space. Let  $o \in M$ ,  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$  be the corresponding Cartan decomposition, and

$V$  a Lie triple system in  $\mathfrak{m}$  such that its orthogonal complement  $V^\perp$  is also a Lie triple system in  $\mathfrak{m}$ . Since  $V$  is a Lie triple system, there exists a connected complete totally geodesic submanifold  $F$  of  $M$  with  $o \in F$  and  $T_oF = V$ . In fact,  $F$  is the orbit through  $o$  of the connected Lie subgroup of  $G$  with Lie algebra  $[V, V] \oplus V \subset \mathfrak{k} \oplus \mathfrak{m} = \mathfrak{g}$ . The fact that  $V^\perp$  is a Lie triple system means geometrically there exists an open neighborhood  $U$  of  $F$  in  $M$  such that the geodesic reflection of  $U$  in  $F$  is an isometry. If, in addition,  $M$  is simply connected, this local geodesic reflection can be extended to a globally well-defined isometry on  $M$ , since every local isometry on a simply connected, complete, real analytic Riemannian manifold can be extended to a global isometry. We thus have proved

**Corollary.** *Let  $S$  be a symmetric submanifold of a symmetric space  $M$ . Then, at each point  $o \in S$ , the tangent space  $T_oS$  and the normal space  $\nu_oS$  is a Lie triple system. If, in addition,  $M$  is simply connected, then there exists a reflective submanifold  $F$  of  $M$  with  $o \in F$  and  $T_oF = T_oS$ , where  $o \in S$  is an arbitrary point. In particular, a complete totally geodesic submanifold of a simply connected symmetric space is a symmetric submanifold if and only if it is a reflective submanifold.*

Reflective submanifolds of simply connected irreducible symmetric spaces of compact type have been classified by Leung in [7] and [8], where also an explicit list can be found. Using duality between symmetric spaces of compact type and of noncompact type one can use Leung's classification to derive the corresponding classification for the noncompact case.

We now discuss briefly some classification results of symmetric submanifolds of Euclidean space  $\mathbb{R}^n$ , the sphere  $S^n$ , the complex projective space  $\mathbb{C}P^n$ , the quaternionic projective space  $\mathbb{H}P^n$ , the real hyperbolic space  $\mathbb{R}H^n$ , the complex hyperbolic space  $\mathbb{C}H^n$ , and the quaternionic hyperbolic space  $\mathbb{H}H^n$ . For more details we refer to the survey by Naitoh and Takeuchi [18].

### 3. SYMMETRIC SUBMANIFOLDS OF $\mathbb{R}^n$

An  $R$ -space or *real flag manifold* is an orbit of the isotropy representation of a simply connected symmetric space  $M$  of compact type. Note that the isotropy representation of a symmetric space of noncompact type is the same as the one of the corresponding dual symmetric space of compact type. Thus, in order to classify and study  $R$ -spaces it is sufficient to consider just one type of symmetric spaces. An  $R$ -space which is a symmetric space is called a *symmetric  $R$ -space*. A symmetric  $R$ -space coming from the isotropy representation of an irreducible symmetric space is called an *irreducible symmetric  $R$ -space*. Every symmetric  $R$ -space is the Riemannian product of irreducible symmetric  $R$ -spaces.

The classification of the symmetric  $R$ -spaces was established by Kobayashi and Nagano [5]. It follows from their classification and a result by Takeuchi [21] that the symmetric  $R$ -spaces consist of the Hermitian symmetric spaces of compact type and their real forms. A real form  $M^{\mathbb{R}}$  of a Hermitian symmetric space  $M$  is a connected, complete, totally real, totally geodesic submanifold of  $M$  with  $\dim_{\mathbb{R}} M^{\mathbb{R}} = \dim_{\mathbb{C}} M$ . These real forms were classified by Takeuchi [21] and independently by Leung [9].

One can show that every irreducible symmetric  $R$ -space is a symmetric submanifold of the corresponding tangent space (which is a Euclidean space). Surprisingly, as was proved by Ferus [3], there are essentially no other symmetric submanifolds in Euclidean spaces. More precisely, we have

**Theorem (Ferus).** *Let  $S$  be a symmetric submanifold of  $\mathbb{R}^n$ . Then there exist non-negative integers  $n_0, n_1, \dots, n_k$  with  $n = n_0 + \dots + n_k$  and irreducible symmetric  $R$ -spaces  $S_1 \subset \mathbb{R}^{n_1}, \dots, S_k \subset \mathbb{R}^{n_k}$  such that  $S$  is isometric to  $\mathbb{R}^{n_0} \times S_1 \times \dots \times S_k$  and the embedding of  $S$  into  $\mathbb{R}^n$  is the product embedding of  $\mathbb{R}^{n_0} \times S_1 \times \dots \times S_k$  into  $\mathbb{R}^n = \mathbb{R}^{n_0} \times \dots \times \mathbb{R}^{n_k}$ .*

In the following table we list all symmetric submanifolds  $S$  of  $\mathbb{R}^n$  which are isometric to irreducible symmetric  $R$ -spaces. The spaces in the upper box of the table arise from adjoint representations of compact Lie groups and are the Hermitian symmetric spaces among the irreducible symmetric  $R$ -spaces. The spaces in the lower box are the real forms of Hermitian symmetric spaces.

$S$	$n$	Remarks
$G_2^+(\mathbb{R}^m) = SO(m)/SO(2)SO(m-2)$	$m(m-1)/2$	$m \geq 5$
$SO(2m)/U(m)$	$m(2m-1)$	$m \geq 3$
$G_p(\mathbb{C}^m) = SU(m)/S(U(p)U(m-p))$	$m^2 - 1$	$m \geq 2, 1 \leq p \leq \lfloor \frac{m}{2} \rfloor$
$Sp(m)/U(m)$	$m(2m+1)$	$m \geq 2$
$E_6/Spin(10)U(1)$	78	
$E_7/E_6U(1)$	133	
$G_p(\mathbb{R}^m) = SO(m)/S(O(p)O(m-p))$	$(m+2)(m-1)/2$	$m \geq 3, 1 \leq p \leq \lfloor \frac{m}{2} \rfloor$
$G_p(\mathbb{H}^m) = Sp(m)/Sp(p)Sp(m-p)$	$(2m+1)(m-1)$	$m \geq 2, 1 \leq p \leq \lfloor \frac{m}{2} \rfloor$
$U(m)$	$2m^2$	$m \geq 2$
$(S^{p-1} \times S^{m-p-1})/\mathbb{Z}_2$	$p(m-p)$	$m \geq 3, 1 \leq p \leq \lfloor \frac{m}{2} \rfloor$
$SO(m)$	$m^2$	$m \geq 5$
$U(2m)/Sp(m)$	$2m(2m-1)$	$m \geq 3$
$U(m)/SO(m)$	$m(m+1)$	$m \geq 3$
$Sp(m)$	$4m^2$	$m \geq 2$
$G_2(\mathbb{H}^4)/\mathbb{Z}_2 = (Sp(4)/Sp(2)Sp(2))/\mathbb{Z}_2$	42	
$OP^2 = F_4/Spin(9)$	26	
$(SU(8)/Sp(4))/\mathbb{Z}_2$	70	
$S^1 \times E_6/F_4$	54	

#### 4. SYMMETRIC SUBMANIFOLDS OF $S^n$ AND $\mathbb{R}H^n$

If  $S$  is a symmetric submanifold of  $S^n$ , then it is also a symmetric submanifold of  $\mathbb{R}^{n+1}$ . From the classification of symmetric submanifolds of Euclidean spaces we therefore get easily the classification of symmetric submanifolds of  $S^n$ .

**Theorem.** *Let  $S$  be a symmetric submanifold of  $S^n$ . Then there exist nonnegative integers  $n_1, \dots, n_k$  with  $n+1 = n_1 + \dots + n_k$  and irreducible symmetric  $R$ -spaces  $S_1 \subset \mathbb{R}^{n_1}, \dots, S_k \subset \mathbb{R}^{n_k}$  such that  $S$  is isometric to  $S_1 \times \dots \times S_k$  and the embedding of  $S$  into  $S^n$  is the product embedding of  $S_1 \times \dots \times S_k$  into  $S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$ .*

The classification of symmetric submanifolds of  $\mathbb{R}H^n$  can be deduced in a similar fashion by using the Lorentzian space  $\mathbb{L}^{n+1}$  instead of the Euclidean space  $\mathbb{R}^{n+1}$ . Consider the standard model of the real hyperbolic space in  $\mathbb{L}^{n+1}$ .

**Theorem.** *Let  $S$  be a symmetric submanifold of  $\mathbb{R}H^n$ . Then there exist nonnegative integers  $n_0, n_1, \dots, n_k$  with  $n = n_0 + \dots + n_k$  and irreducible symmetric  $R$ -spaces  $S_1 \subset \mathbb{R}^{n_1}, \dots, S_k \subset \mathbb{R}^{n_k}$  such that  $S$  is isometric to  $\mathbb{R}H^{n_0} \times S_1 \times \dots \times S_k$  and the embedding of  $S$  into  $\mathbb{R}H^n$  is the product embedding of  $\mathbb{R}H^{n_0} \times S_1 \times \dots \times S_k$  into  $\mathbb{L}^{n+1} = \mathbb{L}^{n_0+1} \times \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$ .*

In both theorems one has to consider suitable normalizations so that each embedding really lives in the sphere or in the real hyperbolic space.

### 5. SYMMETRIC COMPLEX SUBMANIFOLDS OF $\mathbb{C}P^n$ AND $\mathbb{C}H^n$

The symmetric complex submanifolds of the complex projective space  $\mathbb{C}P^n$  have been classified by Nakagawa and Takagi in [19].

**Theorem (Nakagawa-Takagi).** *A complete complex submanifold of  $\mathbb{C}P^n$ ,  $n \geq 2$ , is a symmetric submanifold if and only if it is congruent to one of the following models:*

1. *the totally geodesic subspace  $\mathbb{C}P^k \subset \mathbb{C}P^n$  for some  $k \in \{1, \dots, n-1\}$ ;*
2. *the second Veronese embedding of  $\mathbb{C}P^m$  into  $\mathbb{C}P^n$ , where  $n = (m+2)(m+1)/2 - 1$  and  $m \geq 2$ , given by the set of all rank one matrices (up to complex scalars) in the  $(n+1)$ -dimensional complex vector space of all symmetric  $(m+1) \times (m+1)$ -matrices with complex coefficients;*
3. *the Segre embedding of  $\mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2}$  into  $\mathbb{C}P^n$ , where  $n = (m_1+1)(m_2+1) - 1$  and  $m_1, m_2 \geq 1$ , given by*

$$([z_0 : \dots : z_{m_1}], [w_0 : \dots : w_{m_2}]) \mapsto [z_0 w_0 : \dots : z_\nu w_\mu : \dots : z_{m_1} w_{m_2}] ;$$

4. *the complex quadric  $Q^{n-1} = G_2^+(\mathbb{R}^{n+1})$  in  $\mathbb{C}P^n$ , where  $n \geq 2$ , given by*

$$Q^{n-1} = \{[z_0 : \dots : z_n] \in \mathbb{C}P^n \mid z_0^2 + \dots + z_n^2 = 0\} ;$$

5. *the Plücker embedding of the complex Grassmannian  $G_2(\mathbb{C}^m)$  into  $\mathbb{C}P^n$ , where  $n = m(m-1)/2 - 1$  and  $m \geq 3$ , which is induced by the orbit through  $e_1 \wedge e_2$  of the canonical action of  $SU(m)$  on  $\Lambda^2 \mathbb{C}^m$ .*
6. *the embedding of the Hermitian symmetric space  $SO(10)/U(5)$  into  $\mathbb{C}P^{15}$  which is induced by the orbit through the highest weight vector of a 16-dimensional irreducible spin representation of  $SO(10)$ ;*
7. *the embedding of the Hermitian symmetric space  $E_6/Spin(10)U(1)$  into  $\mathbb{C}P^{26}$  which is induced by the orbit through the highest weight vector of the 27-dimensional irreducible representation of the exceptional Lie group  $E_6$ .*

Kon [6] proved that every symmetric complex submanifold of the complex hyperbolic space  $\mathbb{C}H^n$ ,  $n \geq 2$ , is totally geodesic.

6. SYMMETRIC TOTALLY REAL SUBMANIFOLDS OF  $\mathbb{C}P^n$  AND  $\mathbb{C}H^n$ 

The classification of  $n$ -dimensional symmetric totally real submanifolds of  $\mathbb{C}P^n$  was established by Naitoh [11] (for the irreducible case) and by Naitoh and Takeuchi [17] (for the general case). Consider first the natural action of  $SL(m, \mathbb{C})$  on  $J_m(\mathbb{R}) \otimes \mathbb{C}$ , the complexification of the real Jordan algebra  $J_m(\mathbb{R})$  of all symmetric  $m \times m$ -matrices with real coefficients, given by  $(A, X) \mapsto AXA^T$  for  $A \in SL(m, \mathbb{C})$  and  $X \in J_m(\mathbb{R}) \otimes \mathbb{C}$ . The complex dimension of  $J_m(\mathbb{R}) \otimes \mathbb{C}$  is  $m(m+1)/2$ , and hence this action induces an action of  $SL(m, \mathbb{C})$  on  $\mathbb{C}P^n$  with  $n = m(m+1)/2 - 1$ . This action has exactly  $m$  orbits which are parametrized by the rank of the matrices. The subgroup of  $SL(m, \mathbb{C})$  preserving complex conjugation on  $\mathbb{C}P^n$  is  $SL(m, \mathbb{R})$ . Now fix a maximal compact subgroup  $SO(m)$  of  $SL(m, \mathbb{R})$ . The restriction to  $SO(m, \mathbb{C})$  of the action of  $SL(m, \mathbb{C})$  on  $J_m(\mathbb{R}) \otimes \mathbb{C}$  splits off a one-dimensional trivial factor corresponding to the trace. This means that  $SO(m, \mathbb{C})$ , and hence  $SO(m)$ , fixes the point  $o$  in  $\mathbb{C}P^n$  given by complex scalars of the identity matrix in  $J_m(\mathbb{R}) \otimes \mathbb{C}$ . The maximal compact subgroup  $SO(m)$  of  $SL(m, \mathbb{R})$  determines a maximal compact subgroup  $SU(m)$  of  $SL(m, \mathbb{C})$ . The orbit of the action of  $SU(m)$  through  $o$  gives an embedding of  $SU(m)/SO(m)$  in  $\mathbb{C}P^n$  as a symmetric totally real submanifold of real dimension  $n$ . Using the real Jordan algebras  $J_m(\mathbb{C})$ ,  $J_m(\mathbb{H})$  and  $J_3(\mathbb{O})$ , and corresponding subgroups according to the following table,

$SL(m, \mathbb{C})$	$SL(m, \mathbb{C}) \times SL(m, \mathbb{C})$	$SL(2m, \mathbb{C})$	$E_6(\mathbb{C})$
$SL(m, \mathbb{R})$	$SL(m, \mathbb{C})$	$SL(m, \mathbb{H})$	$E_6^{-26}$
$SO(m)$	$SU(m)$	$Sp(m)$	$F_4$
$SU(m)$	$SU(m) \times SU(m)$	$SU(2m)$	$E_6$

we can construct further symmetric totally real submanifolds of  $\mathbb{C}P^n$ . Naitoh proved in [11] that all  $n$ -dimensional irreducible symmetric totally real submanifolds of  $\mathbb{C}P^n$  which are not totally geodesic can be obtained in this way.

**Theorem (Naitoh).** *An  $n$ -dimensional complete irreducible totally real submanifold of  $\mathbb{C}P^n$ ,  $n \geq 2$ , is a symmetric submanifold if and only if it is congruent to one of the following embeddings:*

1. the totally geodesic subspace  $\mathbb{R}P^n$  of  $\mathbb{C}P^n$ ;
2. the embedding of  $SU(m)/SO(m)$  into  $\mathbb{C}P^n$  via the Jordan algebra  $J_m(\mathbb{R})$ , where  $m \geq 3$  and  $n = (m+1)m/2 - 1$ ;
3. the embedding of  $SU(m)$  into  $\mathbb{C}P^n$  via the Jordan algebra  $J_m(\mathbb{C})$ , where  $m \geq 3$  and  $n = m^2 - 1$ ;
4. the embedding of  $SU(2m)/Sp(m)$  into  $\mathbb{C}P^n$  via the Jordan algebra  $J_m(\mathbb{H})$ , where  $m \geq 3$  and  $n = (2m+1)(m-1)$ ;
5. the embedding of  $E_6/F_4$  into  $\mathbb{C}P^{26}$  via the Jordan algebra  $J_3(\mathbb{O})$ .

Naitoh and Takeuchi proved in [17] that each  $n$ -dimensional symmetric totally real submanifold of  $\mathbb{C}P^n$  is basically a product of the irreducible submanifolds discussed above and a flat torus. A suitable product of  $n+1$  circles in  $S^{2n+1}$  projects via the Hopf map to a flat torus  $T^n$  embedded in  $\mathbb{C}P^n$  as a symmetric totally real submanifold. Naitoh and Takeuchi gave in [17] a unifying description of all  $n$ -dimensional symmetric totally real

submanifolds of  $\mathbb{C}P^n$  using the Shilov boundary of symmetric bounded domains of tube type. The classification of  $n$ -dimensional symmetric totally real submanifolds of  $\mathbb{C}H^n$  has been obtained by Naitoh in [12].

### 7. SYMMETRIC TOTALLY COMPLEX SUBMANIFOLDS OF $\mathbb{H}P^n$ AND $\mathbb{H}H^n$

The  $2n$ -dimensional symmetric totally complex submanifolds of the quaternionic projective space  $\mathbb{H}P^n$  have been classified by Tsukada in [22]. A basic tool for the classification is the twistor map  $\mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$ . Consider  $\mathbb{H}^{n+1}$  as a (right) vector space and pick a unit quaternion, say  $i$ , which turns  $\mathbb{H}^{n+1}$  into a complex vector space  $\mathbb{C}^{2n+2}$ . The twistor map  $\mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$  maps a complex line in  $\mathbb{C}^{2n+2}$  to the quaternionic line in  $\mathbb{H}^{n+1}$  spanned by it. The fiber over each point is a complex projective line  $\mathbb{C}P^1 \subset \mathbb{C}P^{2n+1}$ . Alternatively, the set of all almost Hermitian structures in the quaternionic Kähler structure of a quaternionic Kähler manifold  $M$  forms the so-called twistor space  $Z$  of  $M$ , and the natural projection  $Z \rightarrow M$  is the so-called twistor map onto  $M$ . In the case of  $\mathbb{H}P^n$  the twistor space is just  $\mathbb{C}P^{2n+1}$ .

Now let  $S$  be a  $2n$ -dimensional non-totally geodesic symmetric totally complex submanifold of  $\mathbb{H}P^n$ . The first step in the classification is to show that  $S$  is a Hermitian symmetric space with respect to a Kähler structure which is induced from the quaternionic Kähler structure of  $\mathbb{H}P^n$ . Then one shows that  $S$  can be lifted to a Kähler immersion into the twistor space  $\mathbb{C}P^{2n+1}$ . The main part of the proof is then to show, using representation theory of complex semisimple Lie algebras, that this lift is one of the following embeddings in  $\mathbb{C}P^{2n+1}$ :

1. the embedding of  $Sp(3)/U(3)$  into  $\mathbb{C}P^{13}$ ;
2. the embedding of  $G_3(\mathbb{C}^6)$  into  $\mathbb{C}P^{19}$ ;
3. the embedding of  $SO(12)/U(6)$  into  $\mathbb{C}P^{31}$ ;
4. the embedding of  $E_7/E_6U(1)$  into  $\mathbb{C}P^{55}$ ;
5. the embedding of  $\mathbb{C}P^1 \times G_2^+(\mathbb{R}^m)$  into  $\mathbb{C}P^{2n+1}$ , where  $m \geq 3$  and  $n = m - 1$ .

Note that in the last case the submanifold is isometric to  $\mathbb{C}P^1 \times \mathbb{C}P^1$  for  $m = 3$  and isometric to  $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$  for  $m = 4$ . The embedding of  $G_3(\mathbb{C}^6)$  into  $\mathbb{C}P^{19}$  is the Plücker embedding. The image of each of these embeddings under the Hopf map  $\mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$  is indeed a  $2n$ -dimensional symmetric totally complex submanifold of  $\mathbb{H}P^n$ , and Tsukada proved

**Theorem (Tsukada).** *A  $2n$ -dimensional complete irreducible totally complex submanifold of  $\mathbb{H}P^n$ ,  $n \geq 2$ , is a symmetric submanifold if and only if it is congruent to one of the following embeddings:*

1. the totally geodesic subspace  $\mathbb{C}P^n$  of  $\mathbb{H}P^n$ ;
2. the embedding of  $Sp(3)/U(3)$  into  $\mathbb{H}P^6$ ;
3. the embedding of  $G_3(\mathbb{C}^6)$  into  $\mathbb{H}P^9$ ;
4. the embedding of  $SO(12)/U(6)$  into  $\mathbb{H}P^{15}$ ;
5. the embedding of  $E_7/E_6U(1)$  into  $\mathbb{H}P^{27}$ ;
6. the embedding of  $\mathbb{C}P^1 \times G_2^+(\mathbb{R}^m)$  into  $\mathbb{H}P^n$ , where  $m \geq 3$  and  $n = m - 1$ .

Tsukada proved in [22] that every symmetric totally complex submanifold of the quaternionic hyperbolic space  $\mathbb{H}H^n$ ,  $n \geq 2$ , is totally geodesic.

## 8. GRASSMANN GEOMETRIES

Let  $M$  be an  $m$ -dimensional connected Riemannian manifold. For each integer  $k \in \{1, \dots, m-1\}$  the identity component  $G$  of the isometry group of  $M$  acts canonically on the Grassmann bundle  $G_k(TM)$  of all  $k$ -dimensional linear subspaces of  $T_pM$ ,  $p \in M$ . For  $V \in G_k(TM)$  we denote by  $G \cdot V$  the orbit of  $G$  through  $V$ . By  $\mathfrak{G}(V)$  we denote the set of all connected submanifolds  $S$  of  $M$  with the property that all its tangent spaces belong to  $G \cdot V$ . The set  $\mathfrak{G}(V)$  is called the *Grassmann geometry associated to  $V$* . If  $S$  is a homogeneous submanifold of  $M$ , then all its tangent spaces lie in the same orbit of  $G$ , and hence the *Grassmann geometry  $\mathfrak{G}(S)$  associated to  $S$*  is well-defined. We give some examples.

1. Let  $S^n$  be the  $n$ -dimensional sphere and  $\mathbb{R}H^n$  the  $n$ -dimensional real hyperbolic space, and  $k \in \{1, \dots, n-1\}$ . The Grassmann geometry  $\mathfrak{G}(S^k)$  (respectively  $\mathfrak{G}(\mathbb{R}H^k)$ ) associated to a totally geodesic  $S^k \subset S^n$  (respectively  $\mathbb{R}H^k \subset \mathbb{R}H^n$ ) is the set of all  $k$ -dimensional connected submanifolds of  $S^n$  (respectively  $\mathbb{R}H^n$ ).
2. Let  $\mathbb{C}P^n$  be the  $n$ -dimensional complex projective space and  $\mathbb{C}H^n$  the  $n$ -dimensional complex hyperbolic space, and  $k \in \{1, \dots, n-1\}$ . The Grassmann geometry  $\mathfrak{G}(\mathbb{C}P^k)$  (respectively  $\mathfrak{G}(\mathbb{C}H^k)$ ) associated to a totally geodesic  $\mathbb{C}P^k \subset \mathbb{C}P^n$  (respectively  $\mathbb{C}H^k \subset \mathbb{C}H^n$ ) is the set of all  $k$ -dimensional connected complex submanifolds of  $\mathbb{C}P^n$  (respectively  $\mathbb{C}H^n$ ).
3. The Grassmann geometry  $\mathfrak{G}(\mathbb{R}P^n)$  (respectively  $\mathfrak{G}(\mathbb{R}H^n)$ ) associated to a totally geodesic  $\mathbb{R}P^n \subset \mathbb{C}P^n$  (respectively  $\mathbb{R}H^n \subset \mathbb{C}H^n$ ) is the set of all  $n$ -dimensional connected totally real submanifolds of  $\mathbb{C}P^n$  (respectively  $\mathbb{C}H^n$ ).
4. Let  $\mathbb{H}P^n$  be the  $n$ -dimensional quaternionic projective space and  $\mathbb{H}H^n$  the  $n$ -dimensional quaternionic hyperbolic space, and  $k \in \{1, \dots, n-1\}$ . The Grassmann geometry  $\mathfrak{G}(\mathbb{C}P^n)$  (respectively  $\mathfrak{G}(\mathbb{C}H^n)$ ) associated to a totally geodesic  $\mathbb{C}P^n \subset \mathbb{H}P^n$  (respectively  $\mathbb{C}H^n \subset \mathbb{H}H^n$ ) is the set of all  $n$ -dimensional connected totally complex submanifolds of  $\mathbb{H}P^n$  (respectively  $\mathbb{H}H^n$ ).

We can now rephrase the corollary from Section 2 in the following way.

**Corollary.** *Let  $S$  be a symmetric submanifold of a simply connected Riemannian symmetric space  $M$ . Then there exists a reflective submanifold  $F$  of  $M$  such that  $S \in \mathfrak{G}(F)$ .*

This motivates to study the Grassmann geometries of reflective submanifolds in more detail. We say that a Grassmann geometry associated to a reflective submanifold  $F$  is trivial if it contains only totally geodesic submanifolds. Otherwise the Grassmann geometry  $\mathfrak{G}(F)$  is said to be nontrivial. For simply connected Riemannian symmetric space of compact type Naitoh determined in [13], [14], [15], [16] all nontrivial Grassmann geometries associated to reflective submanifolds. Using duality between Riemannian symmetric spaces of compact type and of noncompact type his results can easily be transferred also to the noncompact case. For the irreducible case his main result is



**Theorem (Naitoh).** *Let  $F$  be a reflective submanifold of a simply connected irreducible Riemannian symmetric space. Then the Grassmann geometry  $\mathfrak{G}(F)$  associated to  $F$  is nontrivial if and only if it is one of the Grassmann geometries in the above examples or if it is a Grassmann geometry associated to an irreducible symmetric  $R$ -space (see next section).*

This result might be viewed as a generalization of the classical result by Alekseevsky [1] that a quaternionic submanifold of  $\mathbb{H}P^n$  or  $\mathbb{H}H^n$  is totally geodesic, as the Grassmann geometries associated to  $\mathbb{H}P^k$  and  $\mathbb{H}H^k$  are trivial according to Naitoh's result. We now have to explain the Grassmann geometries associated to irreducible symmetric  $R$ -spaces.

## 9. GRASSMANN GEOMETRIES ASSOCIATED TO IRREDUCIBLE SYMMETRIC $R$ -SPACES

Let  $S$  be an irreducible symmetric  $R$ -space coming from the isotropy representation of the simply connected irreducible Riemannian symmetric space  $M = G/K$ . Then  $S$  is the orbit of the action of  $K$  on  $\mathfrak{m}$  through a nonzero tangent vector  $X \in T_oM \cong \mathfrak{m}$ . The vector  $X$  determines a closed geodesic  $\gamma$  in  $M$ . The antipodal point  $o'$  of  $o$  on  $\gamma$  is a pole of  $o$ , that is,  $o'$  is a fixed point of the action of  $K$  on  $M$ . Let  $p$  and  $p'$  be the two midpoints on  $\gamma$  between  $o$  and  $o'$ . Then the orbit  $F = K \cdot p = K \cdot p'$  is isometric to the irreducible symmetric  $R$ -space  $S$ . Moreover,  $F$  is a reflective submanifold of  $M$ . The Grassmann geometry  $\mathfrak{G}(F)$  associated to  $F$  is what we call the Grassmann geometry associated to the irreducible symmetric  $R$ -space  $S$ . The submanifold  $F$  is also known as the centrosome of the two poles  $o$  and  $o'$ . Moreover, the orbit of  $K$  through any point on the geodesic different from  $o, o', p, p'$  is a symmetric submanifold of  $M$  that is not totally geodesic.

We illustrate this with a simple example. Consider the sphere  $S^n = SO(n+1)/SO(n)$ , where  $SO(n)$  is the isotropy subgroup of  $SO(n+1)$  at  $o \in S^n$ . Then  $SO(n)$  has exactly one other fixed point  $o'$  in  $S^n$ , namely the antipodal point of  $o$  in  $S^n$ . Let  $\gamma$  be a closed geodesic through  $o$  and  $o'$ . Then the orbit  $F$  of  $SO(n)$  through any of the two midpoints on  $\gamma$  between  $o$  and  $o'$  is an equator, that is, a totally geodesic hypersphere  $S^{n-1}$ , which obviously is a reflective submanifold. The orbits through the other points on  $\gamma$  are totally umbilical hyperspheres and clearly symmetric submanifolds of the sphere  $S^n$ .

**Theorem (Naitoh).** *Let  $\mathfrak{G}(F)$  be the Grassmann geometry associated to the irreducible symmetric  $R$ -space  $F$  and assume that the rank of  $M$  is greater than one. Then every symmetric submanifold in  $\mathfrak{G}(F)$  that is not totally geodesic arises in the way described above.*

We now turn to the noncompact case. We start with recalling the theory of symmetric  $R$ -spaces from another viewpoint (see Kobayashi and Nagano [5], Nagano [10], and Takeuchi [20] for details). Let  $(\mathfrak{g}, \sigma)$  be a positive definite symmetric graded Lie algebra, that is,  $\mathfrak{g}$  is a real semisimple Lie algebra with a gradation  $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  so that  $\mathfrak{g}_{-1} \neq \{0\}$  and the adjoint action of  $\mathfrak{g}_0$  on the vector space  $\mathfrak{g}_{-1}$  is effective, and a Cartan involution  $\sigma$  satisfying  $\sigma(\mathfrak{g}_p) = \mathfrak{g}_{-p}$  for all  $p \in \{-1, 0, 1\}$ . The positive definite symmetric graded Lie algebras have been completely classified (see [5], [20]).

By defining  $\tau(X) = (-1)^p X$  for  $X \in \mathfrak{g}_p$  we obtain an involutive automorphism  $\tau$  of  $\mathfrak{g}$  which satisfies  $\sigma\tau = \tau\sigma$ . Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition induced by  $\sigma$ .

Then we have  $\tau(\mathfrak{k}) = \mathfrak{k}$  and  $\tau(\mathfrak{p}) = \mathfrak{p}$ . Let  $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-$  and  $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$  be the  $\pm 1$ -eigenspace decompositions of  $\mathfrak{k}$  and  $\mathfrak{p}$  with respect to  $\tau$ . Obviously, we have  $\mathfrak{k}_+ = \mathfrak{k} \cap \mathfrak{g}_0$ ,  $\mathfrak{k}_- = \mathfrak{k} \cap (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)$ ,  $\mathfrak{p}_+ = \mathfrak{p} \cap \mathfrak{g}_0$  and  $\mathfrak{p}_- = \mathfrak{p} \cap (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)$ . Since  $\mathfrak{g}$  is a semisimple Lie algebra, there is a unique element  $\nu \in \mathfrak{g}_0$  so that  $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid \text{ad}(\nu)X = pX\}$  for all  $p \in \{-1, 0, 1\}$ . It can be easily seen that  $\nu \in \mathfrak{p}$  and hence  $\nu \in \mathfrak{p}_+$ .

We denote by  $B$  the Killing form of  $\mathfrak{g}$ . The restriction of  $B$  to  $\mathfrak{p} \times \mathfrak{p}$  is a positive definite inner product on  $\mathfrak{p}$  and will be denoted by  $\langle \cdot, \cdot \rangle$ . This inner product is invariant under the adjoint action of  $\mathfrak{k}$  on  $\mathfrak{p}$  and under the involution  $\tau|_{\mathfrak{p}}$ . In particular,  $\mathfrak{p}_+$  and  $\mathfrak{p}_-$  are perpendicular to each other. Let  $G$  be the simply connected Lie group with Lie algebra  $\mathfrak{g}$  and  $K$  be the connected Lie subgroup of  $G$  corresponding to  $\mathfrak{k}$ , and define the homogeneous space  $M = G/K$ . Let  $\pi : G \rightarrow M$  be the natural projection, and put  $o = \pi(e)$ , where  $e \in G$  is the identity. The restriction to  $\mathfrak{p}$  of the differential  $\pi_{*e} : \mathfrak{g} \rightarrow T_oM$  of  $\pi$  at  $e$  yields a linear isomorphism  $\mathfrak{p} \rightarrow T_oM$ . In the following we will always identify  $\mathfrak{p}$  and  $T_oM$  via this isomorphism. From the  $\text{Ad}(K)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p} \cong T_oM$  we get a  $G$ -invariant Riemannian metric on  $M$ . Then  $M = G/K$  is the Riemannian symmetric space of noncompact type which is associated with  $(\mathfrak{g}, \sigma, \langle \cdot, \cdot \rangle)$ .

We put  $K'_+ = \{k \in K \mid \text{Ad}(k)\nu = \nu\}$ . Then  $K'_+$  is a closed Lie subgroup whose Lie algebra is  $\mathfrak{k}_+$ . The homogeneous space  $M' = K/K'_+$  is diffeomorphic to the orbits  $\text{Ad}(K) \cdot \nu \subset \mathfrak{p}$  and  $K \cdot \pi(\exp \nu) \subset M$ , where  $\exp : \mathfrak{g} \rightarrow G$  denotes the Lie exponential map from  $\mathfrak{g}$  into  $G$ . We equip  $M'$  with the induced Riemannian metric from  $M$ . Then  $M'$  is a compact Riemannian symmetric space associated to the orthogonal symmetric Lie algebra  $(\mathfrak{k}, \tau|_{\mathfrak{k}})$ , where  $\tau|_{\mathfrak{k}}$  is the restriction of  $\tau$  to  $\mathfrak{k}$ . The symmetric spaces  $M'$  arising in this manner are precisely the symmetric R-spaces. If  $\mathfrak{g}$  is simple, then  $M'$  is an irreducible symmetric R-space.

The subspace  $\mathfrak{p}_-$  is a Lie triple system in  $\mathfrak{p} = T_oM$  and  $[\mathfrak{p}_-, \mathfrak{p}_-] \subset \mathfrak{k}_+$ . Thus there exists a complete totally geodesic submanifold  $F$  of  $M$  with  $o$  and  $T_oF = \mathfrak{p}_-$ . Since  $F$  is the image of  $\mathfrak{p}_-$  under the exponential map of  $M$  at  $o$ , we see that  $F$  is simply connected. We define a Lie subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  by  $\mathfrak{h} = \mathfrak{k}_+ \oplus \mathfrak{p}_-$  and denote by  $H$  the connected Lie subgroup of  $G$  which corresponds to  $\mathfrak{h}$ . Then, by construction,  $F$  is the  $H$ -orbit through  $o$ . We denote by  $K_+$  the isotropy subgroup at  $o$  of the action of  $H$  on  $M$ . The Lie algebra of  $K_+$  is  $\mathfrak{k}_+$ . Since  $F = H/K_+$  is simply connected,  $K_+$  is connected. The restriction  $\tau|_{\mathfrak{h}}$  of  $\tau$  to  $\mathfrak{h}$  is an involutive automorphism of  $\mathfrak{g}$  and  $(\mathfrak{h}, \tau|_{\mathfrak{h}})$  is an orthogonal symmetric Lie algebra dual to  $(\mathfrak{k}, \tau|_{\mathfrak{k}})$ . Moreover,  $F$  is a Riemannian symmetric space of noncompact type associated with  $(\mathfrak{h}, \tau|_{\mathfrak{h}})$ . Since both  $\mathfrak{p}_-$  and  $\mathfrak{p}_+$  are Lie triple systems,  $F$  is a reflective submanifold of  $M$ . The corresponding Grassmann geometry  $\mathfrak{G}(F)$  is a geometry according to Naitoh's Theorem.

We will construct a one-parameter family of symmetric submanifolds in  $M$  consisting of submanifolds belonging to that Grassmann geometry, and which contains the totally geodesic submanifold  $F$  and the symmetric R-space  $M'$ . For each  $c \in \mathbb{R}$  we define a subspace  $\mathfrak{p}_c$  of  $\mathfrak{p}_- \oplus \mathfrak{k}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$  by  $\mathfrak{p}_c = \{X + c \text{ad}(\nu)X \mid X \in \mathfrak{p}_-\}$ . In particular,  $\mathfrak{p}_1 = \mathfrak{g}_1$  and  $\mathfrak{p}_{-1} = \mathfrak{g}_{-1}$  are Abelian subalgebras of  $\mathfrak{g}$ . Then  $\mathfrak{h}_c = \mathfrak{k}_+ \oplus \mathfrak{p}_c$  is a  $\tau$ -invariant Lie subalgebra of  $\mathfrak{g}$  and  $(\mathfrak{h}_c, \tau|_{\mathfrak{h}_c})$  is an orthogonal symmetric Lie algebra. We denote by

$H_c$  the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{h}_c$  and by  $S_c$  the orbit of the origin  $o$  by  $H_c$  in  $M$ .

**Proposition.** *For each  $c \in \mathbb{R}$  the orbit  $S_c$  is a symmetric submanifold of  $M$  belonging to the Grassmann geometry  $\mathfrak{G}(F)$  on  $M$ . The submanifolds  $S_c$  and  $S_{-c}$  are congruent via the geodesic symmetry  $s_o$  of  $M$  at  $o$ . The submanifolds  $S_c$ ,  $0 \leq c < 1$ , form a family of noncompact symmetric submanifolds which are homothetic to the reflective submanifold  $F$ . The submanifolds  $S_c$ ,  $1 < c < \infty$ , form a family of compact symmetric submanifolds which are homothetic to the symmetric  $R$ -space  $M'$ . The submanifold  $M_1$  is a flat symmetric space which is isometric to a Euclidean space. The second fundamental form  $\alpha_c$  of  $S_c$  is given by  $\alpha_c(X, Y) = c[\text{ad}(\nu)X, Y] \in \mathfrak{p}_+ = T_o^\perp S_c$  for all  $X, Y \in \mathfrak{p}_- = T_o S_c$ . In particular, all submanifolds  $S_c$ ,  $0 \leq c < \infty$ , are pairwise noncongruent.*

It was proved in [2] that every symmetric submanifold of an irreducible Riemannian symmetric space of noncompact type and rank  $\geq 2$  arises in this way.

**Theorem (Berndt-Eschenburg-Naitoh-Tsukada).** *Let  $M$  be an irreducible Riemannian symmetric space of noncompact type and rank  $\geq 2$ , and let  $\mathfrak{G}(F)$  be the Grassmann geometry on  $M$  associated to a reflective submanifold  $F$  whose compact dual is a Grassmann geometry associated to an irreducible symmetric  $R$ -space. Then every complete submanifold in  $\mathfrak{G}(F)$  is congruent to  $F$  or to a symmetric submanifold  $S_c$  as constructed above.*

We list below the symmetric spaces  $M$  and reflective submanifolds that are relevant for the above theorem.

$M$	$F$	Remarks
$SO(n, \mathbb{C})/SO(n)$	$SO^o(2, n-2)/SO(2)SO(n-2)$	$n \geq 5$
$SO(2n, \mathbb{C})/SO(2n)$	$SO(n, \mathbb{H})/U(n)$	$n \geq 3$
$SL(n, \mathbb{C})/SU(n)$	$SU(p, n-p)/S(U(p)U(n-p))$	$n \geq 2$ , $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$
$Sp(n, \mathbb{C})/Sp(n)$	$Sp(n, \mathbb{R})/U(n)$	$n \geq 2$
$E_6^{\mathbb{C}}/E_6$	$E_6^{-14}/Spin(10)U(1)$	
$E_7^{\mathbb{C}}/E_7$	$E_7^{-25}/E_6U(1)$	
$SL(n, \mathbb{R})/SO(n)$	$SO^o(p, n-p)/SO(p)SO(n-p)$	$n \geq 3$ , $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$
$SL(n, \mathbb{H})/Sp(n)$	$Sp(p, n-p)/Sp(p)Sp(n-p)$	$n \geq 2$ , $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$
$SU(n, n)/S(U(n)U(n))$	$\mathbb{R} \times SL(n, \mathbb{C})/SU(n)$	$n \geq 2$
$SO^o(p, n-p)/SO(p)SO(n-p)$	$\mathbb{R}H^{p-1} \times \mathbb{R}H^{n-p-1}$	$n \geq 3$ , $2 \leq p \leq \lfloor \frac{n}{2} \rfloor$
$SO^o(n, n)/SO(n)SO(n)$	$SO(n, \mathbb{C})/SO(n)$	$n \geq 5$
$SO(2n, \mathbb{H})/U(2n)$	$\mathbb{R} \times SL(n, \mathbb{H})/Sp(n)$	$n \geq 3$
$Sp(n, \mathbb{R})/U(n)$	$\mathbb{R} \times SL(n, \mathbb{R})/SO(n)$	$n \geq 3$
$Sp(n, n)/Sp(n)Sp(n)$	$Sp(n, \mathbb{C})/Sp(n)$	$n \geq 2$
$E_6^6/Sp(4)$	$Sp(2, 2)/Sp(2)Sp(2)$	
$E_6^{-26}/F_4$	$F_4^{-20}/Spin(9)$	
$E_7^7/SU(8)$	$SL(4, \mathbb{H})/Sp(4)$	
$E_7^{-25}/E_6U(1)$	$\mathbb{R} \times E_6^{-26}/F_4$	

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