LIE GROUP ACTIONS ON MANIFOLDS

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1. Introduction

One of the most successful approaches to geometry is the one suggested by Felix Klein. According to Klein, a geometry is a G-space M, that is, a set M together with a group G of transformations of M. This approach provides a powerful link between geometry and algebra. Of particular importance is the situation when the group G acts transitively on M, that is, for any two points p and q in M there exists a transformation in G which maps p to q. In this situation M is called a homogeneous G-space. Basic examples of homogeneous geometries are Euclidean geometry, affine geometry, projective geometry and elliptic geometry. In the homogeneous situation many geometric problems can be reformulated in algebraic terms which are often easier to solve. For instance, Einstein's equations in general relativity form a complicated system of nonlinear partial differential equations, but in the special case of a homogeneous manifold these equations reduce to algebraic equations which can be solved explicitly in many cases.

The situation for inhomogeneous geometries is much more complicated. Nevertheless, one special case is currently of particular importance. This special case is when the action of the transformation group G has an orbit of codimension one in M, in which case the action is said to be of cohomogeneity one and M is called a cohomogeneity one G-space. In this situation the above mentioned Einstein equations reduce to an ordinary differential equation which can also be solved in many cases. A fundamental problem is to investigate and to classify all cohomogeneity one G-spaces satisfying some given properties.

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2. Riemannian geometry

In this section we summarize some of the basics of Riemannian geometry that is used in this course. Some modern introductions to Riemannian geometry can be found in the books by Chavel [21], Gallot-Hulin-Lafontaine [29], Jost [37], Petersen [58] and Sakai [59].

2.1. **Riemannian manifolds.** Let M be an m-dimensional smooth manifold. By smooth we always mean C^∞, and manifolds are always assumed to satisfy the second countability axiom and hence are paracompact. For each $p \in M$ we denote by T_pM the tangent space of M at p. The tangent bundle of M is denoted by TM.

Suppose each tangent space T_pM is equipped with an inner product $\langle \cdot, \cdot \rangle_p$. If the function $p \mapsto \langle X_p, Y_p \rangle_p$ is smooth for any two smooth vector fields X, Y on M, then this family of inner products is called a *Riemannian metric*, or *Riemannian structure*, on M. Usually we denote a Riemannian metric, and each of the inner products it consists of, by $\langle \cdot, \cdot \rangle$. Paracompactness implies that any smooth manifold admits a Riemannian structure. A smooth manifold equipped with a Riemannian metric is called a *Riemannian manifold*.

2.2. **Length, distance, and completeness.** The presence of an inner product on each tangent space allows one to measure the length of tangent vectors, by which we can define the length of curves and a distance function. For the latter one we have to assume that M is connected. If $c : [a, b] \to M$ is any smooth curve into a Riemannian manifold M, the *length* $L(c)$ of c is defined by

$$L(c) := \int_a^b \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle} dt ,$$

where $\dot{c}$ denotes the tangent vector field of c. The length $L(c)$ of a piecewise smooth curve $c : [a, b] \to M$ is then defined in the usual way by means of a suitable subdivision of $[a, b]$. The *distance* $d(p, q)$ between two points $p, q \in M$ is defined as the infimum over all $L(c)$, where $c : [a, b] \to M$ is a piecewise smooth curve in M with $c(a) = p$ and $c(b) = q$. The distance function $d : M \times M \to \mathbb{R}$ turns M into a metric space. The topology on M induced by this metric coincides with the underlying manifold topology. A *complete Riemannian manifold* is a Riemannian manifold M which is complete when considered as a metric space, that is, if every Cauchy sequence in M converges in M.

2.3. **Isometries.** Let M and N be Riemannian manifolds with Riemannian metrics $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_N$, respectively. A smooth diffeomorphism $f : M \to N$ is called an *isometry* if $\langle f_*X, f_*Y \rangle_N = \langle X, Y \rangle_M$ for all $X, Y \in T_pM$, $p \in M$, where f_* denotes the differential of f at p. If M is connected, a surjective continuous map $f : M \to M$ is an isometry if and only if it preserves the distance function d on M, that is, if $d(f(p), f(q)) = d(p, q)$ for all $p, q \in M$. An isometry of a connected Riemannian manifold is completely determined by both its value and its differential at some point. In particular, an isometry which fixes a point and whose differential at this point is the identity is the identity map. If M is a connected, simply connected, complete, real analytic Riemannian manifold, then every local isometry of M can be extended to a global isometry of M.

The isometries of a Riemannian manifold form a group in an obvious manner, which we shall denote by $I(M)$ and call the *isometry group* of M. We consider this group always as a topological group equipped with the compact-open topology. With respect to this topology it carries the structure of a Lie group acting on M as a Lie transformation group. We usually denote by $I^o(M)$ the identity component of $I(M)$, that is, the connected component of $I(M)$ containing the identity transformation of M.

2.4. **Riemannian products and covering spaces.** Let M_1 and M_2 be Riemannian manifolds. At each point $(p_1, p_2) \in M_1 \times M_2$ the tangent space $T_{(p_1,p_2)}(M_1 \times M_2)$ is canonically isomorphic to the direct sum $T_{p_1}M_1 \oplus T_{p_2}M_2$. The inner products on $T_{p_1}M_1$ and $T_{p_2}M_2$ therefore induce an inner product on $T_{(p_1,p_2)}(M_1 \times M_2)$. In this way we get a Riemannian metric on $M_1 \times M_2$. The product manifold $M_1 \times M_2$ equipped with this Riemannian metric is called the *Riemannian product* of M_1 and M_2. For each connected Riemannian manifold M there exists a connected, simply connected Riemannian manifold $\widetilde{M}$ and an isometric covering map $\widetilde{M} \to M$. Such a manifold $\widetilde{M}$ is unique up to isometry and is called the *Riemannian universal covering space* of M. A Riemannian manifold M is called *reducible* if its Riemannian universal covering space $\widetilde{M}$ is isometric to the Riemannian product of at least two Riemannian manifolds of dimension ≥ 1. Otherwise M is called *irreducible*. A Riemannian manifold M is said to be *locally reducible* if for each point $p \in M$ there exists an open neighborhood of p in M which is isometric to the Riemannian product of at least two Riemannian manifolds of dimension ≥ 1. Otherwise M is said to be *locally irreducible*.

2.5. **Connections.** There is a natural way to differentiate smooth functions on a smooth manifold, but there is no natural way to differentiate smooth vector fields on a smooth manifold. The theory that consists of studying the various possibilities for such a differentiation process is called the theory of connections, or covariant derivatives. A *connection* on a smooth manifold M is an operator ∇ assigning to two vector fields X, Y on M another vector field $\nabla_X Y$ and satisfying the following axioms:

(i) ∇ is $\mathbb{R}$-bilinear;
(ii) $\nabla_{fX} Y = f\nabla_X Y$;
(iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$.

Here X and Y are vector fields on M, f is any smooth function on M and $Xf = df(X)$ is the derivative of f in direction X. If M is a Riemannian manifold it is important to consider connections that are compatible with the metric, that is to say, connections satisfying

(iv) $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$.

A connection ∇ satisfying (iv) is called *metric*. A connection ∇ is called *torsion-free* if it satisfies

(v) $\nabla_X Y - \nabla_Y X = [X, Y]$.

On a Riemannian manifold there exist is a unique torsion-free metric connection, i.e. a connection satisfying properties (iv) and (v). This connection is usually called the *Riemannian connection* or *Levi Civita connection* of the Riemannian manifold M. If not stated otherwise, ∇ usually denotes the Levi Civita connection of a Riemannian manifold. Explicitly, from these properties, the Levi Civita connection can be computed by the well-known Koszul formula

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle \ .$$

2.6. **Parallel vector fields and parallel transport.** Given a piecewise differentiable curve $c : I \to M$ defined on an interval I there is a covariant derivative operator along c which maps smooth tangent vector fields X of M along c to smooth tangent vector fields X' of M along c. The covariant derivative of vector fields along a curve c is completely determined by the following properties:

(i) $(Z_1 + Z_2)'(t) = Z_1'(t) + Z_2'(t)$ for all vector fields Z_1, Z_2 along c;
(ii) $(fZ)'(t) = f'(t)Z(t) + f(t)Z'(t)$ for all vector fields Z along c and all smooth functions $f : I \to \mathbb{R}$;
(iii) $(Y \circ c)'(t) = \nabla_{\dot{c}(t)} Y$ for all vector fields Y on M.

Since ∇ is metric we have

$$\langle X, Y\rangle'(t) = \langle X'(t), Y(t)\rangle + \langle X(t), Y'(t)\rangle$$

for all vector fields X, Y along c. We remark that if $c \equiv p$ is a constant curve and X is a vector field along c, i.e. for all t we have $X(t) \in T_pM$, then $X'(t)$ is the usual derivative in the vector space T_pM.

A vector field X along c is called *parallel* if $X' = 0$. The above equality implies that $\langle X, Y\rangle$ is constant if both vector fields are parallel along c. From the theory of ordinary differential equations one can easily see that for each $v \in T_{c(t_o)}M$, $t_o \in I$, there exists a unique parallel vector field X_v along c such that $X_v(t_o) = v$. For each $t \in I$ there is then a well-defined linear isometry $\tau^c(t) : T_{c(t_o)} \to T_{c(t)}$, called the *parallel transport* along c, given by

$$\tau^c(t)(v) = X_v(t) \ .$$

The covariant derivative operator and parallel transport along $c(t)$ are related by

$$X'(t) = \left.\frac{d}{dh}\right|_{h=0} (\tau^c(t+h))^{-1} X(t+h) \ .$$

Note that the parallel transport does not depend on the parametrization of the curve.

2.7. **Killing vector fields.** A vector field X on a Riemannian manifold M is called a *Killing vector field* if the local diffeomorphisms $\Phi_t^X : U \to M$ are isometries into M. This just means that the Lie derivative of the Riemannian metric of M with respect to X vanishes. A useful characterization of Killing vector fields is that a vector field X on a Riemannian manifold is a Killing vector field if and only if its covariant derivative ∇X is a skew-symmetric tensor field on M. A Killing vector field is completely determined by its value and its covariant derivative at any given point. In particular, a Killing vector field X for which $X_p = 0$ and $(\nabla X)_p = 0$ at some point $p \in M$ vanishes at each point of M. For a complete Killing vector field X on a Riemannian manifold M the corresponding one-parameter group (Φ_t^X) consists of isometries of M. Conversely, suppose we have a one-parameter group Φ_t of isometries on a Riemannian manifold M. Then

$$X_p := \left.\frac{d}{dt}\right|_{t=0} (t \mapsto \Phi_t(p))$$

defines a complete Killing vector field X on M with $\Phi_t^X = \Phi_t$ for all $t \in \mathbb{R}$. If X is a Killing vector field on M and $X_p = 0$ then

$$\left.\frac{d}{dt}\right|_{t=0} (t \mapsto (\Phi_t^X)_{*p}) = (\nabla X)_p$$

for all $t \in \mathbb{R}$.

2.8. **Distributions and the Frobenius Theorem.** A *distribution* on a Riemannian manifold M is a smooth vector subbundle H of the tangent bundle TM. A distribution H on M is called *integrable* if for any $p \in M$ there exists a connected submanifold L_p of M such that $T_qL_p = H_q$ for all $q \in L_p$. Such a submanifold L_p is called an *integral manifold* of H. The *Frobenius Theorem* states that H is integrable if and only if it is involutive, that is, if the Lie bracket of any two vector fields tangent to H is also a vector field tangent to H. If H is integrable, there exists through each point $p \in M$ a maximal integral manifold of H containing p. Such a maximal integral manifold is called the *leaf of H through p*. A distribution H on M is called *autoparallel* if $\nabla_H H \subset H$, that is, if for any two vector fields X, Y tangent to H the vector field $\nabla_X Y$ is also tangent to H. By the Frobenius Theorem every autoparallel distribution is integrable. An integrable distribution is autoparallel if and only if its leaves are totally geodesic submanifolds of the ambient space. A distribution H on M is called *parallel* if $\nabla_X H \subset H$ for any vector field X on M. Obviously, any parallel distribution is autoparallel. Since ∇ is a metric connection, for each parallel distribution H on M its orthogonal complement $H^\perp$ in TM is also a parallel distribution on M.

2.9. **Geodesics.** Of great importance in Riemannian geometry are the curves that minimize the distance between two given points. Of course, given two arbitrary points such curves do not exist in general. But they do exist provided the manifold is connected and complete. Distance-minimizing curves γ are solutions of a variational problem. The corresponding first variation formula shows that any such curve γ satisfies $\dot\gamma' = 0$. A smooth curve γ satisfying this equation is called a *geodesic*. Every geodesic is locally distance-minimizing, but not globally, as a great circle on a sphere illustrates. The basic theory of ordinary differential equations implies that for each point $p \in M$ and each tangent vector $X \in T_pM$ there exists a unique geodesic $\gamma : I \to M$ with $0 \in I$, $\gamma(0) = p$, $\dot\gamma(0) = X$, and such that for any other geodesic $\alpha : J \to M$ with $0 \in J$, $\alpha(0) = p$ and $\dot\alpha(0) = X$ we have $J \subset I$. This curve γ is often called the *maximal geodesic* in M through p tangent to X, and we denote it sometimes by γ_X. The *Hopf-Rinow Theorem* states that a Riemannian manifold is complete if and only if γ_X is defined on $\mathbb{R}$ for each $X \in TM$.

2.10. **Exponential map and normal coordinates.** Of great importance is the exponential map exp of a Riemannian manifold. To define it we denote by $\widetilde{TM} \subset TM$ the set of all tangent vectors for which $\gamma_X(1)$ is defined. This is an open subset of TM containing the zero section. A Riemannian manifold is complete if and only if $\widetilde{TM} = TM$. The map

$$\exp : \widetilde{TM} \to M \ , \ X \mapsto \gamma_X(1)$$

is called the *exponential map* of M. For each $p \in M$ we denote the restriction of exp to $T_pM \cap \widetilde{TM}$ by $\exp_p$. The map $\exp_p$ is a diffeomorphism from some open neighborhood of $0 \in T_pM$ onto some open neighborhood of $p \in M$. If we choose an orthonomal basis $e_1, \ldots, e_m$ of T_pM, then the map

$$(x_1, \ldots, x_m) \mapsto \exp_p \left(\sum_{i=1}^m x_i e_i \right)$$

defines local coordinates of M in some open neighborhood of p. Such coordinates are called *normal coordinates*.

2.11. **Riemannian curvature tensor, Ricci curvature, scalar curvature.** The major concept of Riemannian geometry is curvature. There are various notions of curvature which are of great interest. All of them can be deduced from the so-called *Riemannian curvature tensor*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ .$$

The Riemannian curvature tensor has the properties

$$\begin{aligned} \langle R(X,Y)Z,W \rangle &= -\langle R(Y,X)Z,W \rangle \ , \\ \langle R(X,Y)Z,W \rangle &= -\langle R(X,Y)W,Z \rangle \ , \\ \langle R(X,Y)Z,W \rangle &= \langle R(Z,W)X,Y \rangle \ , \end{aligned}$$

and

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \ .$$

These equations are often called the *algebraic curvature identities* of R, the latter one also the *algebraic Bianchi identity* or *first Bianchi identity.* Moreover, R satisfy the equation

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0 \ ,$$

which is known as the *differential Bianchi identity* or *second Bianchi identity.*

Let $p \in M$, $X, W \in T_pM$, and denote by $\mathrm{ric}_p(X,W)$ the real number which is obtained by contraction of the bilinear map

$$T_pM \times T_pM \to \mathbb{R} \ , \ (Y,Z) \mapsto \langle R(X,Y)Z,W \rangle \ .$$

The algebraic curvature identities show that ric_p is a symmetric bilinear map on T_pM. The tensor field ric is called the *Ricci tensor* of M. The corresponding selfadjoint tensor field of type (1,1) is denoted by Ric. A Riemannian manifold for which the Ricci tensor satisfies

$$\mathrm{ric} = f \langle \cdot , \cdot \rangle$$

with some smooth function f on M is called an *Einstein manifold.*

The weakest notion of curvature on a Riemannian manifold is the *scalar curvature*. This is the smooth function on M which is obtained by contracting the Ricci tensor.

2.12. **Sectional curvature.** The perhaps most geometric interpretation of the Riemannian curvature tensor arises via the sectional curvature. Consider a 2-dimensional linear subspace σ of T_pM, $p \in M$, and choose an orthonormal basis X, Y of σ. Since $\exp_p$ is a local diffeomorphism near 0 in T_pM, it maps an open neighborhood of 0 in σ onto some 2-dimensional surface S in M. Then the Gaussian curvature of S at p, which we denote by $K(\sigma)$, satisfies

$$K(\sigma) = \langle R(X,Y)Y, X\rangle \ .$$

Let $G_2(TM)$ be the Grassmann bundle over M consisting of all 2-dimensional linear subspaces $\sigma \subset T_pM$, $p \in M$. The map

$$K : G_2(TM) \to \mathbb{R} \ , \ \sigma \mapsto K(\sigma)$$

is called the *sectional curvature function* of M, and $K(\sigma)$ is called the *sectional curvature* of M with respect to σ. It is worthwhile to mention that one can reconstruct the Riemannian curvature tensor from the sectional curvature function by using the curvature identities.

A Riemannian manifold M is said to have *constant curvature* if the sectional curvature function is constant. If $\dim M \geq 3$, the second Bianchi identity and Schur's Lemma imply the following well-known result: *if the sectional curvature function depends only on the point p then M has constant curvature.* A space of constant curvature is also called a *space form.* The Riemannian curvature tensor of a space form with constant curvature κ is given by

$$R(X,Y)Z = \kappa(\langle Y, Z\rangle X - \langle X, Z\rangle Y) \ .$$

Every connected three-dimensional Einstein manifold is a space form. It is an algebraic fact (i.e. does not involve the second Bianchi identity) that a Riemannian manifold M has constant sectional curvature equal to zero if and only if M is flat, i.e. the Riemannian curvature tensor of M vanishes.

A connected, simply connected, complete Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold.* The *Hadamard Theorem* states that for each point p in a Hadamard manifold M the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism. More generally, if M is a connected, complete Riemannian manifold of nonpositive sectional curvature, then the exponential map $\exp_p : T_pM \to M$ is a covering map for each $p \in M$.

2.13. **Holonomy.** A Riemannian manifold M is said to be *flat* if its curvature tensor vanishes. This implies that locally the parallel transport does not depend on the curve used for joining two given points. If the curvature tensor does not vanish the parallel transport depends on the curve. A way of measuring how far the space deviates from being flat is given by the *holonomy group.* Let $p \in M$ and $\Omega(p)$ the set of all piecewise smooth curves $c : [0,1] \to M$ with $c(0) = c(1) = p$. Then the parallel translation along any curve $c \in \Omega(p)$ from $c(0)$ to $c(1)$ is an orthogonal transformation of T_pM. The set of all these parallel translations forms in an obvious manner a subgroup $\mathrm{Hol}_p(M)$ of the orthogonal group $O(T_pM)$, which is called the *holonomy group of M at p*. As a subset of $O(T_pM)$ it carries a natural topology. With respect to this topology, the identity component $\mathrm{Hol}^o_p(M)$ of $\mathrm{Hol}_p(M)$ is called the *restricted holonomy group* of M at p. The restricted holonomy group consists of all those transformations arising from null homotopic curves in $\Omega(p)$. If M is connected then all (restricted) holonomy groups are congruent to each other, and in this situation one speaks of the (restricted) holonomy group of the manifold M, which we will then denote by $\mathrm{Hol}(M)$ resp. $\mathrm{Hol}^o(M)$. The connected Lie group $\mathrm{Hol}^o(M)$ is always compact, whereas $\mathrm{Hol}(M)$ is in general not closed in the orthogonal group. A reduction of the holonomy group corresponds to an additional geometric structure on M. For instance, $\mathrm{Hol}(M)$ is contained in $SO(T_pM)$ for some $p \in M$ if and only if M is orientable. An excellent introduction to holonomy groups can be found in the book by Salamon [60].

2.14. **The de Rham Decomposition Theorem.** The *de Rham Decomposition Theorem* states that a connected Riemannian manifold M is locally reducible if and only if T_pM is reducible as a $\mathrm{Hol}^o(M)$-module for some, and hence for every, point $p \in M$. Since $\mathrm{Hol}^o(M)$ is compact there exists a decomposition

$$T_pM = V_0 \oplus V_1 \oplus \ldots \oplus V_k$$

of T_pM into $\mathrm{Hol}^o(M)$-invariant subspaces of T_pM, where $V_0 \subset T_pM$ is the fixed point set of the action of $\mathrm{Hol}^o(M)$ on T_pM and $V_1, \ldots, V_k$ are irreducible $Hol^o(M)$-modules. It might happen that $V_0 = T_pM$, for instance when $M = \mathbb{R}^n$, or $V_0 = \{0\}$, for instance when M is the sphere S^n, $n > 1$. The above decomposition is unique up to order of the factors and determines integrable distributions $V_0, \ldots, V_k$ on M. Then there exists an open neighborhood of p in M which is isometric to the Riemannian product of sufficiently small integral manifolds of these distributions through p. The global version of the de Rham decomposition theorem states that a connected, simply connected, complete Riemannian manifold M is reducible if and only if T_pM is reducible as a $\mathrm{Hol}^o(M)$-module. If M is reducible and $T_pM = V_0 \oplus \ldots \oplus V_k$ is the decomposition of T_pM as described above, then M is isometric to the Riemannian product of the maximal integral manifolds $M_0, \ldots, M_k$ through p of the distributions $V_0, \ldots, V_k$. In this situation $M = M_0 \times \ldots \times M_k$ is called the *de Rham decomposition* of M. The Riemannian manifold M_0 is isometric to a, possibly zero-dimensional, Euclidean space. If $\dim M_0 > 0$ then M_0 is called the *Euclidean factor* of M. A connected, complete Riemannian manifold M is said to have *no Euclidean factor* if the de Rham decomposition of the Riemannian universal covering space $\widetilde{M}$ of M has no Euclidean factor.

2.15. **Jacobi vector fields.** Let $\gamma : I \to M$ be a geodesic parametrized by arc length. A vector field Y along γ is called a *Jacobi vector field* if it satisfies the second order differential equation

$$Y'' + R(Y, \dot{\gamma})\dot{\gamma} = 0 \ .$$

Standard theory of ordinary differential equations implies that the Jacobi vector fields along a geodesic form a $2n$-dimensional vector space. Every Jacobi vector field is uniquely determined by the initial values $Y(t_0)$ and $Y'(t_0)$ at a fixed number $t_0 \in I$. The Jacobi vector fields arise geometrically as infinitesimal variational vector fields of geodesic variations. Jacobi vector fields may be used to describe the differential of the exponential map. Indeed, let $p \in M$ and $\exp_p$ the exponential map of M restricted to T_pM. For each $X \in T_pM$ we identify $T_X(T_pM)$ with T_pM in the canonical way. Then for each $Z \in T_pM$ we have

$$exp_{p*X} Z = Y_Z(1) \ ,$$

where Y_Z is the Jacobi vector field along γ_X with initial values $Y_Z(0) = 0$ and $Y'_Z(0) = Z$.

2.16. **Kähler manifolds.** An *almost complex structure* on a smooth manifold M is a tensor field J of type (1,1) on M satisfying $J^2 = -\mathrm{id}_{TM}$. An *almost complex manifold* is smooth manifold equipped with an almost complex structure. Each tangent space of an almost complex manifold is isomorphic to a complex vector space, which implies that the dimension of an almost complex manifold is an even number. An *Hermitian metric* on an almost complex manifold M is a Riemannian metric $\langle \cdot, \cdot \rangle$ for which the almost complex structure J on M is orthogonal, that is,

$$\langle JX, JY \rangle = \langle X, Y \rangle$$

for all $X, Y \in T_pM$, $p \in M$. An orthogonal almost complex structure on a Riemannian manifold is called an *almost Hermitian structure.*

Every complex manifold M has a canonical almost complex structure. In fact, if $z = x + iy$ is a local coordinate on M, define

$$J\frac{\partial}{\partial x_\nu} = \frac{\partial}{\partial y_\nu} \ , \ J\frac{\partial}{\partial y_\nu} = -\frac{\partial}{\partial x_\nu} \ .$$

These local almost complex structures are compatible on the intersection of any two coordinate neighborhoods and hence induce an almost complex structure, which is called the *induced complex structure* of M. An almost complex structure J on a smooth manifold M is *integrable* if M can be equipped with the structure of a complex manifold so that J is the induced complex structure. A famous result by Newlander-Nirenberg says that the almost complex structure J of an almost complex manifold M is integrable if and only if

$$[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$$

for all $X, Y \in T_pM$, $p \in M$. A *Hermitian manifold* is an almost Hermitian manifold with an integrable almost complex structure. The almost Hermitian structure of an Hermitian manifold is called an *Hermitian structure.*

The 2-form ω on an Hermitian manifold M defined by

$$\omega(X, Y) = \langle X, JY \rangle$$

is called the *Kähler form* of M. A *Kähler manifold* is an Hermitian manifold whose Kähler form is closed. A Hermitian manifold M is a Kähler manifold if and only if its Hermitian structure J is parallel with respect to the Levi Civita connection ∇ of M, that is, if $\nabla J = 0$. The latter condition characterizes the Kähler manifolds among all Hermitian manifolds by the geometric property that parallel translation along curves commutes with the Hermitian structure J. A $2m$-dimensional connected Riemannian manifold M can be equipped with the structure of a Kähler manifold if and only if its holonomy group $\mathrm{Hol}(M)$ is contained in the unitary group $U(m)$.

2.17. **Quaternionic Kähler manifolds.** A *quaternionic Kähler structure* on a Riemannian manifold M is a rank three vector subbundle $\mathfrak{J}$ of the endomorphism bundle $\mathrm{End}(TM)$ over M with the following properties: (1) For each p in M there exist an open neighborhood U of p in M and sections J_1, J_2, J_3 of $\mathfrak{J}$ over U so that J_ν is an almost Hermitian structure on U and

$$J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu \quad \text{(index modulo three)}$$

for all $\nu = 1, 2, 3$; (2) $\mathfrak{J}$ is a parallel subbundle of $\mathrm{End}(TM)$, that is, if J is a section in $\mathfrak{J}$ and X a vector field on M, then $\nabla_X J$ is also a section in $\mathfrak{J}$. Each triple J_1, J_2, J_3 of the above kind is called a *canonical local basis* of $\mathfrak{J}$, or, if restricted to the tangent space T_pM of M at p, a *canonical basis* of $\mathfrak{J}_p$. A *quaternionic Kähler manifold* is a Riemannian manifold equipped with a quaternionic Kähler structure. The canonical bases of a quaternionic Kähler structure turn the tangent spaces of a quaternionic Kähler manifold into quaternionic vector spaces. Therefore, the dimension of a quaternionic Kähler manifold is $4m$ for some $m \in \mathbb{N}$. A $4m$-dimensional connected Riemannian manifold M can be equipped with a quaternionic Kähler structure if and only if its holonomy group $\mathrm{Hol}(M)$ is contained in $Sp(m) \cdot Sp(1)$.

3. Lie groups and Lie algebras

Lie groups were introduced by Sophus Lie in the framework of his studies on differential equations as local transformation groups. The global theory of Lie groups was developed by Hermann Weyl and Élie Cartan. Lie groups are both groups and manifolds. This fact allows us to use concepts both from algebra and analysis to study these objects. Some modern books on this topic are Adams [1], Carter-Segal-Macdonald [20], Knapp [39], Varadarajan [66]. Foundations on Lie theory may also be found in Onishchik [55], and the structure of Lie groups and Lie algebras is discussed in Onishchik-Vinberg [56]. A good introduction to the exceptional Lie groups may be found in Adams [2].

3.1. **Lie groups.** A *real Lie group*, or briefly *Lie group*, is an abstract group G which is equipped with a smooth manifold structure such that $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ and $G \to G$, $g \mapsto g^{-1}$ are smooth maps. For a *complex Lie group* G one requires that G is equipped with a complex analytic structure and that multiplication and inversion are holomorphic maps.

Two Lie groups G and H are *isomorphic* if there exists a smooth isomorphism $G \to H$, and they are *locally isomorphic* if there exist open neighborhoods of the identities in G and H and a smooth isomorphism between these open neighborhoods.

Examples: 1. $\mathbb{R}^n$ equipped with its additive group structure is an Abelian (or commutative) Lie group.

2. Denote by $\mathbb{F}$ the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, or the skewfield $\mathbb{H}$ of quaternionic numbers. The group $GL(n, \mathbb{F})$ of all nonsingular $n \times n$-matrices

with coefficients in $\mathbb{F}$ is a Lie group, a so-called *general linear group* (over $\mathbb{F}$). Moreover, $GL(n,\mathbb{C})$ is a complex Lie group.

Proof: For $\mathbb{F}=\mathbb{R}$: We have $GL(n,\mathbb{R})=\{A\in Mat(n,n,\mathbb{R})\mid \det(A)\neq 0\}$, which shows that $GL(n,\mathbb{R})$ is an open subset of $Mat(n,n,\mathbb{R})\cong\mathbb{R}^{n^2}$ and hence a manifold. The smoothness of matrix multiplication is clear, and the smoothness of the inverse map follows from Cramer's rule. The proof for $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$ is similar.

3. The isometry group $I(M)$ of a connected Riemannian manifold is a Lie group.

3.2. **Lie subgroups.** An important way to obtain Lie groups is to consider certain subgroups of Lie groups. A subgroup H of a Lie group G is called a *Lie subgroup* if H is a Lie group and if the inclusion $H\to G$ is a smooth map.

Examples: 1. For every Lie group G the connected component of G containing the identity of G is called the identity component of G. We denote this component usually by G^o. Then G^o is a Lie subgroup of G.

2. Every closed subgroup of a Lie group is a Lie subgroup. This is a very important and useful criterion! The proof is nontrivial and will be omitted. We just give some applications below. A closed subgroup of $GL(n,\mathbb{F})$ is also called a *closed linear group.*

3. The *special linear group*

$$SL(n,\mathbb{F})=\{A\in GL(n,\mathbb{F})\mid \det A=1\}$$

is a closed subgroup of $GL(n,\mathbb{F})$. The group $SL(n,\mathbb{C})$ is a complex Lie group.

4. For $A\in GL(n,\mathbb{F})$ we denote by A^* the matrix which is obtained from A by conjugation and transposing, that is, $A^*=\bar{A}^t$. By I_n we denote the $n\times n$-identity matrix. Then we get the following closed subgroups of $GL(n,\mathbb{F})$: The *orthogonal group*

$$O(n)=\{A\in GL(n,\mathbb{R})\mid A^*A=I_n\}\ ,$$

the *unitary group*

$$U(n)=\{A\in GL(n,\mathbb{C})\mid A^*A=I_n\}\ ,$$

and the *symplectic group*

$$Sp(n)=\{A\in GL(n,\mathbb{H})\mid A^*A=I_n\}\ .$$

We denote by $\langle\cdot,\cdot\rangle$ the Hermitian form on $\mathbb{F}^n\times\mathbb{F}^n$ given by

$$\langle x,y\rangle=\sum_{\nu=1}^{n}x_\nu\bar{y}_\nu\ .$$

Then $O(n),U(n),Sp(n)$ is precisely the group of all $A\in GL(n,\mathbb{F})$ preserving this Hermitian form. The orthogonal group has two connected components, corresponding to the determinant ±1. The identity component

$$SO(n)=\{A\in O(n)\mid \det A=1\}$$

is called the *special orthogonal group.* The determinant of a unitary matrix is a complex number of modulus one. The subgroup

$$SU(n)=\{A\in U(n)\mid \det A=1\}$$

is called the *special unitary group*. Every symplectic matrix has determinant one.

5. Let m, n be positive integers and consider the Hermitian form on $\mathbb{F}^{m+n} \times \mathbb{F}^{m+n}$ given by

$$\langle x, y \rangle = x_1\bar{y}_1 + \ldots + x_m\bar{y}_m - x_{m+1}\bar{y}_{m+1} - \ldots - x_{m+n}\bar{y}_{m+n} \ .$$

The group of all $A \in GL(m+n, \mathbb{F})$ leaving this Hermitian form invariant is denoted by $O(m,n)$, $U(m,n)$ and $Sp(m,n)$, respectively. Alternatively, if we denote by $I_{m,n}$ the matrix

$$I_{m,n} = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} \ ,$$

we have

$$O(m,n) = \{A \in GL(m+n, \mathbb{R}) \mid A^* I_{m,n} A = I_{m,n}\} \ ,$$
$$U(m,n) = \{A \in GL(m+n, \mathbb{C}) \mid A^* I_{m,n} A = I_{m,n}\} \ ,$$

and

$$Sp(m,n) = \{A \in GL(m+n, \mathbb{H}) \mid A^* I_{m,n} A = I_{m,n}\} \ .$$

The subgroups of determinant one matrices are denoted by

$$SO(m,n) = \{A \in O(m,n) \mid \det A = 1\}$$

and

$$SU(m,n) = \{A \in U(m,n) \mid \det A = 1\} \ .$$

Every matrix in $Sp(m,n)$ has determinant one. All the above subgroups are closed.

6. The *complex special orthogonal group*

$$SO(n, \mathbb{C}) = \{A \in SL(n, \mathbb{C}) \mid A^t A = I_n\}$$

and the *complex symplectic group*

$$Sp(n, \mathbb{C}) = \{A \in SL(2n, \mathbb{C}) \mid A^t J_n A = J_n\}$$

are complex Lie groups. Here,

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \ .$$

The real version of the latter group is the *real symplectic group*

$$Sp(n, \mathbb{R}) = \{A \in SL(2n, \mathbb{R}) \mid A^t J_n A = J_n\} \ .$$

Finally, we define

$$SO^*(2n) = \{A \in SU(n,n) \mid A^t K_n A = K_n\} \ ,$$

where

$$K_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \ .$$

Remark. All the classical groups $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $Sp(n, \mathbb{C})$ and $SO(n)$, $SU(n)$, $Sp(n)$, $SL(n, \mathbb{R})$, $SL(n, \mathbb{H})$, $SU(m,n)$, $Sp(m,n)$, $Sp(n, \mathbb{R})$ and $SO^*(2n)$ are connected, and $SO(m,n)$ has two connected components. We will see later that these groups play a fundamental role in Lie group theory, since they essentially provide all classical simple real and complex Lie groups.

7. The holonomy group of a connected Riemannian manifold is in general not a Lie group (in general it is not closed in the orthogonal group), but its identity component, the restricted holonomy group, is always a Lie group (since it is compact and hence a closed subgroup of the special orthogonal group).

3.3. **Abelian Lie groups.** Let Γ be a lattice in $\mathbb{R}^n$, that is, Γ is a discrete subgroup of rank n of the group of translations of $\mathbb{R}^n$. Then Γ is a normal subgroup of $\mathbb{R}^n$ and hence the quotient $T^n = \mathbb{R}^n/\Gamma$ is also an Abelian group. Since T^n and $\mathbb{R}^n$ are locally isomorphic, T^n is also a Lie group. One can see easily that T^n is compact. Hence T^n is a compact Abelian Lie group, a so-called n-dimensional *torus*. One can show that every Abelian Lie group is isomorphic to the direct product $\mathbb{R}^n \times T^k$ for some nonnegative integers $n, k \geq 0$.

3.4. **Direct products and semidirect products of Lie groups.** Let G and H be Lie groups. The *direct product* $G \times H$ of G and H is the smooth product manifold $G \times H$ equipped with the multiplication and inversion

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2) \quad , \quad (g, h)^{-1} = (g^{-1}, h^{-1}) \ .$$

Let τ be a homomorphism from H into the group $\mathrm{Aut}(G)$ of automorphisms of G. The *semidirect product* $G \times_\tau H$ is the smooth manifold $G \times H$ equipped with the group structure

$$(g_1, h_1)(g_2, h_2) = (g_1\tau(h_1)g_2, h_1h_2) \quad , \quad (g, h)^{-1} = (\tau(h^{-1})g^{-1}, h^{-1}) \ .$$

If τ is the trivial homomorphism, then $G \times_\tau H$ is just the direct product $G \times H$.

Examples: 1. The isometry group of $\mathbb{R}^n$ is the semidirect product $\mathbb{R}^n \times_\tau O(n)$, where $\mathbb{R}^n$ acts on itself isometrically by translations and $\tau : O(n) \to \mathrm{Aut}(\mathbb{R}^n)$ is given by $\tau(A)x = Ax$ for $A \in O(n)$ and $x \in \mathbb{R}^n$.

2. $O(n) = SO(n) \times_\tau \mathbb{Z}_2$ with $\tau : \mathbb{Z}_2 \to \mathrm{Aut}(SO(n))$ given by $\tau(x)A = XAX^{-1}$, where X is the diagonal matrix with entries $x, 1, \ldots, 1$ and $x \in \mathbb{Z}_2 = \{\pm 1\}$. Similarily, $U(n) = SU(n) \times_\tau U(1)$ with $\tau : U(1) \to \mathrm{Aut}(SU(n))$ given by $\tau(x)A = XAX^{-1}$, where X is the diagonal matrix with entries $x, 1, \ldots, 1$ and $x \in U(1) = \{x \in \mathbb{C} \mid |x| = 1\}$. In particular, this shows that as a manifold $U(n)$ is diffeomorphic to $SU(n) \times S^1$, where S^1 is the one-dimensional sphere.

3.5. **Universal covering groups.** Let G be a connected Lie group, and let $\tilde{G}$ be the universal covering space of G with covering $\pi : \tilde{G} \to G$. Let $\tilde{e} \in \pi^{-1}(\{e_G\})$, where e_G is the identity of G. Then there exists a unique Lie group structure on $\tilde{G}$ such that $\tilde{e}$ is the identity of $\tilde{G}$ and π is a Lie group homomorphism. The Lie group $\tilde{G}$ is called the *universal covering group* of G. It is unique up to isomorphism.

Example: The fundamental group of $SO(n)$ is $\mathbb{Z}_2$ for $n \geq 3$. The universal covering group of $SO(n)$ is the so-called *Spin group* $Spin(n)$. It can be explicitly constructed from Clifford algebras. For $n = 3$ we have $Spin(3) = SU(2) = Sp(1)$. If we identify $\mathbb{R}^3$ with the imaginary part of $\mathbb{H}$, then $Sp(1)$ acts isometrically on $\mathbb{R}^3$ by conjugation. This induces the covering map $Sp(1) \to SO(3)$, which is fundamental in physics for describing rotations and angular momenta.

3.6. **Left and right translations, inner automorphisms.** For each $g \in G$ the smooth diffeomorphisms

$$L_g : G \to G \ , \ g' \mapsto gg' \ \text{ and } \ R_g : G \to G \ , \ g' \mapsto g'g$$

are called the *left translation* and *right translation* on G with respect to g, respectively. A vector field X on G is called *left-invariant* resp. *right-invariant* if it is invariant under any left translation resp. right translation, i.e. if $L_{g*}X = X \circ L_g$ resp. $R_{g*}X = X \circ R_g$ for all $g \in G$. The smooth diffeomorphism

$$I_g = L_g \circ R_{g^{-1}} : G \to G \ , \ g' \mapsto gg'g^{-1}$$

is called an *inner automorphism* of G.

3.7. **Lie algebras and subalgebras.** A (real or complex) *Lie algebra* is a (real or complex) vector space $\mathfrak{g}$ equipped with a skew-symmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. The latter identity is called the *Jacobi identity. We will always assume that a Lie algebra is finite-dimensional.*

Two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic* if there exists an algebra isomorphism $\mathfrak{g} \to \mathfrak{h}$.

Example: The real vector space $\mathfrak{gl}(n,\mathbb{R})$ of all $n \times n$-matrices with real coefficients together with the product $[A,B] = AB - BA$ is a Lie algebra. Analogously, the complex vector space $\mathfrak{gl}(n,\mathbb{C})$ of all $n \times n$-matrices with complex coefficients together with the product $[A,B] = AB - BA$ is a complex Lie algebra.

A *subalgebra* of a Lie algebra $\mathfrak{g}$ is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$. An *ideal* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ with $[\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h}$. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then the vector space $\mathfrak{g}/\mathfrak{h}$ becomes a Lie algebra by means of $[X+\mathfrak{h}, Y+\mathfrak{h}] = [X,Y]+\mathfrak{h}$. This is the so-called *quotient algebra* of $\mathfrak{g}$ and $\mathfrak{h}$. A subalgebra $\mathfrak{h}$ is *Abelian* if $[\mathfrak{h},\mathfrak{h}] = 0$.

3.8. **The Lie algebra of a Lie group.** To every Lie group G there is associated a Lie algebra $\mathfrak{g}$, namely the vector space of all left-invariant vector fields equipped with the bilinear map arising from the commutator of vector fields. Since each left-invariant vector field is uniquely determined by its value at the identity $e \in G$, $\mathfrak{g}$ is isomorphic as a vector space to T_eG. In particular, we have $\dim \mathfrak{g} = \dim G$. If G and H are locally isomorphic Lie groups then their Lie algebras are isomorphic.

In case G is a closed linear group, the Lie algebra $\mathfrak{g}$ of G can be determined in the following way. Consider smooth curves $c : \mathbb{R} \to G$ with $c(0) = e_G$. Then $T_{e_G}G = \{c'(0)\}$ forms a set of matrices which is closed under the bracket $[X,Y] = XY - YX$. In this way we see that $\mathfrak{gl}(n,\mathbb{R})$ is the Lie algebra of $GL(n,\mathbb{R})$ and $\mathfrak{gl}(n,\mathbb{C})$ is the Lie algebra of $GL(n,\mathbb{C})$. The Lie algebras of the classical complex Lie groups we discussed above are

$$\begin{aligned}
\mathfrak{sl}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \mathrm{tr}X = 0\} \ , \\
\mathfrak{so}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X + X^t = 0\} \ , \\
\mathfrak{sp}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(2n,\mathbb{C}) \mid X^tJ_n + J_nX = 0\} \ .
\end{aligned}$$

From this we can easily calculate the dimensions of the classical complex Lie groups,

$$\dim_{\mathbb{C}} SL(n,\mathbb{C}) = n^2-1\ ,\ \dim_{\mathbb{C}} SO(n,\mathbb{C}) = n(n-1)/2\ ,\ \dim_{\mathbb{C}} Sp(n,\mathbb{C}) = 2n^2+n\ .$$

For low dimensions there are some isomorphisms,

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sp}(1,\mathbb{C})\ ,\ \mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(2,\mathbb{C})\ ,\ \mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C})\ .$$

Moreover,

$$\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})\ .$$

For the classical compact real Lie groups we get the following Lie algebras

$$\begin{aligned} \mathfrak{o}(n) = \mathfrak{so}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^* = 0\}\ , \\ \mathfrak{u}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X + X^* = 0\}\ , \\ \mathfrak{su}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X + X^* = 0\ ,\ \mathrm{tr}X = 0\}\ , \\ \mathfrak{sp}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{H}) \mid X + X^* = 0\}\ . \end{aligned}$$

Here we have the analogous isomorphisms

$$\mathfrak{so}(3) \cong \mathfrak{su}(2) \cong \mathfrak{sp}(1)\ ,\ \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)\ ,\ \mathfrak{so}(5) \cong \mathfrak{sp}(2)\ ,\ \mathfrak{so}(6) \cong \mathfrak{su}(4)\ .$$

For the remaining classical Lie groups we get

$$\begin{aligned} \mathfrak{sl}(n,\mathbb{R}) &= \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \mathrm{tr}X = 0\}\ , \\ \mathfrak{sl}(n,\mathbb{H}) &= \{X \in \mathfrak{gl}(n,\mathbb{H}) \mid \mathrm{Re}(\mathrm{tr}X) = 0\}\ , \\ \mathfrak{so}(m,n) &= \{X \in \mathfrak{gl}(m+n,\mathbb{R}) \mid X^* I_{m,n} + I_{m,n} X = 0\}\ , \\ \mathfrak{su}(m,n) &= \{X \in \mathfrak{sl}(m+n,\mathbb{C}) \mid X^* I_{m,n} + I_{m,n} X = 0\}\ , \\ \mathfrak{sp}(m,n) &= \{X \in \mathfrak{gl}(m+n,\mathbb{H}) \mid X^* I_{m,n} + I_{m,n} X = 0\}\ , \\ \mathfrak{sp}(n,\mathbb{R}) &= \{X \in \mathfrak{gl}(2n,\mathbb{R}) \mid X^t J_n + J_n X = 0\}\ , \\ \mathfrak{so}^*(2n) &= \{X \in \mathfrak{su}(n,n) \mid X^t K_n + K_n X = 0\}\ . \end{aligned}$$

For low dimensions there are the following isomorphisms:

$$\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1) \cong \mathfrak{so}(2,1) \cong \mathfrak{sp}(1,\mathbb{R})\ ,\ \mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{so}(3,3)\ ,\ \mathfrak{sl}(2,\mathbb{H}) \cong \mathfrak{so}(5,1)\ ,$$
$$\mathfrak{sp}(2,\mathbb{R}) \cong \mathfrak{so}(3,2)\ ,\ \mathfrak{so}^*(4) \cong \mathfrak{su}(2) \oplus \mathfrak{sl}(2,\mathbb{R})\ ,\ \mathfrak{so}^*(6) \cong \mathfrak{su}(3,1)\ ,\ \mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)\ ,$$
$$\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})\ ,\ \mathfrak{so}(4,1) \cong \mathfrak{sp}(1,1)\ ,\ \mathfrak{so}(4,2) \cong \mathfrak{su}(2,2)\ .$$

3.9. **Complexifications and real forms.** Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ be the complexification of $\mathfrak{g}$ considered as a vector space. By extending the Lie algebra structure on $\mathfrak{g}$ complex linearly to $\mathfrak{g}^{\mathbb{C}}$ we turn $\mathfrak{g}^{\mathbb{C}}$ into a complex Lie algebra, the *complexification* of $\mathfrak{g}$. Any complex Lie algebra $\mathfrak{h}$ can be considered canonically as a real Lie algebra $\mathfrak{h}^{\mathbb{R}}$ by restricting the scalar multiplication to $\mathbb{R} \subset \mathbb{C}$. If $\mathfrak{g}$ is a real Lie algebra and $\mathfrak{h}$ is a complex Lie algebra so that $\mathfrak{h}$ is isomorphic to $\mathfrak{g}^{\mathbb{C}}$, then $\mathfrak{g}$ is a *real form* of $\mathfrak{h}$.

Examples: $\mathfrak{gl}(n,\mathbb{R})^{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{gl}(n,\mathbb{C})^{\mathbb{R}} \cong \mathfrak{gl}(2n,\mathbb{R})$, $\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}} \cong \mathfrak{so}(3,1)$.

$\mathfrak{su}(n)$ and $\mathfrak{sl}(n,\mathbb{R})$ are real forms of $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{H})$ is a real form of $\mathfrak{sl}(2n,\mathbb{C})$, $\mathfrak{su}(m,n)$ is a real form of $\mathfrak{sl}(m+n,\mathbb{C})$, $\mathfrak{so}(n)$ is a real form of $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{so}(m,n)$ is a real form of $\mathfrak{so}(m+n,\mathbb{C})$, $\mathfrak{so}^*(2n)$ is a real form of $\mathfrak{so}(2n,\mathbb{C})$, $\mathfrak{sp}(n)$ and $\mathfrak{sp}(n,\mathbb{R})$ are real forms of

$\mathfrak{sp}(n,\mathbb{C})$, and $\mathfrak{sp}(m+n)$ is a real form of $\mathfrak{sp}(m+n,\mathbb{C})$. Since the real dimension of a real form $\mathfrak{g}$ of a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is equal to the complex dimension of $\mathfrak{g}^{\mathbb{C}}$, we easily get the dimensions of the above classical real Lie algebras.

3.10. **The differential of smooth Lie group homomorphisms.** Let G and H be Lie groups and $\Phi : G \to H$ be a smooth homomorphism. Then the differential ϕ of Φ at the identity e_G of G is a linear map from the tangent space $T_{e_G}G$ into the tangent space $T_{e_H}H$, and hence from $\mathfrak{g}$ into $\mathfrak{h}$ by means of our identification of the Lie algebra of a Lie group with the tangent space at the identity. If X is a left-invariant vector field on G, and Y is the left-invariant vector field on H with $Y_{e_H} = \phi(X_{e_G})$, then we have $d_g\Phi(X_g) = Y_{\Phi(g)}$ for all $g \in G$. It follows that $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. This also shows that, if G is connected, the Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ uniquely determines the Lie group homomorphism $\Phi : G \to H$. The image of Φ is a Lie subgroup of H, and the map Φ from G into this Lie subgroup is smooth.

3.11. **Direct sums and semidirect sums of Lie algebras.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. The *direct sum* $\mathfrak{g} \oplus \mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{h}$ is the vector space $\mathfrak{g} \oplus \mathfrak{h}$ (direct sum) equipped with the bracket operation such that $\mathfrak{g}$ brackets with $\mathfrak{g}$ as before, $\mathfrak{h}$ with $\mathfrak{h}$ as before, and $[\mathfrak{g},\mathfrak{h}] = 0$.

A *derivation* on $\mathfrak{g}$ is an endomorphism D of $\mathfrak{g}$ satisfying

$$D[X,Y] = [DX,Y] + [X,DY]$$

for all $X,Y \in \mathfrak{g}$. The vector space $\mathrm{Der}(\mathfrak{g})$ of all derivations on $\mathfrak{g}$ is a Lie algebra with respect to the usual bracket for endomorphisms. One can prove that the Lie algebra of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of all derivations on $\mathfrak{g}$.

Let π be a homomorphism from $\mathfrak{h}$ into the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of derivations on $\mathfrak{g}$. The *semidirect sum* $\mathfrak{g} \oplus_\pi \mathfrak{h}$ is the vector space $\mathfrak{g} \oplus \mathfrak{h}$ (direct sum) equipped with the bracket operation such that $\mathfrak{g}$ brackets with $\mathfrak{g}$ as before, $\mathfrak{h}$ with $\mathfrak{h}$ as before, and $[X,Y] = \pi(X)Y$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$. If π is the trivial homomorphism, then $\mathfrak{g} \oplus_\pi \mathfrak{h}$ is just the direct sum $\mathfrak{g} \oplus \mathfrak{h}$.

Let $G \times_\tau H$ be a semidirect of G and H. Then we have a homomorphism $\tau : H \to \mathrm{Aut}(G)$. The differential of τ at the identity is a Lie algebra homomorphism π from $\mathfrak{h}$ into the Lie algebra of $\mathrm{Aut}(G)$, which is isomorphic to the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$, and hence into $\mathrm{Der}(\mathfrak{g})$. One can show that the Lie algebra of $G \times_\tau H$ is $\mathfrak{g} \oplus_\pi \mathfrak{h}$.

3.12. **Lie exponential map.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Any $X \in \mathfrak{g}$ is a left-invariant vector field on G and hence determines a flow $\Phi^X : \mathbb{R} \times G \to G$. The smooth map

$$\mathrm{Exp} : \mathfrak{g} \to G\ ,\ X \mapsto \Phi^X(1,e)$$

is called the *Lie exponential map* of $\mathfrak{g}$ or G. For each $X \in \mathfrak{g}$ the curve $t \mapsto \mathrm{Exp}(tX)$ is a one-parameter subgroup of G and we have $\Phi^X(t,g) = R_{\mathrm{Exp}(tX)}(g)$ for all $g \in G$ and $t \in \mathbb{R}$. The Lie exponential map is crucial when studying the interplay between Lie groups and Lie algebras. It is a diffeomorphism of some open neighborhood of $0 \in \mathfrak{g}$ onto some open neighborhood of $e_G \in G$. If $\mathfrak{g}$ is a matrix Lie algebra, then Exp is the usual exponential map for matrices. The Lie exponential map is neither injective nor surjective in general.

If G is a Lie group and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then there exists a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$. The subgroup H is the smallest Lie subgroup in G containing $\mathrm{Exp}(\mathfrak{h})$.

If $\Phi : G \to H$ is a Lie group homomorphism, then the differential ϕ of Φ at e_G is a Lie algebra homomorphism from $\mathfrak{g}$ into $\mathfrak{h}$ with the property $\Phi \circ \mathrm{Exp}_{\mathfrak{g}} = \mathrm{Exp}_{\mathfrak{h}} \circ \phi$.

Let G and H be Lie groups and $\phi : \mathfrak{g} \to \mathfrak{h}$ a Lie algebra homomorphism. If G is simply connected, then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that ϕ is the differential of Φ at e_G.

3.13. **The Lie algebra of the isometry group of a Riemannian manifold.** Let M be a connected Riemannian manifold. The Lie algebra $\mathfrak{i}(M)$ of the isometry group $I(M)$ can be identified with the Lie algebra $\mathfrak{k}(M)$ of all Killing vector fields on M in the following way. The Lie bracket on $\mathfrak{k}(M)$ is the usual commutator of vector fields. For $X \in \mathfrak{i}(M)$ we define a vector field X^* on M by

$$X^*_p = \left.\frac{d}{dt}\right|_{t=0} (t \mapsto \mathrm{Exp}(tX)(p))$$

for all $p \in M$. Then the map

$$\mathfrak{i}(M) \to \mathfrak{k}(M)\ ,\ X \mapsto X^*$$

is a vector space isomorphism satisfying

$$[X, Y]^* = -[X^*, Y^*]\ .$$

In other words, if one would define the Lie algebra $\mathfrak{i}(M)$ of $I(M)$ by using right-invariant vector fields instead of left-invariant vector fields, then the map $\mathfrak{i}(M) \to \mathfrak{k}(M)\ ,\ X \mapsto X^*$ would be a Lie algebra isomorphism.

3.14. **Adjoint representation.** The inner automorphisms I_g of G determine the so-called *adjoint representation of* G by

$$\mathrm{Ad} : G \to GL(\mathfrak{g})\ ,\ g \mapsto I_{g*e}\ ,$$

where I_{g*e} denotes the differential of I_g at e and we identify T_eG with $\mathfrak{g}$ by means of the vector space isomorphism

$$\mathfrak{g} \to T_eG\ ,\ X \mapsto X_e\ .$$

The kernel of Ad is the center $Z(G)$ of G,

$$Z(G) = \{g \in G \mid \forall h \in G : gh = hg\}\ .$$

In general, a representation of G is a homomorphism $\pi : G \to GL(V)$, where V is a real or complex vector space.

The *adjoint representation of* $\mathfrak{g}$ is the homomorphism

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})\ ,\ X \mapsto (\mathfrak{g} \to \mathfrak{g}\ ,\ Y \mapsto [X, Y])\ .$$

The kernel of ad is the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$,

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g} : [X,Y] = 0\} .$$

In general, a representation of $\mathfrak{g}$ is a homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a real or complex vector space. The image $\mathrm{ad}(\mathfrak{g})$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$, the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$. The connected Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$ is denoted by $\mathrm{Int}(\mathfrak{g})$, and the elements in $\mathrm{Int}(\mathfrak{g})$ are called *inner automorphisms of* $\mathfrak{g}$.

The homomorphism ad can be obtained from Ad by means of

$$\mathrm{ad}(X)Y = \left.\frac{d}{dt}\right|_{t=0} (t \mapsto \mathrm{Ad}(\mathrm{Exp}(tX))Y) .$$

The relation between Ad and ad is described by

$$\mathrm{Ad}(\mathrm{Exp}(X)) = e^{\mathrm{ad}(X)} ,$$

where $e^{\cdot}$ denotes the exponential map for endomorphisms of the vector space $\mathfrak{g}$.

3.15. **Cartan-Killing form.** The symmetric bilinear form B on $\mathfrak{g}$ defined by

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$$

for all $X, Y \in \mathfrak{g}$ is called the *Cartan-Killing form* of $\mathfrak{g}$. Every automorphism σ of $\mathfrak{g}$ has the property

$$B(\sigma X, \sigma Y) = B(X,Y)$$

for all $X, Y \in \mathfrak{g}$.

Proof. Since $[\sigma X, Y] = \sigma[X, \sigma^{-1}Y]$ we have $\mathrm{ad}(\sigma X) = \sigma \mathrm{ad}(X)\sigma^{-1}$. Since $\mathrm{tr}(AC) = \mathrm{tr}(CA)$ this implies $B(\sigma X, \sigma Y) = \mathrm{tr}(\mathrm{ad}(\sigma X)\mathrm{ad}(\sigma Y)) = \mathrm{tr}(\sigma \mathrm{ad}(X)\sigma^{-1}\sigma \mathrm{ad}(Y)\sigma^{-1}) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) = B(X,Y)$.

This implies that

$$B(\mathrm{ad}(Z)X, Y) + B(X, \mathrm{ad}(Z)Y) = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Proof. Differentiate $B(\mathrm{Ad}(\mathrm{Exp}(tZ))X, \mathrm{Ad}(\mathrm{Exp}(tZ))Y) = B(X,Y)$ at $t = 0$.

3.16. **Solvable and nilpotent Lie algebras and Lie groups.** Let $\mathfrak{g}$ be a Lie algebra. The *commutator ideal*, or *derived subalgebra*, $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is the ideal in $\mathfrak{g}$ generated by all vectors in $\mathfrak{g}$ of the form $[X,Y]$, $X, Y \in \mathfrak{g}$. The *commutator series of* $\mathfrak{g}$ is the decreasing sequence

$$\mathfrak{g}^0 = \mathfrak{g} \ , \ \mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0] \ , \ \mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1] \ , \ \ldots$$

of ideals of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is *solvable* if this sequence is finite, that is, if $\mathfrak{g}^k = 0$ for some $k \in \mathbb{N}$. The *lower central series of* $\mathfrak{g}$ is the decreasing sequence

$$\mathfrak{g}_0 = \mathfrak{g} \ , \ \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}_0] \ , \ \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1] \ , \ \ldots$$

of ideals in $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is *nilpotent* if this sequence is finite, that is, if $\mathfrak{g}_k = 0$ for some $k \in \mathbb{N}$. One can show by induction that $\mathfrak{g}^k \subset \mathfrak{g}_k$ for all k. Hence every nilpotent Lie algebra is solvable.

Examples: 1. The Lie algebra of all upper triangular $n \times n$-matrices is solvable.

2. The Lie algebra of all upper triangular $n \times n$-matrices with zeroes in the diagonal is nilpotent. For $n = 3$ this is the so-called *Heisenberg algebra*.

Every subalgebra of a solvable (resp. nilpotent) Lie algebra is also solvable (resp. nilpotent). A Lie algebra is solvable if and only if its derived subalgebra is nilpotent.

A Lie group G is solvable or nilpotent if and only if its Lie algebra $\mathfrak{g}$ is solvable or nilpotent, respectively.

The Lie exponential map $\mathrm{Exp} : \mathfrak{n} \to N$ of a simply connected nilpotent Lie group N is a diffeomorphism from $\mathfrak{n}$ onto N. Thus N is diffeomorphic to $\mathbb{R}^n$ with $n = \dim N$.

3.17. **Cartan's criterion for solvability.** A Lie algebra $\mathfrak{g}$ is solvable if and only if its Cartan-Killing form B satisfies $B(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

3.18. **Simple and semisimple Lie algebras and Lie groups.** Let $\mathfrak{g}$ be a Lie algebra. There exists a unique solvable ideal in $\mathfrak{g}$ which contains all solvable ideals in $\mathfrak{g}$, the so-called *radical* $\mathrm{rad}(\mathfrak{g})$ of $\mathfrak{g}$. If this radical is trivial the Lie algebra is called *semisimple*.

A semisimple Lie algebra $\mathfrak{g}$ is called *simple* if it contains no ideals different from $\{0\}$ and $\mathfrak{g}$. A Lie group is semisimple or simple if and only if its Lie algebra is semisimple or simple, respectively.

A Lie algebra $\mathfrak{g}$ is called *reductive* if for each ideal $\mathfrak{a}$ in $\mathfrak{g}$ there exists an ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. One can show that a Lie algebra is reductive if and only if it is the direct sum of a semisimple Lie algebra and an Abelian Lie algebra. A Lie group is reductive if and only if its Lie algebra is reductive.

3.19. **Cartan's criterion for semisimplicity.** A Lie algebra is semisimple if and only if its Cartan-Killing form is nondegenerate. Recall that B is nondegenerate if $B(X, Y) = 0$ for all $Y \in \mathfrak{g}$ implies $X = 0$.

3.20. **The Levi-Malcev decomposition.** Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Then there exists a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ and a homomorphism $\pi : \mathfrak{s} \to \mathrm{Der}(\mathrm{rad}(\mathfrak{g}))$ such that $\mathfrak{g}$ is isomorphic to the semidirect sum $\mathrm{rad}(\mathfrak{g}) \oplus_\pi \mathfrak{s}$.

3.21. **Structure theory of semisimple complex Lie algebras.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra and B its Cartan-Killing form. A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ so that all endomorphisms $\mathrm{ad}(H)$, $H \in \mathfrak{h}$, are simultaneously diagonalizable. There always exists a Cartan subalgebra in $\mathfrak{g}$, and any two of them are conjugate by an inner automorphism of $\mathfrak{g}$. The common value of the dimension of these Cartan subalgebras is called the *rank* of $\mathfrak{g}$.

Any semisimple complex Lie algebra can be decomposed into the direct sum of simple complex Lie algebras, which were classified by Elie Cartan: *The simple complex Lie algebras are*

$$A_n = \mathfrak{sl}(n+1, \mathbb{C}) \ , \ B_n = \mathfrak{so}(2n+1, \mathbb{C}) \ , \ C_n = \mathfrak{sp}(n, \mathbb{C}) \ , \ D_n = \mathfrak{so}(2n, \mathbb{C})(n \geq 3) \ ,$$

which are the simple complex Lie algebras of classical type, and

$$G_2 \ , \ F_4 \ , \ E_6 \ , \ E_7 \ , \ E_8 \ ,$$

which are the simple complex Lie algebras of exceptional type. Here, the index refers to the rank of the Lie algebra. Note that there are isomorphisms $A_1 = B_1 = C_1$, $B_2 = C_2$ and $A_3 = D_3$. The Lie algebra $D_2 = \mathfrak{so}(4,\mathbb{C})$ is not simple since $D_2 = A_1 \oplus A_1$.

Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple complex Lie algebra $\mathfrak{g}$. For each one-form α in the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$ we define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \operatorname{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\} .$$

If $\mathfrak{g}_\alpha$ is nontrivial and α is nonzero, α is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\mathfrak{g}_\alpha$ is called the *root space* of $\mathfrak{g}$ with respect to α. The complex dimension of $\mathfrak{g}_\alpha$ is always one. We denote by Δ the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

is called the *root space decomposition* of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$.

The Cartan-Killing form B restricted to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate. Thus there exists for each $\alpha \in \Delta$ a vector $H_\alpha \in \mathfrak{h}$ such that $\alpha(H) = B(H_\alpha, H)$ for all $H \in \mathfrak{h}$. Let $\mathfrak{h}_0$ be the real span of all vectors H_α, $\alpha \in \Delta$. Then $\mathfrak{h}_o$ is a real form of the Cartan subalgebra $\mathfrak{h}$ and Δ forms a reduced abstract root system on the real vector space $\mathfrak{h}_0^*$.

We recall that an *abstract root system* on a finite-dimensional real vector space V with an inner product $\langle\cdot,\cdot\rangle$ is a finite set Δ of nonzero elements of V such that Δ spans V, the orthogonal transformations

$$s_\alpha : V \to V \ , \ v \mapsto v - \frac{2\langle v,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$$

map Δ to itself for all $\alpha \in \Delta$, and

$$\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$$

for all $\alpha, \beta \in \Delta$. An abstract root system is *reduced* if $2\alpha \notin \Delta$ for all $\alpha \in \Delta$. The maps s_α are orthogonal reflections in hyperplanes of V, and the group generated by these reflections is called the *Weyl group* of Δ.

We now fix a notion of positivity on $\mathfrak{h}_o^*$, for instance by means of a lexicographic ordering. We fix a basis $v_1, \ldots, v_n$ of $\mathfrak{h}_0$ and say that $\alpha > 0$ if there exists an index k such that $\alpha(v_i) = 0$ for all $i \in \{1, \ldots, k-1\}$ and $\alpha(v_k) > 0$. A root $\alpha \in \Delta$ is called *simple* if it is positive and if it cannot be written as the sum of two positive roots. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of Δ. The $n \times n$-matrix A with coefficients

$$A_{ij} = \frac{2\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \in \mathbb{Z}$$

is called the *Cartan matrix* of Δ and Π. The Cartan matrix depends on the enumeration of Π, but different enumerations lead to Cartan matrices that are conjugate to each other by a permutation matrix. We now associate to Π a diagram in the following way. For each simple root α_i we draw a vertex. We connect the vertices α_i and α_j by $A_{ij}A_{ji}$ edges. If $|\alpha_i| > |\alpha_j|$ we draw an arrow pointing from α_i to α_j. The resulting diagram is called the *Dynkin diagram* of the root system Δ or of the Lie algebra $\mathfrak{g}$.

Example: $G = SL(n+1,\mathbb{C})$. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by all diagonal matrices with trace zero. We denote by ϵ_i the one-form on $\mathfrak{h}$ given by $\epsilon_i(H) = x_i$, where $H = \mathrm{Diag}(x_1,\dots,x_{n+1}) \in \mathfrak{h}$. Let E_{ij} be the $(n+1)\times(n+1)$-matrix with 1 in the i-th row and j-th column, and zeroes everywhere else. Then we have

$$\mathrm{ad}(H)E_{ij} = [H, E_{ij}] = (x_i - x_j)E_{ij} = (\epsilon_i - \epsilon_j)(H)E_{ij}$$

for all $H \in \mathfrak{h}$ and $i \neq j$. It follows that

$$\Delta = \{\epsilon_i - \epsilon_j \mid i \neq j \ , \ i,j \in \{1,\dots,n+1\}\} \ .$$

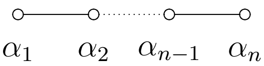
The resulting root space decomposition is

$$\mathfrak{sl}(n+1,\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{i\neq j} \mathbb{C}E_{ij} \ .$$

A set of simple roots is given by

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \ , \ i \in \{1,\dots,n\} \ ,$$

and the resulting Dynkin diagram is



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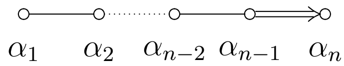
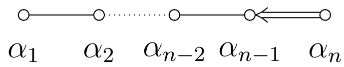
In a similar way one can calculate explicitly the Dynkin diagrams of the other simple complex Lie algebras of classical type:

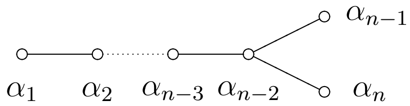
$\mathfrak{so}(2n+1,\mathbb{C})$:



$\mathfrak{sp}(n,\mathbb{C})$:

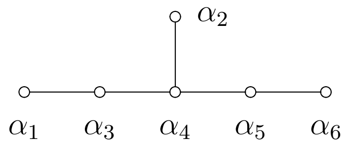


$\mathfrak{so}(2n,\mathbb{C})$:

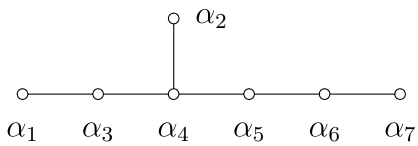


The Dynkin diagrams of the complex simple Lie algebras of exceptional type are:

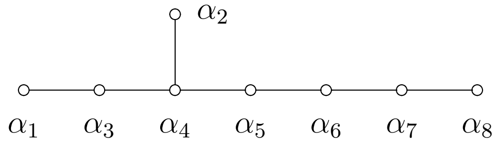
E_6:



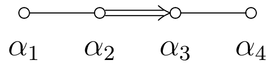
E_7:



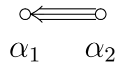
E_8:



F_4:



G_2:



The exceptional complex Lie algebras are related to algebraic structures that are constructed from the octonions (or Cayley numbers).

One can reconstruct the simple complex Lie algebra from its Dynkin diagram. The basic idea for the classification of the simple complex Lie algebras is to show that there are no other Dynkin diagrams (or equivalently, no other reduced root systems).

3.22. **Structure theory of compact real Lie groups.** Let G be a connected compact real Lie group. The Lie algebra $\mathfrak{g}$ of G admits an inner product so that each $\mathrm{Ad}(g)$, $g \in G$, acts as an orthogonal transformation on $\mathfrak{g}$ and each $\mathrm{ad}(X)$, $X \in \mathfrak{g}$, is a skew-symmetric endomorphism of $\mathfrak{g}$. This yields the direct sum decomposition

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}] \ ,$$

where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is the commutator ideal in $\mathfrak{g}$, which is always semisimple. The Cartan-Killing form of $\mathfrak{g}$ is negative semidefinite. If, in addition, $\mathfrak{g}$ is semisimple, or equivalently if $\mathfrak{z}(\mathfrak{g}) = 0$, then its Cartan-Killing form B is negative definite and hence $-B$ induces an $\mathrm{Ad}(G)$-invariant Riemannian metric on G. This metric is biinvariant, that is, all left and right translations are isometries of G. Let $Z(G)^o$ be the identity component of the center $Z(G)$ of G and G^s the connected Lie subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Both $Z(G)^o$ and G^s are closed subgroups of G, G^s is semisimple and has finite center, and G is isomorphic to the direct product $Z(G)^o \times G^s$.

A *torus* in G is a connected Abelian Lie subgroup T of G. The Lie algebra $\mathfrak{t}$ of a torus T in G is an Abelian Lie subalgebra of $\mathfrak{g}$. A torus T in G which is not properly contained in any other torus in G is called a *maximal torus*. Analogously, an Abelian Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ which is not properly contained in any other Abelian Lie subalgebra of $\mathfrak{g}$ is called a *maximal Abelian subalgebra*. There is a natural correspondence between the maximal tori in G and the maximal Abelian subalgebras of $\mathfrak{g}$. Any maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is of the form

$$\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{t}^s \ ,$$

where $\mathfrak{t}^s$ is some maximal Abelian subalgebra of the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.

Any two maximal Abelian subalgebras of $\mathfrak{g}$ are conjugate via $\mathrm{Ad}(g)$ for some $g \in G$. This readily implies that any two maximal tori in G are conjugate. Furthermore, if T is a maximal torus in G, then any $g \in G$ is conjugate to some $t \in T$. Any two elements in

T are conjugate in G if and only if they are conjugate via the Weyl group $W(G,T)$ of G with respect to T. The *Weyl group* of G with respect to T is defined by

$$W(G,T) = N_G(T)/Z_G(T) \ ,$$

where $N_G(T)$ is the normalizer of T in G and $Z_G(T) = T$ is the centralizer of T in G. In particular, the conjugacy classes in G are parametrized by $T/W(G,T)$. The common dimension of the maximal tori of G (resp. of the maximal Abelian subalgebras of $\mathfrak{g}$) is called the *rank* of G (resp. the *rank* of $\mathfrak{g}$). Let $\mathfrak{t}$ be a maximal Abelian subalgebra of $\mathfrak{g}$. Then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. For this reason $\mathfrak{t}$ is also called a *Cartan subalgebra* of $\mathfrak{g}$ and the rank of $\mathfrak{g}$ coincides with the rank of $\mathfrak{g}^{\mathbb{C}}$.

We assume from now on that $\mathfrak{g}$ is semisimple, that is, the center of $\mathfrak{g}$ is trivial. Then $\mathfrak{g}$ is called a compact real form of $\mathfrak{g}^{\mathbb{C}}$. Each semisimple complex Lie algebra has a compact real form which is unique up to conjugation by an element in the connected Lie subgroup of the group of real automorphisms of $\mathfrak{g}^{\mathbb{C}}$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. The compact real forms of the simple complex Lie algebras are for the classical complex Lie algebras

$$\mathfrak{su}(n+1) \subset A_n \ , \ \mathfrak{so}(2n+1) \subset B_n \ , \ \mathfrak{sp}(n) \subset C_n \ , \ \mathfrak{so}(2n) \subset D_n \ ,$$

and for the exceptional complex Lie algebras

$$\mathfrak{g}_2 \subset G_2 \ , \ \mathfrak{f}_4 \subset F_4 \ , \ \mathfrak{e}_6 \subset E_6 \ , \ \mathfrak{e}_7 \subset E_7 \ , \ \mathfrak{e}_8 \subset E_8 \ .$$

Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} (\mathfrak{g}^{\mathbb{C}})_\alpha$$

be the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Each root $\alpha \in \Delta$ is imaginary-valued on $\mathfrak{t}$ and real-valued on $i\mathfrak{t}$. The subalgebra $i\mathfrak{t}$ of $\mathfrak{t}^{\mathbb{C}}$ is a real form of $\mathfrak{t}^{\mathbb{C}}$ and we may view each root $\alpha \in \Delta$ as a one-form in the dual space $(i\mathfrak{t})^*$. Since the Cartan-Killing form B of $\mathfrak{g}$ is negative definite, it leads via complexification to a positive definite inner product on $i\mathfrak{t}$, which we also denote by B. For each $\lambda \in (i\mathfrak{t})^*$ there exists a vector $H_\lambda \in i\mathfrak{t}$ such that

$$\lambda(H) = B(H, H_\lambda)$$

for all $H \in i\mathfrak{t}$. The inner product on $i\mathfrak{t}$ induces an inner product $\langle\cdot,\cdot\rangle$ on $(i\mathfrak{t})^*$. For each $\lambda, \mu \in \Delta$ we then have

$$\langle\lambda,\mu\rangle = B(H_\lambda, H_\mu) \ .$$

For each $\alpha \in \Delta$ we define the root reflection

$$s_\alpha(\lambda) = \lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha \quad (\lambda \in (i\mathfrak{t})^*) \ ,$$

which is a transformation on $(i\mathfrak{t})^*$. The Weyl group of G with respect to T is isomorphic to the group generated by all s_α, $\alpha \in \Delta$. Equivalently one might view $W(G,T)$ as the group of transformations on $i\mathfrak{t}$ generated by the reflections in the hyperplanes perpendicular to iH_λ, $\lambda \in \Delta$.

3.23. **Structure theory of semisimple real Lie algebras.** Let G be a connected semisimple real Lie group, $\mathfrak{g}$ its Lie algebra and B its Cartan-Killing form. A *Cartan involution* on $\mathfrak{g}$ is an involutive automorphism θ of $\mathfrak{g}$ so that

$$B_\theta(X,Y) = -B(X,\theta Y)$$

is a positive definite inner product on $\mathfrak{g}$. Each semisimple real Lie algebra has a Cartan involution, and any two of them are conjugate via $\mathrm{Ad}(g)$ for some $g \in G$. Let θ be a Cartan involution on $\mathfrak{g}$. Denoting by $\mathfrak{k}$ the $(+1)$-eigenspace of θ and by $\mathfrak{p}$ the (-1)-eigenspace of θ, we get the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \ .$$

This decomposition is orthogonal with respect to B and B_θ, B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, and

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \ , \ [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \ , \ [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \ .$$

The Lie algebra $\mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$.

Let K be the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. Then there exists a unique involutive automorphism Θ of G whose differential at the identity of G coincides with θ. Then K is the fixed point set of Θ, is closed, and contains the center $Z(G)$ of G. If K is compact then $Z(G)$ is finite, and if $Z(G)$ is finite then K is a maximal compact subgroup of G. Moreover, the map

$$K \times \mathfrak{p} \to G \ , \ (k,X) \mapsto k\mathrm{Exp}(X)$$

is a diffeomorphism onto G. This is known as a *polar decomposition of* G.

Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. Then all $\mathrm{ad}(H)$, $H \in \mathfrak{a}$, form a commuting family of selfadjoint endomorphisms of $\mathfrak{g}$ with respect to the inner product B_θ. For each $\alpha \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\} \ .$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$, then λ is called a *restricted root* and $\mathfrak{g}_\lambda$ a *restricted root space* of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We denote by Σ the set of all restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. The *restricted root space decomposition* of $\mathfrak{g}$ is the direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda \ .$$

We always have

$$[\mathfrak{g}_\lambda,\mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$$

and

$$\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$$

for all $\lambda,\mu \in \Sigma$. Moreover,

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \ ,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We now choose a notion of positivity for $\mathfrak{a}^*$, which leads to a subset Σ^+ of positive restricted roots. Then

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$$

is a nilpotent Lie subalgebra of $\mathfrak{g}$. Any two such nilpotent Lie subalgebras are conjugate via $\mathrm{Ad}(k)$ for some k in the normalizer of $\mathfrak{a}$ in K. The vector space direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is called an *Iwasawa decomposition of* $\mathfrak{g}$. The vector space $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is in fact a solvable Lie subalgebra of $\mathfrak{g}$ with $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{n}$. Let A, N be the Lie subgroups of G with Lie algebra $\mathfrak{a}, \mathfrak{n}$ respectively. Then A and N are simply connected and the map

$$K \times A \times N \to G \ , \ (k, a, n) \mapsto kan$$

is a diffeomorphism onto G, a so-called *Iwasawa decomposition of* G.

Example: If $G = SL(n, \mathbb{R})$, then $K = SO(n)$, A is the Abelian Lie group of all diagonal $n \times n$-matrices with determinant one, and N is the nilpotent Lie group of upper triangular matrices with entries 1 in the diagonal. This decomposition of matrices with determinant one is well-known from Linear Algebra.

If $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathfrak{m}$, then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, that is, $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{g}^{\mathbb{C}})_\alpha \ .$$

Then we have

$$\mathfrak{g}_\lambda = \mathfrak{g} \cap \bigoplus_{\alpha \in \Delta,\ \alpha|\mathfrak{a} = \lambda} (\mathfrak{g}^{\mathbb{C}})_\alpha$$

for all $\lambda \in \Sigma$ and

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta,\ \alpha|\mathfrak{a} = 0} (\mathfrak{g}^{\mathbb{C}})_\alpha \ .$$

In particular, all roots are real on $\mathfrak{a} \oplus i\mathfrak{t}$. Of particular interest are those real forms of $\mathfrak{g}^{\mathbb{C}}$ for which $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. In this case $\mathfrak{g}$ is called a *split real form* of $\mathfrak{g}^{\mathbb{C}}$. Note that $\mathfrak{g}$ is a split real form if and only if $\mathfrak{m}$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, is trivial. The split real form of $\mathfrak{sl}(n, \mathbb{C})$ is $\mathfrak{sl}(n, \mathbb{R})$, the one of $\mathfrak{so}(2n+1, \mathbb{C})$ is $\mathfrak{so}(n+1, n)$, the one of $\mathfrak{sp}(n, \mathbb{C})$ is $\mathfrak{sp}(n, \mathbb{R})$, and the one of $\mathfrak{so}(2n, \mathbb{C})$ is $\mathfrak{so}(n, n)$.

The classification of real simple Lie algebras is difficult, and we just mention the result. Every simple real Lie algebra is isomorphic to one of the following Lie algebras:

1. the Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is a simple complex Lie algebra (see Section 3.21);
2. the compact real form of a simple complex Lie algebra (see Section 3.22);
3. the classical real simple Lie algebras $\mathfrak{so}(m, n)$, $\mathfrak{su}(m, n)$, $\mathfrak{sp}(m, n)$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{H})$, $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{so}^*(2n)$;
4. the exceptional real simple Lie algebras $\mathfrak{e}_6^6$, $\mathfrak{e}_6^2$, $\mathfrak{e}_6^{-14}$, $\mathfrak{e}_6^{-24}$, $\mathfrak{e}_7^7$, $\mathfrak{e}_7^{-5}$, $\mathfrak{e}_7^{-25}$, $\mathfrak{e}_8^8$, $\mathfrak{e}_8^{-24}$, $\mathfrak{f}_4^4$, $\mathfrak{f}_4^{-20}$, $\mathfrak{g}_2^2$.

All the exceptional real simple Lie algebras are related to algebraic structures constructed from the octonions.

4. Homogeneous spaces

A homogeneous space is a manifold with a transitive group of transformations. Homogeneous spaces provide prominent examples for studying the interplay of analysis, geometry, algebra and topology. A modern introduction to homogeneous spaces can be found in Kawakubo [38]. Further results on Lie transformation groups may be found in [55].

4.1. **Transformation groups.** Let G be a group and M be a set. We say that G is a *transformation group* on M if there exists a map

$$G \times M \to M \ , \ (g,p) \mapsto gp$$

such that $ep = p$ for all $p \in M$, where e is the identity of G, and $g_2(g_1p) = (g_2g_1)p$ for all $g_1, g_2 \in G$ and $p \in M$. Such a map is also called a *G-action* on M. If $p \in M$, then $G \cdot p = \{gp \mid g \in G\}$ is *the orbit of G through p* and $G_p = \{g \in G \mid gp = p\}$ is the *isotropy subgroup* or *stabilizer* of G at p. The action is *transitive* if for all $p, q \in M$ there exists a transformation $g \in G$ with $gp = q$, that is, if there exists only one orbit in M. In this situation M is called a *homogeneous G-space*.

4.2. **Closed subgroups of Lie groups.** In the framework of homogeneous spaces, closed subgroups of Lie groups play an important role. For this reason we summarize here some sufficient criteria for a subgroup of a Lie group to be closed. Let G be a connected Lie group and K a connected Lie subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively.

1. (Yosida [71]) If $G = GL(n,\mathbb{C})$ and $\mathfrak{k}$ is semisimple, then K is closed in G.
2. (Chevalley [23]) If G is simply connected and solvable, then K is simply connected and closed in G.
3. (Malcev [47]) If the rank of K is equal to the rank of G, then K is closed in G.
4. (Chevalley [24]) If G is simply connected and $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, then K is closed in G.
5. (Goto [30]) If $\mathrm{Exp}(\mathfrak{k})$ is closed in G, then K is closed in G.
6. (Mostow [52]) If G is simply connected or compact, and if $\mathfrak{k}$ is semisimple, then K is closed in G.
7. (Borel-Lichnerowicz [14]) If $G = SO(n)$ and K acts irreducibly on $\mathbb{R}^n$, then K is closed in G.

4.3. **The quotient space G/K.** Let G be a Lie group and K a closed subgroup of G. By G/K we denote the set of left cosets of K in G,

$$G/K = \{gK \mid g \in G\} \ ,$$

and by π the canonical projection

$$\pi : G \to G/K \ , \ g \mapsto gK \ .$$

We equip G/K with the quotient topology relative to π. Then π is a continuous map and, since K is closed in G, a Hausdorff space. There exists exactly one smooth manifold structure on G/K (which is even real analytic) so that π becomes a smooth map and local smooth sections of G/K in G exist. If K is a normal subgroup of G, then G/K becomes a Lie group with respect to the multiplication $g_1K \cdot g_2K = (g_1g_2)K$.

If K is a closed subgroup of a Lie group G, then

$$G \times G/K \to G/K \ , \ (g_1, g_2K) \mapsto (g_1g_2)K$$

is a transitive smooth action of G on G/K. In fact, the smooth structure on G/K can be characterized by the property that this action is smooth. Conversely, suppose we have a transitive smooth action

$$G \times M \to M \ , \ (g, p) \mapsto gp$$

of a Lie group G on a smooth manifold M. Let p be a point in M and

$$G_p = \{g \in G \mid gp = p\}$$

the isotropy subgroup of G at p. If q is another point in M and $g \in G$ with $gp = q$, then $G_q = gG_pg^{-1}$. Thus the isotropy subgroups of G are all conjugate to each other. The isotropy group G_p is obviously closed in G. Thus we may equip G/G_p with a smooth manifold structure as described above. With respect to this structure the map

$$G/G_p \to M \ , \ gG_p \mapsto gp$$

is a smooth diffeomorphism. In this way we will always identify the smooth manifold M with the coset space G/K. In this situation $\pi : G \to G/K$ is a principal fiber bundle with fiber and structure group K, where K acts on G by multiplication from the right.

In the following we will always assume that M is a smooth manifold and G is a Lie group acting transitively on M, so that $M = G/K$ with $K = G_o$ for some point $o \in M$.

4.4. **Connected homogeneous spaces.** If M is a connected homogeneous G-space, then also the identity component G^o of G acts transitively on M. This allows us to reduce many problems on connected homogeneous spaces to connected Lie groups and thereby to Lie algebras.

Proof: Since $\pi : G \to G/K$ is an open map and $G/K \to M$ is a homeomorphism, the orbit of each connected component of G through a point $p \in M$ is an open subset of M. If M is connected this implies that each of these open subsets is M itself, because orbits are either disjoint or equal.

4.5. **Compact homogeneous spaces.** If $M = G/K$ is a compact homogeneous G-space with G and K connected, then there exists a compact subgroup of G acting transitively on M (Montgomery [50]). This provides the possibility to use the many useful features of compact Lie groups for studying compact homogeneous spaces.

4.6. **The fundamental group.** Let $M = G/K$ be a homogeneous G-space and assume that G is connected. If K is connected, then the first fundamental group $\pi_1(G/K)$ of $M = G/K$ is a subgroup of the first fundamental group $\pi_1(G)$ of G and hence Abelian. If K is not connected, then $\pi_1(G/K^o)$ is an Abelian normal subgroup of $\pi_1(G/K)$ with index $\#(K/K^o)$.

4.7. **The Euler characteristic.** Recall that for a compact smooth manifold M the Euler characteristic $\chi(M)$ vanishes if and only if there exists a nowhere vanishing smooth vector field on M. Since any compact Lie group is parallelizable, the Euler characteristic of any Lie group vanishes. For homogeneous spaces Hopf and Samelson [33] proved:

Let $M = G/K$ be a homogeneous G-space with G compact. Then $\chi(M) \geq 0$, and $\chi(M) > 0$ if and only if G and K have the same rank.

Recall that the rank of a compact Lie group is the dimension of a maximal torus in it, and that any two maximal tori are conjugate to each other by an inner automorphism of the group.

As an application of the Hopf-Samelson result we see that a compact surface M of genus $g \geq 2$ cannot be a homogeneous space with respect to any compact Lie group, since $\chi(M) = 2 - 2g < 0$.

The result by Hopf and Samelson naturally leads to the question: What are the homogeneous spaces with a given Euler characteristic? For G/K simply connected, Wang [67] proved:

1. If $\chi(G/K) = 1$, then G/K is a point.
2. If $\chi(G/K) = 2$, then G/K is diffeomorphic to the sphere S^{2n} for some $n \in \mathbb{N}$.

Wang also classified all simply connected compact homogeneous spaces for which the Euler characteristic is a prime number. This is a rather short list.

4.8. **Effective actions.** Let M be a homogeneous G-space and $\phi : G \to \mathrm{Diff}(M)$ be the homomorphism from G into the diffeomorphism group of M assigning to each $g \in G$ the diffeomorphism

$$\varphi_g : M \to M \ , \ p \mapsto gp \ .$$

One says that the action of G on M is *effective* if $\ker \phi = \{e\}$, where e denotes the identity in G. In other words, an action is effective if just the identity of G acts as the identity transformation on M. Writing $M = G/K$, we may characterize $\ker \phi$ as the largest normal subgroup of G which is contained in K. Thus $G/\ker \phi$ is a Lie group with an effective transitive action on M.

4.9. **Reductive decompositions.** Let $M = G/K$ be a homogeneous G-space. We denote by e the identity of G and put $o = eK \in M$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. As usual we identify the tangent space of a Lie group at the identity with the corresponding Lie algebra. We choose a linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary to $\mathfrak{k}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ (direct sum of vector spaces). Then the differential π_{*e} at e of the projection $\pi : G \to G/K$ gives rise to an isomorphism

$$\pi_{*e}|\mathfrak{m} : \mathfrak{m} \to T_oM \ .$$

One of the basic tools in studying homogeneous spaces is to use this isomorphism to identify tangent vectors of M at o with elements in the Lie algebra $\mathfrak{g}$. But there are many choices of complementary subspaces $\mathfrak{m}$, and certain ones turn out to be quite useful. We will describe this now.

Let $\mathrm{Ad}_G : G \to GL(\mathfrak{g})$ be the adjoint representation of G. The subspace $\mathfrak{m}$ is said to be $\mathrm{Ad}_G(K)$-invariant if $\mathrm{Ad}_G(k)\mathfrak{m} \subset \mathfrak{m}$ for all $k \in K$. If $\mathfrak{m}$ is $\mathrm{Ad}_G(K)$-invariant and $k \in K$, the differential φ_{k*o} at o of the diffeomorphism $\varphi_k : M \to M \ , \ p \mapsto kp$ has the simple expression

$$\varphi_{k*o} = \mathrm{Ad}_G(k)|\mathfrak{m} \ .$$

For this reason one is interested in finding $\mathrm{Ad}_G(K)$-invariant linear subspaces $\mathfrak{m}$ of $\mathfrak{g}$. Unfortunately, not every homogeneous space admits such subspaces. A homogeneous space G/K is called *reductive* if there exists an $\mathrm{Ad}_G(K)$-invariant linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. In this situation $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is called a *reductive decomposition* of $\mathfrak{g}$.

We list below a few sufficient criteria for a homogeneous space G/K to admit a reductive decomposition:

1. K is compact.
2. K is connected and semisimple.
3. K is a discrete subgroup of G.

4.10. **Isotropy representations and invariant metrics.** The homomorphism

$$\chi : K \to GL(T_oM) \ , \ k \mapsto \varphi_{k*o}$$

is called the *isotropy representation* of the homogeneous space G/K, and the image $\chi(K) \subset GL(T_oM)$ is called the *linear isotropy group* of G/K. In case G/K is reductive and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive decomposition, the isotropy representation of G/K coincides with the adjoint representation $\mathrm{Ad}_G|K : K \to GL(\mathfrak{m})$ (via the identification $\mathfrak{m} = T_oM$).

The linear isotropy group contains the information whether a homogeneous space G/K can be equipped with a G-invariant Riemannian structure. A *G-invariant Riemannian metric* $\langle \cdot, \cdot \rangle$ on $M = G/K$ is a Riemannian metric so that φ_g is an isometry of M for each $g \in G$, that is, if G acts on M by isometries. A homogeneous space $M = G/K$ can be equipped with a G-invariant Riemannian metric if and only if the linear isotropy group $\chi(K)$ is a relative compact subset of the topological space $\mathfrak{gl}(T_oM)$ of all endomorphisms $T_oM \to T_oM$. It follows that every homogeneous space G/K with K compact admits a G-invariant Riemannian metric. Each Riemannian homogeneous space is reductive.

If G/K is reductive and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive decomposition, then there is a one-to-one correspondence between the G-invariant Riemannian metrics on G/K and the positive definite $\mathrm{Ad}_G(K)$-invariant symmetric bilinear forms on $\mathfrak{m}$. Any such bilinear form defines a Riemannian metric on M by requiring that each φ_g is an isometry. The $\mathrm{Ad}_G(K)$-invariance of the bilinear form ensures that the inner product on each tangent space is well-defined. In particular, if $K = \{e\}$, that is, $M = G$ is a Lie group, then the G-invariant Riemannian metrics on M are exactly the left-invariant Riemannian metrics on G.

4.11. **Levi Civita connection of Riemannian homogeneous spaces.** There is an explicit formula for the Levi Civita connection of a Riemannian homogeneous space. This of course allows us to investigate the Riemannian geometry of Riemannian homogeneous spaces in great detail. We will describe the Levi Civita connection now.

Let $M = G/K$ be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and assume that $\langle \cdot, \cdot \rangle$ is an $Ad_G(K)$-invariant inner product on $\mathfrak{m}$. We denote by g the induced G-invariant Riemannian metric on M. For each $X \in \mathfrak{g}$ we obtain a Killing vector field X^* on M by means of

$$X_p^* = \frac{d}{dt}\Big|_{t=0} (t \mapsto \operatorname{Exp}(tX)p)$$

for all $p \in M$. Then

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_o^* = 0\} \ ,$$

and

$$\mathfrak{m} \to T_oM \ , \ X \mapsto X_o^*$$

is a vector space isomorphism. A simple calculation yields

$$[X, Y]^* = -[X^*, Y^*]$$

for all $X, Y \in \mathfrak{g}$. For a vector $X \in \mathfrak{g}$ we denote by $X_\mathfrak{m}$ the $\mathfrak{m}$-component of X with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We define a symmetric bilinear map $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X]_\mathfrak{m}, Y\rangle + \langle X, [Z, Y]_\mathfrak{m}\rangle \ , \ X, Y, Z \in \mathfrak{m} \ .$$

Then the Levi Civita connection ∇ of (M, g) is given by

$$(\nabla_{X^*} Y^*)_o = \left(-\frac{1}{2}[X, Y]_\mathfrak{m} + U(X, Y)\right)_o^*$$

for all $X, Y \in \mathfrak{m}$.

4.12. **Naturally reductive Riemannian homogeneous spaces.** The condition $U \equiv 0$, with U as in the previous section, characterizes the so-called naturally reductive Riemannian homogeneous spaces. More precisely, a Riemannian homogeneous space M is said to be *naturally reductive* if there exists a connected Lie subgroup G of the isometry group $I(M)$ of M which acts transitively and effectively on M and a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{k}$ is the Lie algebra of the isotropy subgroup K of G at some point $o \in M$, such that

$$\langle [X, Z]_\mathfrak{m}, Y\rangle + \langle Z, [X, Y]_\mathfrak{m}\rangle = 0$$

for all $X, Y, Z \in \mathfrak{m}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathfrak{m}$ which is induced by the Riemannian metric on M. Any such decomposition is called a *naturally reductive decomposition* of $\mathfrak{g}$. The above algebraic condition is equivalent to the geometric condition that every geodesic in M through o is of the form $\operatorname{Exp}(tX)o$ for some $X \in \mathfrak{m}$.

Note that the definition of natural reductivity depends on the choice of the subgroup G. A useful criterion for natural reductivity was proved by Kostant [42]:

Let $M = G/K$ be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. An $\mathrm{Ad}_G(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ induces a naturally reductive Riemannian metric on M if and only if on the ideal $\mathfrak{g}' = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{g}$ there exists a nondegenerate symmetric bilinear form q such that $q(\mathfrak{g}' \cap \mathfrak{k}, \mathfrak{m}) = 0$, $\langle \cdot, \cdot \rangle = q|(\mathfrak{m} \times \mathfrak{m})$, and q is $\mathrm{Ad}_G(G')$-invariant, where G' is the connected Lie subgroup of G with Lie algebra $\mathfrak{g}'$.

4.13. **Normal Riemannian homogeneous spaces.** A homogeneous space $M = G/K$ with a G-invariant Riemannian metric g is called *normal homogeneous* if there exists an $\mathrm{Ad}_G(G)$-invariant inner product q on $\mathfrak{g}$ such that g is the induced Riemannian metric from $q|(\mathfrak{m} \times \mathfrak{m})$, where $\mathfrak{m} = \mathfrak{k}^\perp$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to q. It follows immediately from Kostant's result in the previous section that each normal homogeneous space is naturally reductive.

It is well-known that there exists an $\mathrm{Ad}_G(G)$-invariant inner product on the Lie algebra $\mathfrak{g}$ of a Lie group G if and only if G is compact. Thus every normal homogeneous space is compact. If G is compact and semisimple, then the Cartan-Killing form B is negative definite. Thus we may choose $q = -B$, in which case $M = G/K$ is called a *standard Riemannian homogeneous space*, and the induced Riemannian metric on M is called the *standard homogeneous metric* or *Cartan-Killing metric* on M. Every standard Riemannian homogeneous space is normal homogeneous and hence also naturally reductive. To summarize, if G is a compact and semisimple Lie group and K is a closed subgroup of G, then the Cartan-Killing form of G induces a G-invariant Riemannian metric g on the homogeneous space $M = G/K$ such that (M, g) is normal homogeneous.

If, in addition, there exists an involutive automorphism σ on G such that $(G^\sigma)^o \subset K \subset G^\sigma$, where G^σ denotes the fixed point set of σ, then the standard homogeneous space $M = G/K$ is a Riemannian symmetric space of compact type (see next section for more details on symmetric spaces).

An example of a naturally reductive Riemannian homogeneous space which is not normal is the 3-dimensional Heisenberg group with any left-invariant Riemannian metric.

4.14. **Curvature of naturally reductive Riemannian homogeneous spaces.** Let $M = G/K$ be a naturally reductive Riemannian homogeneous space with naturally reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Using the explicit expression for the Levi Civita connection of M, a straightforward lengthy calculation leads to the following expression for the Riemannian curvature tensor R of M:

$$R_o(X,Y)Z = -[[X,Y]_\mathfrak{k}, Z] - \frac{1}{2}[[X,Y]_\mathfrak{m}, Z]_\mathfrak{m} - \frac{1}{4}[[Z,X]_\mathfrak{m}, Y]_\mathfrak{m} + \frac{1}{4}[[Z,Y]_\mathfrak{m}, X]_\mathfrak{m}$$

for all $X, Y, Z \in \mathfrak{m} \cong T_oM$.

From the curvature tensor one can easily calculate the sectional curvature. If $X, Y \in \mathfrak{m}$ are orthonormal, we denote by $K_{X,Y}$ the sectional curvature with respect to the 2-plane spanned by X and Y. If M is normal homogeneous, then

$$K_{X,Y} = ||[X,Y]_\mathfrak{k}||^2 + \frac{1}{4}||[X,Y]_\mathfrak{m}||^2 \geq 0 \ .$$

Thus every normal Riemannian homogeneous space has nonnegative sectional curvature.

One can easily deduce formulae for the Ricci curvature and the scalar curvature from the above expression for the Riemannian curvature tensor.

5. Symmetric spaces and flag manifolds

Symmetric spaces form a subclass of the homogeneous spaces and were studied intensely and also classified by Elie Cartan [17], [18]. The fundamental books on this topic are Helgason [31] and Loos [46]. Another nice introduction may be found in [64]. Flag manifolds are homogeneous spaces which are intimately related to symmetric spaces.

5.1. **(Locally) symmetric spaces.** Let M be a Riemannian manifold, $p \in M$, and $r \in \mathbb{R}_+$ sufficiently small so that normal coordinates are defined on the open ball $B_r(p)$ consisting of all points in M with distance less than r to p. Denote by $\exp_p : T_pM \to M$ the exponential map of M at p. The map

$$s_p : B_r(p) \to B_r(p) \ , \ \exp(tv) \mapsto \exp(-tv)$$

reflects in p the geodesics of M through p and is called a *local geodesic symmetry* at p. A connected Riemannian manifold is called a *locally symmetric space* if at each point p in M there exists an open ball $B_r(p)$ such that the corresponding local geodesic symmetry s_p is an isometry. A connected Riemannian manifold is called a *symmetric space* if at each point $p \in M$ such a local geodesic symmetry extends to a global isometry $s_p : M \to M$. This is equivalent to saying that there exists an involutive isometry s_p of M such that p is an isolated fixed point of s_p. In such a case one calls s_p the *symmetry* of M in p. If M is a symmetric space, then the symmetries s_p, $p \in M$, generate a group of isometric transformations which acts transitively on M. Hence every symmetric space is a Riemannian homogeneous space.

Let M be a Riemannian homogeneous space and suppose there exists a symmetry of M at some point $p \in M$. Let q be any point in M and g an isometry of M with $g(p) = q$. Then $s_q := gs_pg^{-1}$ is a symmetry of M at q. In order to show that a Riemannian homogeneous space is symmetric it therefore suffices to construct a symmetry at one point. Using this we can easily describe some examples of symmetric spaces. The Euclidean space $\mathbb{R}^n$ is symmetric with $s_0 : \mathbb{R}^n \to \mathbb{R}^n \ , \ p \mapsto -p$. The map

$$S^n \to S^n \ , \ (p_1, \ldots, p_n, p_{n+1}) \mapsto (-p_1, \ldots, -p_n, p_{n+1})$$

is a symmetry of the sphere S^n at $(0, \ldots, 0, 1)$. In a similar way, using the hyperboloid model of the real hyperbolic space $\mathbb{R}H^n$ in Lorentzian space $\mathbb{L}^{n+1}$, one can show that $\mathbb{R}H^n$ is a symmetric space. Let G be a connected compact Lie group. Any $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ extends to a biinvariant Riemannian metric on G. With respect to such a Riemannian metric the inverse map $s_e : G \to G \ , \ g \mapsto g^{-1}$ is a symmetry of G at e. Thus any connected compact Lie group is a symmetric space.

We recall some basic features of (locally) symmetric spaces. A Riemannian manifold is locally symmetric if and only if its Riemannian curvature tensor is parallel, that is, $\nabla R = 0$. If M is a connected, complete, locally symmetric space, then its Riemannian universal covering is a symmetric space. Note that there are complete locally symmetric spaces which are not symmetric, even not homogeneous. For instance, let M be a compact Riemann surface with genus ≥ 2 and equipped with a Riemannian metric of constant curvature -1. It is known that the isometry group of M is finite, whence M is not homogeneous and therefore also not symmetric. On the other hand, M is locally isometric to the real hyperbolic plane $\mathbb{R}H^2$ and hence locally symmetric.

5.2. **Cartan decomposition and Riemannian symmetric pairs.** To each symmetric space one can associate a Riemannian symmetric pair. We first recall the definition of a Riemannian symmetric pair. Let G be a connected Lie group and s a nontrivial involutive automorphism of G. We denote by $G_s \subset G$ the set of fixed points of s and by G_s^o the connected component of G_s containing the identity e of G. Let K be a closed subgroup of G with $G_s^o \subset K \subset G_s$. Then $\sigma := s_{*e}$ is an involutive automorphism of $\mathfrak{g}$ and

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \sigma X = X\} \ .$$

The linear subspace

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \sigma X = -X\}$$

of $\mathfrak{g}$ is called the *standard complement* of $\mathfrak{k}$ in $\mathfrak{g}$. Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ (direct sum of vector spaces) and

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \ , \ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \ .$$

This particular decomposition of $\mathfrak{g}$ is called the *Cartan decomposition* or *standard decomposition* of $\mathfrak{g}$ with respect to σ. In this situation, the pair (G, K) is called a *Riemannian symmetric pair* if $\mathrm{Ad}_G(K)$ is a compact subgroup of $GL(\mathfrak{g})$ and $\mathfrak{m}$ is equipped with some $\mathrm{Ad}_G(K)$-invariant inner product.

Suppose (G, K) is a Riemannian symmetric pair. The inner product on $\mathfrak{m}$ determines a G-invariant Riemannian metric on the homogeneous space $M = G/K$, and the map

$$M \to M \ , \ gK \mapsto s(g)K \ ,$$

where s is the involutive automorphism on G, is a symmetry of M at $o = eK \in M$. Thus M is a symmetric space. Conversely, suppose M is a symmetric space. Let G be the identity component of the full isometry group M, o any point in M, s_o the symmetry of M at o, and K the isotropy subgroup of G at o. Then

$$s : G \to G \ , \ g \mapsto s_o g s_o$$

is an involutive automorphism of G with $G_s^o \subset K \subset G_s$, and the inner product on the standard complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is $\mathrm{Ad}_G(K)$-invariant (using our usual identification $\mathfrak{m} = T_oM$). In this way the symmetric space M determines a Riemannian symmetric pair (G, K). This Riemannian symmetric pair is *effective*, that is, each normal subgroup of G which is contained in K is trivial. In the way described here there is a one-to-one correspondence between symmetric spaces and effective Riemannian symmetric pairs.

5.3. **Riemannian geometry of symmetric spaces.** Since the Cartan decomposition is naturally reductive, everything that has been said about the Riemannian geometry of naturally reductive spaces holds also for symmetric spaces. We summarize here a few basic facts.

Let M be a symmetric space, $o \in M$, $G = I^o(M)$, K the isotropy group at o and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the corresponding Cartan decomposition of $\mathfrak{g}$. For each $X \in \mathfrak{g}$ we have a one-parameter group $\mathrm{Exp}(tX)$ of isometries of M. We denote the corresponding complete Killing vector field on M by X^*. As usual, we identify $\mathfrak{m}$ and T_oM by means of the isomorphism $\mathfrak{m} \to T_oM$, $X \mapsto X^*_o$. The Levi Civita connection of M is given by

$$(\nabla_{X^*}Y^*)_o = 0$$

for all $X, Y \in \mathfrak{m}$. For each $X \in \mathfrak{m}$ the geodesic $\gamma_X : \mathbb{R} \to M$ with $\gamma_X(0) = o$ and $\dot\gamma_X(0) = X$ is the curve $t \mapsto \mathrm{Exp}(tX)o$. Let Φ^{X^*} be the flow of X^*. Then the parallel translation along γ_X from $o = \gamma_X(0)$ to $\gamma_X(t)$ is given by

$$(\Phi^{X^*}_t)_{*o} : T_oM \to T_{\gamma_X(t)}M \ .$$

The Riemannian curvature tensor R_o of M at o is given by the simple formula

$$R_o(X,Y)Z = -[[X,Y],Z]$$

for all $X, Y, Z \in \mathfrak{m} = T_oM$.

5.4. **Semisimple symmetric spaces, rank, and duality.** Let M be a symmetric space and $\widetilde{M}$ its Riemannian universal covering space. Let $\widetilde{M}_0 \times \ldots \times \widetilde{M}_k$ be the de Rham decomposition of $\widetilde{M}$, where the Euclidean factor $\widetilde{M}_0$ is isometric to some Euclidean space of dimension ≥ 0. Each $\widetilde{M}_i$, $i > 0$, is a simply connected, irreducible, symmetric space. A *semisimple symmetric space* is a symmetric space for which $\widetilde{M}_0$ has dimension zero. This notion is due to the fact that if $\widetilde{M}_0$ is trivial then $I^o(M)$ is a semisimple Lie group. A symmetric space M is said to be of compact type if M is semisimple and compact, and it is said to be of noncompact type if M is semisimple and each factor of $\widetilde{M}$ is noncompact. Symmetric spaces of noncompact type are always simply connected. An *s-representation* is the isotropy representation of a simply connected, semisimple, symmetric space $M = G/K$ with $G = I^o(M)$.

The *rank* of a semisimple symmetric space $M = G/K$ is the dimension of a maximal Abelian subspace of $\mathfrak{m}$ in some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of $G = I^o(M)$.

Let (G, K) be a Riemannian symmetric pair so that G/K is a simply connected Riemannian symmetric space of compact type or of noncompact type, respectively. Consider the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ of $\mathfrak{g}$ and the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g}$. Then $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{m}$ is a real Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$ with respect to the induced Lie algebra structure. Let G^* be the real Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^*$. Then G^*/K is a simply connected Riemannian symmetric space of noncompact type or of compact type, respectively, with Cartan decomposition $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{m}$. This feature is known as duality between symmetric spaces of compact type and of noncompact type and describes explicitly a one-to-one correspondence between these two types of simply connected symmetric spaces.

5.5. **Classification of symmetric spaces.** Every simply connected symmetric space decomposes into the Riemannian product of a Euclidean space and some simply connected, irreducible, symmetric spaces. Thus the classification problem for simply connected symmetric spaces reduces to the classification of simply connected, irreducible symmetric spaces. Any such space is either of compact type or of noncompact type. The concept of duality enables one to reduce the classification problem to those of noncompact type. The crucial step for deriving the latter classification is to show that every noncompact irreducible symmetric space is of the form $M = G/K$ with some simple noncompact real Lie group G with trivial center and K a maximal compact subgroup of G. If the complexification of $\mathfrak{g}$ is simple as a complex Lie algebra, then M is said to be of type III, otherwise M is said to be of type IV. The corresponding compact irreducible symmetric spaces, which are obtained by duality, are said to be of type I and II, respectively. The complete list of simply connected irreducible symmetric spaces is as follows:

1. Classical types I and III:

Type I (compact)	Type III (noncompact)	Dimension	Rank
$SU(n)/SO(n)$	$SL(n,\mathbb{R})/SO(n)$	$(n-1)(n+2)/2$	$n-1$
$SU(2n)/Sp(n)$	$SL(n,\mathbb{H})/Sp(n)$	$(n-1)(2n+1)$	$n-1$
$SU(p+q)/S(U(p)\times U(q))$	$SU(p,q)/S(U(p)\times U(q))$	$2pq$	$\min\{p,q\}$
$SO(p+q)/SO(p)\times SO(q)$	$SO^o(p,q)/SO(p)\times SO(q)$	pq	$\min\{p,q\}$
$SO(2n)/U(n)$	$SO^*(2n)/U(n)$	$n(n-1)$	$[n/2]$
$Sp(n)/U(n)$	$Sp(n,\mathbb{R})/U(n)$	$n(n+1)$	n
$Sp(p+q)/Sp(p)\times Sp(q)$	$Sp(p,q)/Sp(p)\times Sp(q)$	$4pq$	$\min\{p,q\}$

The symmetric space $SO(p+q)/SO(p)\times SO(q)$ is the Grassmann manifold of all p-dimensional oriented linear subspaces of $\mathbb{R}^{p+q}$ and will often be denoted by $G_p^+(\mathbb{R}^{p+q})$. The Grassmann manifold $G_2^+(\mathbb{R}^4)$ is isometric to the Riemannian product $S^2\times S^2$ and hence reducible. So, strictly speaking, this special case has to be excluded from the above table. Disregarding the orientation of the p-planes we have a natural 2-fold covering map $G_p^+(\mathbb{R}^{p+q}) \to G_p(\mathbb{R}^{p+q})$ onto the Grassmann manifold $G_p(\mathbb{R}^{p+q})$ of all p-dimensional linear subspaces of $\mathbb{R}^{p+q}$, which can be written as the homogeneous space $SO(p+q)/S(O(p)\times O(q))$. Similarily, the symmetric space $SU(p+q)/S(U(p)\times U(q))$ is the Grassmann manifold of all p-dimensional complex linear subspaces of $\mathbb{C}^{p+q}$ and will be denoted by $G_p(\mathbb{C}^{p+q})$. Eventually, the symmetric space $Sp(p+q)/Sp(p)\times Sp(q)$ is the Grassmann manifold of all p-dimensional quaternionic linear subspaces of $\mathbb{H}^{p+q}$ and will be denoted by $G_p(\mathbb{H}^{p+q})$. The Grassmann manifold $G_1^+(\mathbb{R}^{1+q})$ is the q-dimensional sphere S^q. And the Grassmann manifold $G_1(\mathbb{R}^{1+q})$ (resp. $G_1(\mathbb{C}^{1+q})$ or $G_1(\mathbb{H}^{1+q})$) is the q-dimensional real (resp. complex or quaternionic) projective space $\mathbb{R}P^q$ (resp. $\mathbb{C}P^q$ or $\mathbb{H}P^q$). The dual space of the sphere S^q is the real hyperbolic space $\mathbb{R}H^q$. And the dual space of the complex projective space

$\mathbb{C}P^q$ (resp. the quaternionic projective space $\mathbb{H}P^q$) is the complex hyperbolic space $\mathbb{C}H^q$ (resp. the quaternionic hyperbolic space $\mathbb{H}H^q$).

All these spaces have interesting applications in geometry. For instance, if M is a p-dimensional submanifold of $\mathbb{R}^{p+q}$, then the map

$$M \to G_p(\mathbb{R}^{p+q}) \ , \ p \mapsto T_pM$$

is the so-called *Gauss map* of M. It provides an important tool for studying the geometry of submanifolds. For instance, for $p = 2$ and oriented surfaces M^2 in $\mathbb{R}^{2+q}$ the Gauss map takes values in the oriented Grassmann manifold $G_2^+(\mathbb{R}^{2+q})$. This Grassmann manifold is an Hermitian symmetric space (see next section) and hence is equipped with a Kähler structure J. Assume that M^2 is an oriented surface in $\mathbb{R}^{2+q}$ whose Gauss map $f : M \to G_2(\mathbb{R}^{2+q})$ is an immersion. Since M is oriented it has a natural holomorphic structure. Then one can prove that f is a holomorphic map if and only if M is contained in a sphere in $\mathbb{R}^{2+q}$, and f is an anti-holomorphic map if and only if M is a minimal surface.

In low dimensions certain symmetric spaces are isometric to each other (with a suitable normalization of the Riemannian metric):

$$S^2 = \mathbb{C}P^1 = SU(2)/SO(2) = SO(4)/U(2) = Sp(1)/U(1) \ , \ S^4 = \mathbb{H}P^1 \ ,$$
$$S^5 = SU(4)/Sp(2) \ , \ \mathbb{C}P^3 = SO(6)/U(3) \ , \ G_2^+(\mathbb{R}^5) = Sp(2)/U(2) \ ,$$
$$G_2^+(\mathbb{R}^6) = G_2(\mathbb{C}^4) \ , \ G_2^+(\mathbb{R}^8) = SO(8)/U(4) \ , \ G_3^+(\mathbb{R}^6) = SU(4)/SO(4) \ .$$

In the noncompact case one has isometries between the corresponding dual symmetric spaces.

2. Exceptional types I and III:

Type I (compact)	Type III (noncompact)	Dimension	Rank
$E_6/Sp(4)$	$E_6^6/Sp(4)$	42	6
$E_6/SU(6) \times SU(2)$	$E_6^2/SU(6) \times SU(2)$	40	4
$E_6/T \cdot Spin(10)$	$E_6^{-14}/T \cdot Spin(10)$	32	2
E_6/F_4	E_6^{-26}/F_4	26	2
$E_7/SU(8)$	$E_7^7/SU(8)$	70	7
$E_7/SO(12) \times SU(2)$	$E_7^{-5}/SO(12) \times SU(2)$	64	4
$E_7/T \cdot E_6$	$E_7^{-25}/T \cdot E_6$	54	3
$E_8/SO(16)$	$E_8^8/SO(16)$	128	8
$E_8/E_7 \times SU(2)$	$E_8^{-24}/E_7 \times SU(2)$	112	4
$F_4/Sp(3) \times SU(2)$	$F_4^4/Sp(3) \times SU(2)$	28	4
$F_4/Spin(9)$	$F_4^{-20}/Spin(9)$	16	1
$G_2/SO(4)$	$G_2^2/SO(4)$	8	2

Here we denote by E_6, E_7, E_8, F_4, G_2 the connected, simply connected, compact, real Lie group with Lie algebra $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$, respectively. This is the same notation as we used for the corresponding simple complex Lie algebras, but it should always be clear from the context what these symbols represent. The compact real Lie group G_2 can be realized as the automorphism group of the (nonassociative) real division algebra $\mathbb{O}$ of all Cayley numbers (or octonions). The compact real Lie group F_4 can be explicitly realized as the automorphism group of the exceptional Jordan algebra of all 3×3-Hermitian matrices with coefficients in $\mathbb{O}$. The symmetric space $F_4/Spin(9)$ is the Cayley projective plane $\mathbb{O}P^2$ and the dual space $F_4^{-20}/Spin(9)$ is the Cayley hyperbolic plane $\mathbb{O}H^2$. Unlike their counterparts for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the Cayley projective plane and the Cayley hyperbolic plane cannot be realized as a set of lines in a 3-dimensional vector space over $\mathbb{O}$. This is due to the nonassociativity of the Cayley numbers. The exceptional symmetric space $E_6/T \cdot Spin(10)$ is sometimes viewed as the complexification of $\mathbb{O}P^2$ (for more about this see the paper [6] by Atiyah and the author).

3. Classical types II and IV:

Type II (compact)	Type IV (noncompact)	Dimension	Rank
$SU(n+1)$	$SL(n+1,\mathbb{C})/SU(n+1)$	$n(n+2)$	n
$Spin(2n+1)$	$SO(2n+1,\mathbb{C})/SO(2n+1)$	$n(2n+1)$	n
$Sp(n)$	$Sp(n,\mathbb{C})/Sp(n)$	$n(2n+1)$	n
$Spin(2n)$	$SO(2n,\mathbb{C})/SO(2n)$	$n(2n-1)$	n

Since $Spin(2)$ is isomorphic to $U(1)$, and $Spin(4)$ is isomorphic to $SU(2) \times SU(2)$, we have to assume $n \geq 3$ for the spaces in the last row this table. In low dimensions there are the following additional isomorphisms:

$$Spin(3) = SU(2) = Sp(1)\ ,\ Spin(5) = Sp(2)\ ,\ Spin(6) = SU(4)\ .$$

In the noncompact case there are isomorphisms between the corresponding dual spaces.

4. Exceptional types II and IV:

Type II (compact)	Type IV (noncompact)	Dimension	Rank
E_6	$E_6^{\mathbb{C}}/E_6$	78	6
E_7	$E_7^{\mathbb{C}}/E_7$	133	7
E_8	$E_8^{\mathbb{C}}/E_8$	248	8
F_4	$F_4^{\mathbb{C}}/F_4$	52	4
G_2	$G_2^{\mathbb{C}}/G_2$	14	2

5.6. **Hermitian symmetric spaces.** An *Hermitian symmetric space* is a symmetric space which is equipped with a Kähler structure so that the geodesic symmetries are holomorphic maps. The simplest example of an Hermitian symmetric space is the complex vector space $\mathbb{C}^n$. For semisimple symmetric spaces one can easily decide whether it is Hermitian or not. In fact, let (G,K) be the Riemannian symmetric pair of an irreducible semisimple symmetric space M. Then the center of K is either discrete or one-dimensional. The irreducible semisimple Hermitian symmetric spaces are precisely those for which the center of K is one-dimensional. In this case the adjoint action $\mathrm{ad}(Z)$ on $\mathfrak{m}$ of a suitable element Z in the center of $\mathfrak{k}$ induces the almost complex sructure J on M. This gives the list

compact type	noncompact type
$SU(p+q)/S(U(p) \times U(q))$	$SU(p,q)/S(U(p) \times U(q))$
$SO(2+q)/SO(2) \times SO(q)$	$SO^o(2,q)/SO(2) \times SO(q)$
$SO(2n)/U(n)$	$SO^*(2n)/U(n)$
$Sp(n)/U(n)$	$Sp(n,\mathbb{R})/U(n)$
$E_6/T \cdot Spin(10)$	$E_6^{-14}/T \cdot Spin(10)$
$E_7/T \cdot E_6$	$E_7^{-25}/T \cdot E_6$

Note that $SO(4)/SO(2)\times SO(2)$ is isometric to the Riemannian product $S^2 \times S^2$, whence we have to exclude the case $q = 2$ in the second row of the above table. Every semisimple Hermitian symmetric space is simply connected and hence decomposes into the Riemannian product of irreducible semisimple Hermitian symmetric spaces.

5.7. **Complex flag manifolds.** Let G be a connected, compact, semisimple, real Lie group with trivial center and $\mathfrak{g}$ its Lie algebra. Consider the action of G on $\mathfrak{g}$ by the adjoint representation $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$. For each $0 \neq X \in \mathfrak{g}$ the orbit

$$G \cdot X = \{\mathrm{Ad}(g)X \mid g \in G\}$$

is a homogeneous G-space. Let $\mathfrak{t}_X$ be the intersection of all maximal Abelian subalgebras of $\mathfrak{g}$ containing X and T_X the torus in G with Lie algebra $\mathfrak{t}_X$. Then the isotropy subgroup of G at X is $Z_G(T_X)$, the centralizer of T_X in G, and therefore

$$G \cdot X = G/Z_G(T_X) \ .$$

In particular, if X is a regular element of $\mathfrak{g}$, that is, if there is a unique maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ which contains X, then $G \cdot X = G/T$, where T is the maximal torus in G with Lie algebra $\mathfrak{t}$. Any orbit $G \cdot X$ of the adjoint representation of G is called a *complex flag manifold*. In the special case of $G = SU(n)$ one obtains the flag manifolds of all possible flags in $\mathbb{C}^n$ in this way. In particular, when T is a maximal torus of $SU(n)$, then $SU(n)/T$ is the flag manifold of all full flags in $\mathbb{C}^n$, that is, of all possible arrangements $\{0\} \subset V^1 \subset \ldots \subset V^{n-1} \subset \mathbb{C}^n$, where V^k is a k-dimensional complex linear subspace of $\mathbb{C}^n$.

The importance of complex flag manifolds becomes clear from the following facts. Each orbit $G \cdot X$ admits a canonical almost complex structure which is also integrable (Borel [13]). If G is simple there exists a unique (up to homothety) G-invariant Kähler-Einstein metric on $G \cdot X$ with positive scalar curvature and which is compatible with the canonical complex structure on $G \cdot X$ (Koszul [43]). Moreover, any Kähler-Einstein metric on $G \cdot X$ is homogeneous under its own group of isometries and is obtained from a G-invariant Kähler-Einstein metric via some automorphism of the complex structure (Matsushima [48]). Conversely, any simply connected, compact, homogeneous Kähler manifold is isomorphic as a complex homogeneous manifold to some orbit $G \cdot X$ of the adjoint representation of G, where $G = I^o(M)$ and $X \in \mathfrak{g}$. Note that each compact homogeneous Kähler manifold is the Riemannian product of a flat complex torus and a simply connected, compact, homogeneous Kähler manifold (Matsushima [49], Borel and Remmert [15]).

5.8. **Real flag manifolds.** A *real flag manifold* is an orbit of an s-representation. Real flag manifolds are also known as *R-spaces*, a notion that is used more frequently in earlier papers on this topic. Note that the s-representation of a symmetric space of noncompact type is the same as the one of the corresponding dual symmetric space. Thus in order to classify and study real flag manifolds it is sufficient to consider just one type of symmetric spaces.

Let $M = G/K$ be a simply connected semisimple symmetric space of noncompact type with $G = I^o(M)$, $o \in M$ and K the isotropy subgroup of G at o. Note that K is connected as M is assumed to be simply connected and G is connected. We consider the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the semisimple real Lie algebra $\mathfrak{g}$ of G. Let $0 \neq X \in \mathfrak{m}$ and $K \cdot X$ the orbit of K through X via the s-representation. For each $k \in K$ we have $k \cdot X = k_{*o}X = Ad(k)X$ and therefore $K \cdot X = K/K_X$ with $K_X = \{k \in K \mid Ad(k)X = X\}$. Let $\mathfrak{a}_X$ be the intersection of all maximal Abelian subspaces $\mathfrak{a}$ of $\mathfrak{m}$ with $X \in \mathfrak{a}$. We say that X is *regular* if $\mathfrak{a}_X$ is a maximal Abelian subspace of $\mathfrak{m}$, or equivalently, if there exists a unique maximal Abelian subspace of $\mathfrak{m}$ which contains X. Otherwise we call X *singular*. The isotropy subgroup K_X is the centralizer of $\mathfrak{a}_X$ in K. If, in particular, $\mathfrak{g}$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$ and X is regular, then $K \cdot X = K$.

In general, a real flag manifold is not a symmetric space. Consider the semisimple real Lie algebra $\mathfrak{g}$ equipped with the positive definite inner product $B_\sigma(X,Y) = -B(X,\sigma Y)$, where σ is the Cartan involution on $\mathfrak{g}$ coming from the symmetric space structure of G/K. For $0 \neq X \in \mathfrak{m}$ the endomorphism $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is selfadjoint and hence has real eigenvalues. The real flag manifold $K \cdot X$ is a symmetric space if and only if the eigenvalues of $\mathrm{ad}(X)$ are $-c, 0, +c$ for some $c > 0$. Note that not every semisimple real Lie algebra $\mathfrak{g}$ admits such an element X. A real flag manifold which is a symmetric space is called a *symmetric R-space*. If, in addition, $\mathfrak{g}$ is simple, then it is called an *irreducible symmetric R-space*. Decomposing $\mathfrak{g}$ into its simple parts one sees easily that every symmetric R-space is the Riemannian product of irreducible symmetric R-spaces.

The classification of the symmetric R-spaces was established by Kobayashi and Nagano [40]. It follows from their classification and a result by Takeuchi [63] that the symmetric R-spaces consist of the Hermitian symmetric spaces of compact type and their real forms. A real form M of a Hermitian symmetric space $\bar{M}$ is a connected, complete, totally real, totally geodesic submanifold of $\bar{M}$ whose real dimension equals the complex dimension of $\bar{M}$. These real forms were classified by Takeuchi [63] and independently by Leung [44].

Among the irreducible symmetric R-spaces the Hermitian symmetric spaces are precisely those arising from simple complex Lie groups modulo some compact real form. This means that an irreducible symmetric R-space is a Hermitian symmetric space or a real form precisely if the symmetric space G/K is of type IV or III, respectively. The isotropy representation of a symmetric space G/K of noncompact type is the same as the isotropy representation of its dual simply connected compact symmetric space. Thus we may also characterize the Hermitian symmetric spaces among the irreducible symmetric R-spaces as those spaces which arise as an orbit of the adjoint representation of a simply connected, compact, real Lie group G, or equivalently, which is a complex flag manifold. This leads to the following table:

G	$K \cdot X = \mathrm{Ad}(G) \cdot X$	Remarks
$Spin(n)$	$SO(n)/SO(2) \times SO(n-2)$	$n \geq 5$
$Spin(2n)$	$SO(2n)/U(n)$	$n \geq 3$
$SU(n)$	$SU(n)/S(U(p) \times U(n-p))$	$n \geq 2,\ 1 \leq p \leq [\frac{n}{2}]$
$Sp(n)$	$Sp(n)/U(n)$	$n \geq 2$
E_6	$E_6/T \cdot Spin(10)$	
E_7	$E_7/T \cdot E_6$	

The real forms are always non-Hermitian and among the irreducible symmetric R-spaces they are precisely those spaces arising from the isotropy representation of a symmetric space G/K of type I.

G/K	$K \cdot X$	Remarks
$SU(n)/SO(n)$	$G_p(\mathbb{R}^n)$	$n \geq 3,\ 1 \leq p \leq [\frac{n}{2}]$
$SU(2n)/Sp(n)$	$G_p(\mathbb{H}^n)$	$n \geq 2,\ 1 \leq p \leq [\frac{n}{2}]$
$SU(2n)/S(U(n) \times U(n))$	$U(n)$	$n \geq 2$
$SO(n)/SO(p) \times SO(n-p)$	$(S^{p-1} \times S^{n-p-1})/\mathbb{Z}_2$	$n \geq 3,\ 1 \leq p \leq [\frac{n}{2}]$
$SO(2n)/SO(n) \times SO(n)$	$SO(n)$	$n \geq 5$
$SO(4n)/U(2n)$	$U(2n)/Sp(n)$	$n \geq 3$
$Sp(n)/U(n)$	$U(n)/SO(n)$	$n \geq 3$
$Sp(2n)/Sp(n) \times Sp(n)$	$Sp(n)$	$n \geq 2$
$E_6/Sp(4)$	$G_2(\mathbb{H}^4)/\mathbb{Z}_2$	
E_6/F_4	$\mathbb{O}P^2$	
$E_7/SU(8)$	$(SU(8)/Sp(4))/\mathbb{Z}_2$	
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	

The symmetric R-spaces appear in geometry and topology in various contexts. We just mention two examples for geometry.

Every symmetric R-space is a symmetric submanifold of the Euclidean space $\mathfrak{m}$. Here, a submanifold S of $\mathbb{R}^n$ is symmetric if the reflection of $\mathbb{R}^n$ in each normal space of S leaves S invariant. Simple examples of symmetric submanifolds in $\mathbb{R}^n$ are affine subspaces and spheres. It was proved by Ferus [27] that the symmetric submanifolds of Euclidean spaces are essentially given by the symmetric R-spaces. More precisely, Ferus proved: Let S be a symmetric submanifold of $\mathbb{R}^n$. Then there exist nonnegative integers $n_0, n_1, \ldots, n_k$ with $n = n_0 + \ldots + n_k$ and irreducible symmetric R-spaces $S_1 \subset \mathbb{R}^{n_1}, \ldots, S_k \subset \mathbb{R}^{n_k}$ such that S is isometric to $\mathbb{R}^{n_0} \times S_1 \times \ldots \times S_k$ and the embedding of S into $\mathbb{R}^n$ is the product embedding of $\mathbb{R}^{n_0} \times S_1 \times \ldots \times S_k$ into $\mathbb{R}^n = \mathbb{R}^{n_0} \times \ldots \times \mathbb{R}^{n_k}$.

The group of conformal transformations of the sphere S^n, the group of projective transformations of the projective space $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{O}P^2$, and the group of biholomorphic transformations of a Hermitian symmetric space of compact type provides an example of a transformation group that is larger than the isometry group of the space. A natural question is whether every symmetric space of compact type has such a larger transformation group. It was proved by Nagano [53] that just the symmetric R-spaces admit such larger transformation groups.

5.9. **Isotropy irreducible Riemannian homogeneous spaces.** A connected homogeneous space $M = G/K$ is called *isotropy irreducible* if G acts effectively on M, K is compact, and $\mathrm{Ad}_G|K$ acts irreducibly on $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$. It is called *strongly isotropy irreducible* if also $\mathrm{Ad}_G|K^o$ acts irreducibly on $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$. For the classification of strongly isotropy irreducible Riemannian homogeneous spaces see [70], and for isotropy irreducible Riemannian homogeneous spaces see [68].

Let $M = G/K$ be isotropy irreducible. Since K is compact, there exists an $\mathrm{Ad}_G(K)$-invariant inner product on $\mathfrak{m}$. Since $\mathrm{Ad}_G|K$ acts irreducibly on $\mathfrak{m}$, Schur's Lemma implies that this inner product $\langle\cdot,\cdot\rangle$ is unique up to a constant factor. This implies that there is a unique G-invariant Riemannian metric g on M up to homothety. Since the Ricci tensor ric_o at o is also an $\mathrm{Ad}_G(K)$-invariant symmetric bilinear form on $\mathfrak{m}$, Schur's Lemma implies that ric_o is a multiple of $\langle\cdot,\cdot\rangle$. This implies that (M,g) is an Einstein manifold.

One can prove that every noncompact isotropy irreducible Riemannian homogeneous space is a symmetric space (Besse [11]). Thus only the compact case is of interest in this context.

Let $M = G/K$ and G'/K' be simply connected, isotropy irreducible Riemannian homogeneous spaces. If these two spaces are isometric to each other, then one of the following statements holds:

1. There exists an isomorphism $\alpha : G \to G'$ with $\alpha(K) = K'$;
2. M is isometric to the Euclidean space $\mathbb{R}^n$, $n = \dim M$;
3. M is isometric to the 7-dimensional sphere S^7 with its standard metric and the two quotients are $SO(8)/SO(7)$ and $Spin(7)/G_2$;
4. M is isometric to the 6-dimensional sphere S^6 with its standard metric and the two quotients are $SO(7)/SO(6)$ and $G_2/SU(3)$.

This result is a consequence of a classification by Onishchik [54] of closed subgroups G' of G which are still transitive on G/K.

5.10. **The isometry group of an isotropy irreducible Riemannian homogeneous space.** Let M be a Riemannian homogeneous space. A fundamental problem is to determine the isometry group $I(M)$ of M and its identity component $I^o(M)$.

We first describe how one can reduce this problem to the simply connected case. Let $\pi : \widetilde{M} \to M$ be the Riemannian universal covering of M. Then $\widetilde{M}$ is a simply connected Riemannian homogeneous space and $M = \widetilde{M}/\Gamma$, where $\Gamma \subset I(\widetilde{M})$ is the group of deck transformations of the covering. Let $p,q \in M$ and $\tilde{p},\tilde{q} \in \widetilde{M}$ with $\pi(\tilde{p}) = p$ and $\pi(\tilde{q}) = q$. Since $\widetilde{M}$ is simply connected, there exists for each isometry $f \in I(M)$ a unique isometry $\tilde{f} \in I(\widetilde{M})$ with $\tilde{f}(\tilde{p}) = \tilde{q}$ and $\pi \circ \tilde{f} = f \circ \pi$. Clearly, $\tilde{f}$ maps fibers of π to fibers of π. Conversely, any $\tilde{f} \in I(\widetilde{M})$ which maps fibers of π to fibers of π induces a unique isometry $f \in I(M)$ with $\pi \circ \tilde{f} = f \circ \pi$. Obviously, $\tilde{f} \in I(\widetilde{M})$ preserves the fibers of π if and only if $\tilde{f}\Gamma = \Gamma\tilde{f}$, that is, if $\tilde{f}$ is in the normalizer N_Γ of Γ in $I(\widetilde{M})$. It follows that

$$I(M) = N_\Gamma/\Gamma\ .$$

For simply connected symmetric spaces the isometry group has been calculated by E. Cartan. His idea also works for calculating the isometry group in the more general situation of isotropy irreducible Riemannian homogeneous spaces. Let $M = G/K$ be a non-Euclidean simply connected isotropy irreducible Riemannian homogeneous space, where G is connected and acts effectively on M. We also assume that G/K is different from $Spin(7)/G_2$ and $G_2/SU(3)$ (see Section 5.9). Then

$$I^o(M) = G\ .$$

In order to describe the full isometry group $I(M)$ we denote by $\mathrm{Aut}(K)^G$ the group of automorphisms of K that can be extended to automorphisms of G, and by $\mathrm{Inn}(K)^G$ the subgroup of all inner automorphisms in $\mathrm{Aut}(K)^G$, which is a normal subgroup of $\mathrm{Aut}(K)^G$ of finite index r. Hence we can write

$$\mathrm{Aut}(K)^G = \bigcup_{i=1}^{r} k_i \mathrm{Inn}(K)^G\ .$$

We define $\bar{G} = G \cup sG$ and $\bar{K} = K \cup sK$ if $\mathrm{rk}(G) > \mathrm{rk}(K)$ and G/K is a symmetric space with symmetry s, and $\bar{G} = G$ and $\bar{K} = K$ otherwise. Recall that $\mathrm{rk}(G) > \mathrm{rk}(K)$ means that the Euler characteristic of $M = G/K$ is positive. In this case the symmetric space M is often called an *outer symmetric space*. Then the full isometry group $I(M)$ of M is given by

$$I(M) = \bigcup_{i=1}^{r} k_i \bar{G}\ ,$$

and $\cup_{i=1}^{r} k_i \bar{K}$ is the isotropy subgroup at $o = eK$.

5.11. **Ricci-flat Riemannian homogeneous spaces.** Every Riemannian homogeneous space with vanishing Ricci curvature is isometric to the Riemannian product $\mathbb{R}^k \times T^{n-k}$ of the Euclidean space $\mathbb{R}^k$ and a flat torus $T^{n-k} = \mathbb{R}^{n-k}/\Gamma$, where Γ is a lattice in $\mathbb{R}^{n-k}$ (Alekseevsky and Kimelfeld [5]).

Proof. The Cheeger-Gromoll Splitting Theorem states that every connected complete Riemannian manifold with nonnegative Ricci curvature is the Riemannian product of a Euclidean space and a connected complete Riemannian manifold with nonnegative Ricci curvature that does not contain a line (Cheeger and Gromoll [22]). A line is a geodesic that minimizes the distance between any two points on it. Let M be a Riemannian homogeneous space with vanishing Ricci curvature. It follows from the Cheeger-Gromoll Splitting Theorem that the Riemannian universal covering $\widetilde{M}$ of M is isometric to the Riemannian product $\mathbb{R}^n \times N$ with some simply connected Riemannian homogeneous space N with vanishing Ricci curvature. Note that N must be compact since every noncompact Riemannian homogeneous space contains a line. Since N is compact with $\mathrm{ric} = 0$, a result by Bochner [12] implies that the dimension of the isometry group of M is equal to the first Betti number $b_1(N,\mathbb{R})$. But since N is simply connected, $b_1(N,\mathbb{R}) = 0$. Since N is homogeneous this implies that N is a point and hence $\widetilde{M}$ is isometric to the flat Euclidean space $\mathbb{R}^n$. This implies that M is isometric to $\mathbb{R}^k \times T^{n-k}$.

5.12. **Homogeneous quaternionic Kähler manifolds.** We already mentioned the classification of homogeneous Kähler manifolds in the context of our discussion about complex flag manifolds at the end of Section 5.7. We now want to discuss briefly homogeneous quaternionic Kähler manifolds.

It was proved by Alekseevsky [3] that every homogeneous quaternionic Kähler manifold $M = G/K$ of a reductive Lie group G is a symmetric space. Recall that if G is compact then G is reductive. Thus every compact homogeneous quaternionic Kähler manifold is symmetric. The compact symmetric quaternionic Kähler manifolds are the Grassmannians $\mathbb{H}P^n$, $G_2(\mathbb{C}^{n+2})$ and $G_4^+(\mathbb{R}^{n+4})$, and the exceptional symmetric spaces $G_2/SO(4)$, $F_4/Sp(3)Sp(1)$, $E_6/SU(6)Sp(1)$, $E_7/Spin(12)Sp(1)$ and $E_8/E_7Sp(1)$. It is worthwhile to point out that for each compact simple real Lie group G there exists exactly one symmetric space G/K which is quaternionic Kähler. It is still an open problem whether there exist compact quaternionic Kähler manifolds with positive scalar curvature that are different from symmetric spaces.

Alekseevsky [4] constructed many explicit examples of noncompact homogeneous quaternionic Kähler manifolds which are not symmetric. His result, together with a correction by Cortés [25], yields the classification of all noncompact homogeneous quaternionic Kähler manifolds with a transitive solvable group of isometries.

6. Cohomogeneity one actions

Actions of cohomogeneity one are of current interest in the context of various topics: Einstein manifolds, manifolds with special holonomy, manifolds admitting metrics of positive sectional curvature, etcetera. The reason is that by a cohomogeneity one action one can sometimes reduce a system of partial differential equations to an ordinary differential equation. In this section we present some of the basics about cohomogeneity one actions and discuss some classification results.

6.1. **Isometric actions on Riemannian manifolds.** Let M be a Riemannian manifold and G a Lie group acting smoothly on M by isometries. Then we have a Lie group homomorphism

$$\rho : G \to I(M)$$

and a smooth map

$$G \times M \to M\ ,\ (g,p) \mapsto \rho(g)(p) = gp$$

satisfying

$$(gg')p = g(g'p)$$

for all $g, g' \in G$ and $p \in M$. An isometric action of a Lie group G' on a Riemannian manifold M' is said to be *isomorphic* to the action of G on M if there exists a Lie group isomorphism $\psi : G \to G'$ and an isometry $f : M \to M'$ so that $f(gp) = \psi(g)f(p)$ for all $p \in M$ and $g \in G$. For each point $p \in M$ the *orbit* of the action of G through p is

$$G \cdot p := \{gp \mid g \in G\}\ ,$$

and the *isotropy group* at p is

$$G_p := \{g \in G \mid gp = p\} \ .$$

If $G \cdot p = M$ for some $p \in M$, and hence for any $p \in M$, then the action of G is said to be transitive and M is a homogeneous G-space. This has been the topic in the previous sections. Therefore we assume from now on that the action of G is not transitive. Each orbit $G \cdot p$ is a submanifold of M, but in general not an embedded one. For instance, consider the flat torus T^2 obtained from $\mathbb{R}^2$ by factoring out the integer lattice. For each $\omega \in \mathbb{R}_+$ the Lie group $\mathbb{R}$ acts on T^2 isometrically by

$$\mathbb{R} \times T^2 \to T^2 \ , \ (t, [x, y]) \mapsto [x + t, y + \omega t] \ ,$$

where $[x, y]$ denotes the image of $(x, y) \in \mathbb{R}^2$ under the canonical projection $\mathbb{R}^2 \to T^2$. If ω is an irrational number then each orbit of this action is dense in T^2 and hence not an embedded submanifold. Each orbit $G \cdot p$ inherits a Riemannian structure from the ambient space M. With respect to this structure $G \cdot p$ is a Riemannian homogeneous space $G \cdot p = G/G_p$ on which G acts transitively by isometries.

6.2. **The set of orbits.** We denote by M/G the set of orbits of the action of G on M and equip M/G with the quotient topology relative to the canonical projection $M \to M/G \ , \ p \mapsto G \cdot p$. In general M/G is not a Hausdorff space. For instance, when ω is an irrational number in the previous example, then $T^2/\mathbb{R}$ is not a Hausdorff space. This unpleasant behaviour does not occur for so-called proper actions. The action of G on M is called *proper* if for any two distinct points $p, q \in M$ there exist open neighbourhoods U_p and U_q of p and q in M, respectively, so that $\{g \in G \mid gU_p \cap U_q \neq \emptyset\}$ is relative compact in G. This is equivalent to saying that the map

$$G \times M \to M \times M \ , \ (g, p) \mapsto (p, gp)$$

is a proper map, that is, the inverse image of each compact set in $M \times M$ is also compact in $G \times M$. Every action of a compact Lie group is proper, and the action of any closed subgroup of the isometry group of M is proper. If G acts properly on M, then M/G is a Hausdorff space, each orbit $G \cdot p$ is closed in M and hence an embedded submanifold, and each isotropy group G_p is compact.

6.3. **Slices.** A fundamental feature of proper actions is the existence of slices. A submanifold Σ of M is called a *slice* at $p \in M$ if

(Σ_1) $p \in \Sigma$,
(Σ_2) $G \cdot \Sigma := \{gq \mid g \in G \ , \ q \in \Sigma\}$ is an open subset of M,
(Σ_3) $G_p \cdot \Sigma = \Sigma$,
(Σ_4) the action of G_p on Σ is isomorphic to an orthogonal linear action of G_p on an open ball in a Euclidean space,
(Σ_5) the map

$$(G \times \Sigma)/G_p \to M \ , \ G_p \cdot (g, q) \mapsto gq$$

is a diffeomorphism onto $G \cdot \Sigma$, where $(G \times \Sigma)/G_p$ is the space of orbits of the action of G_p on $G \times \Sigma$ given by $k(g, q) := (gk^{-1}, kq)$ for all $k \in G_p$, $g \in G$ and $q \in \Sigma$.

Note that $(G \times \Sigma)/G_p$ is the fiber bundle associated with the principal fiber bundle $G \mapsto G/G_p$ and fiber Σ and hence a smooth manifold.

It was proved by Palais [57] that every proper action admits a slice at each point. One should note that a slice Σ enables us to reduce the study of the action of G on M in some G-invariant open neighborhood of p to the action of G_p on the slice Σ.

6.4. **Orbit types and the cohomogeneity of an action.** The existence of a slice at each point enables us also to define a partial ordering on the set of orbit types. We say that two orbits $G \cdot p$ and $G \cdot q$ have the same orbit type if G_p and G_q are conjugate in G. This defines an equivalence relation among the orbits of G. We denote by $[G \cdot p]$ the corresponding equivalence class, which is called the *orbit type* of $G \cdot p$. By $\mathfrak{O}$ we denote the set of all orbit types of the action of G on M. We then introduce a partial ordering on $\mathfrak{O}$ by saying that $[G \cdot p] \leq [G \cdot q]$ if and only if G_q is conjugate in G to some subgroup of G_p. If Σ is a slice at p, then properties (Σ_4) and (Σ_5) imply that $[G \cdot p] \leq [G \cdot q]$ for all $q \in G \cdot \Sigma$. We assume that M/G is connected. Then there exists a largest orbit type in $\mathfrak{O}$. Each representative of this largest orbit type is called a *principal orbit*. In other words, an orbit $G \cdot p$ is principal if and only if for each $q \in M$ the isotropy group G_p at p is conjugate in G to some subgroup of G_q. The union of all principal orbits is a dense and open subset of M. Each principal orbit is an orbit of maximal dimension. A non-principal orbit with the same dimension as a principal orbit is called an *exceptional orbit*. An orbit whose dimension is less than the dimension of a *principal orbit* is called a *singular orbit*. The *cohomogeneity* of the action is the codimension of a principal orbit. We denote this cohomogeneity by $\mathrm{cohom}(G, M)$.

6.5. **Isotropy representation and slice representation of an action.** We assume from now on that the action of G on M is proper and that M/G is connected. Recall that for each $g \in G$ the map

$$\varphi_g : M \to M \ , \ p \mapsto gp$$

is an isometry of M. If $p \in M$ and $g \in G_p$, then φ_g fixes p. Therefore, at each point $p \in M$, the isotropy group G_p acts on T_pM by

$$G_p \times T_pM \to T_pM \ , \ (g, X) \mapsto g \cdot X := (\varphi_g)_{*p}X \ .$$

But since $g \in G_p$ leaves $G \cdot p$ invariant, this action leaves the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$ of $G \cdot p$ at p invariant. The restriction

$$\chi_p : G_p \times T_p(G \cdot p) \to T_p(G \cdot p) \ , \ (g, X) \mapsto g \cdot X$$

is called the *isotropy representation* of the action at p, and the restriction

$$\sigma_p : G_p \times \nu_p(G \cdot p) \to \nu_p(G \cdot p) \ , \ (g, \xi) \mapsto g \cdot \xi$$

is called the *slice representation* of the action at p. If $(G_p)^o$ is the connected component of the identity in G_p, the restriction of the slice representation to $(G_p)^o$ will be called *connected slice representation.*

6.6. **Geodesic slices.** Let $p \in M$ and $r \in \mathbb{R}_+$ sufficiently small so that the restriction of the exponential map $\exp_p$ of M at p to $U_r(0) \subset \nu_p(G \cdot p)$ is an embedding of $U_r(0)$ into M. Then $\Sigma = \exp_p(U_r(0))$ is a slice at p, a so-called *geodesic slice*. Geometrically, the geodesic slice Σ is obtained by running along all geodesics emanating orthogonally from $G \cdot p$ at p up to the distance r. Since isometries map geodesics to geodesics we see that

$$g\Sigma = \exp_{gp}(g \cdot U_r(0))$$

for all $g \in G$. Thus, $G \cdot \Sigma$ is obtained by sliding Σ along the orbit $G \cdot p$ by means of the group action. Let $q \in \Sigma$ and $g \in G_q$. Then $gq \in \Sigma$ and hence $g\Sigma = \Sigma$. Since $\Sigma \cap G \cdot p = \{p\}$ it follows that $gp = p$ and hence $g \in G_p$. Thus we have proved: If Σ is a geodesic slice at p, then $G_q \subset G_p$ for all $q \in \Sigma$.

Let Σ be a geodesic slice at p. Then $G \cdot \Sigma$ is an open subset of M. As the principal orbits form an open and dense subset of M, the previous result therefore implies that $G \cdot p$ is a principal orbit if and only if $G_q = G_p$ for all $q \in \Sigma$. On the other hand, each $g \in G_q$ fixes both q and p and therefore, assuming the geodesic slice is sufficiently small, the entire geodesic in Σ connecting p and q. Thus G_q fixes pointwise the one-dimensional linear subspace of $\nu_p(G \cdot p)$ corresponding to this geodesic. This implies the following useful characterization of principal orbits: An orbit $G \cdot p$ is principal if and only if the slice representation Σ_p is trivial.

6.7. **Polar and hyperpolar actions.** Let M be a complete Riemannian manifold and let G be a closed subgroup of $I(M)$. A complete embedded closed submanifold Σ of M is called a *section* of the action if Σ intersects each orbit of G in M such that $T_p\Sigma \subset \nu_p(G \cdot p)$ for all $p \in \Sigma$. The action of G is called *polar* if it admits a section. One can prove that every section of a polar action is totally geodesic. A polar action is called *hyperpolar* if it admits a flat section. The hyperpolar actions on simply connected irreducible Riemannian symmetric spaces of compact type have been classified by Kollross [41].

Examples. 1. The isotropy representation of a Riemannian symmetric space is hyperpolar, and a section is given by a maximal Abelian subspace.

2. Let $M = G/K$ and $M' = G/K'$ be Riemannian symmetric spaces of the same semisimple compact Lie group G. Then the action of K' on M is hyperpolar. Such an action often referred to as an *Hermann action* on a symmetric space. Hermann [32] proved that such actions are variationally complete in the sense of Bott and Samelson [16].

6.8. **The orbit space of a cohomogeneity one action.** Let M be a connected complete Riemannian manifold and G a connected closed subgroup of the isometry group $I(M)$ of M acting on M with cohomogeneity one. We denote by M/G the space of orbits of this action and by $\pi : M \to M/G$ the canonical projection that maps a point $p \in M$ to the orbit $G \cdot p$ through p. We equip M/G with the quotient topology relative to π. The following result has been proved by Mostert [51] (for the compact case) and by Bérard Bergery [7] (for the general case): The orbit space M/G is homeomorphic to $\mathbb{R}$, S^1, $[0,1]$, or $[0,\infty[$.

This result implies that a cohomogeneity one action has at most two singular or exceptional orbits, corresponding to the boundary points of M/G. If there is a singular orbit, each principal orbit is geometrically a tube about the singular orbit. If M/G is homeomorphic to $\mathbb{R}$ or S^1, then each orbit is principal and the orbits of the action of G on M form a Riemannian foliation on M. Moreover, since principal orbits are always homeomorphic to each other, the projection $\pi : M \to M/G$ is a fiber bundle. If, in addition, M is simply connected, then M/G cannot be homeomorphic to S^1. This follows from the relevant part of the exact homotopy sequence of a fiber bundle with connected fibers and base space S^1. Every cohomogeneity one action is hyperpolar and a geodesic intersecting an orbit perpendicularly is a section.

Examples: 1. Let G be the group of translations generated by a line in $\mathbb{R}^2$. Then $\mathbb{R}^2/G = \mathbb{R}$.

2. Consider a round cylinder Z in $\mathbb{R}^3$ and let G be the group of translations on Z along its axis. Then $Z/G = S^1$.

3. Let $G = SO(2)$ be the group of rotations around the origin in $\mathbb{R}^2$. Then $\mathbb{R}^2/G = [0, \infty)$.

4. Let $G = SO(2)$ be the isotropy group of the action of $SO(3)$ on the 2-sphere S^2. Then $S^2/G = [0, 1]$.

6.9. **Low-dimensional orbits of cohomogeneity one actions.** The following result shows that "low-dimensional" orbits of isometric cohomogeneity one actions must be totally geodesic. Let G be a connected closed subgroup of the isometry group $I(M)$ of a Riemannian manifold M and $p \in M$. If

$$\dim(G \cdot p) < \frac{1}{2}(\dim M - 1)$$

then $G \cdot p$ is totally geodesic in M.

Proof. We denote again by G_p the isotropy group at p and by $\exp_p$ the exponential map of M at p. First suppose there exists an open neighborhood U of 0 in $T_p(G \cdot p)$ so that $\exp_p(U) \subset G \cdot p$. Then $G \cdot p$ is totally geodesic at p and hence, by homogeneity, totally geodesic at each point. Now we suppose that for each open neighborhood U of 0 in $T_p(G \cdot p)$ the image $\exp_p(U)$ is not contained in $G \cdot p$. We fix an open convex neighborhood C of p in M. Then there exists a point $q \in (G \cdot p) \cap C$ so that the unique geodesic segment $\gamma : [0, a] \to M$ in C connecting $p = \gamma(0)$ and $q = \gamma(a)$, $a = d(p, q)$, is not tangent to $G \cdot p$ at p. Thus there exists a nonzero normal vector $\xi \in \nu_p(G \cdot p)$ and a vector $X \in T_p(G \cdot p)$ so that $\dot\gamma(0) = X + \xi$. For each $g \in G_p$ the curve $\gamma_g = g \circ \gamma$ is the geodesic segment in C with $\gamma_g(0) = p$ and $\gamma_g(a) = g(\gamma(a)) = g(q) \in G \cdot p$. The initial value of γ_g is $\dot\gamma_g(0) = g_* X + g_* \xi$, and since G_p acts on $\nu_p(G \cdot p)$ with cohomogeneity one it follows that

$$\dim(G \cdot p) \geq \dim(G_p \cdot q) \geq \dim \nu_p(G \cdot p) - 1 \ .$$

This implies

$$\dim M = \dim T_p(G \cdot p) + \dim \nu_p(G \cdot p) \leq 1 + 2\dim(G \cdot p) < \dim M \ ,$$

which is a contradiction. Thus $G \cdot p$ is totally geodesic.

6.10. **Minimality of singular and exceptional orbits of cohomogeneity one actions.** We now show that a singular or exceptional orbit of a cohomogeneity one action is minimal. Let F be a singular or exceptional orbit of a cohomogeneity one action on a Riemannian manifold M. Then F is a minimal submanifold of M.

Proof. Let G be a connected closed subgroup of $I(M)$ acting on M isometrically with cohomogeneity one and suppose there exists a singular or exceptional orbit F. Let α be the second fundamental form of F and H the mean curvature vector of F at some point $p \in F$. We choose an orthonormal basis $e_1, \ldots, e_m$ of T_pF with $m = \dim F$. Since the action is of cohomogeneity one, the isotropy group G_p at p acts transitively on vectors of the same length in the normal space of F at p. Thus there exists some isometry $g \in G$ with $g_*H = -H$, and we get

$$mH = \sum_{i=1}^{m} \alpha(e_i, e_i) = \sum_{i=1}^{m} \alpha(g_*e_i, g_*e_i) = \sum_{i=1}^{m} g_*\alpha(e_i, e_i) = mg_*H = -mH \ .$$

This shows that $H = 0$ and hence F is minimal in M.

Recall from Section 6.9 that if $2 \dim F < \dim M - 1$ then F is not only minimal but also totally geodesic.

6.11. **Cohomogeneity one actions on spheres.** The cohomogeneity one actions on spheres have been classified by Hsiang and Lawson [34]: Every cohomogeneity one action on S^n is orbit equivalent to the isotropy representation of an $(n+1)$-dimensional Riemannian symmetric space of rank two.

In fact, Hsiang and Lawson derived an explicit list of groups acting with cohomogeneity one on spheres. It is just an observation that all the actions arising from their list correspond to symmetric spaces of rank two. It is still an open problem to prove this fact directly.

6.12. **Cohomogeneity one actions on complex projective spaces.** An interesting fact is that in complex projective spaces the theories of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures are different. In fact, Wang [69] showed that certain inhomogeneous isoparametric hypersurfaces in spheres project via the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ to isoparametric hypersurfaces in complex projective spaces with non-constant principal curvatures. It is still an open problem whether any hypersurface with constant principal curvatures in $\mathbb{C}P^n$ is isoparametric or homogeneous. The classification of homogeneous hypersurfaces in $\mathbb{C}P^n$ was achieved by Takagi [62]. It is easy to see that every homogeneous hypersurface in $\mathbb{C}P^n$ is the projection of a homogeneous hypersurface in S^{2n+1}. But not every homogeneous hypersurface in S^{2n+1} is invariant under the S^1-action and hence does not project to a homogeneous hypersurface in $\mathbb{C}P^n$. In fact, Takagi proved that those which do project are precisely those which arise from isotropy representations of *Hermitian* symmetric spaces of rank two. In detail, this gives the following classification: Every cohomogeneity one action on $\mathbb{C}P^n$ is orbit equivalent to the projectivized isotropy representation of an $(n+1)$-dimensional Hermitian symmetric space of rank two.

The corresponding Hermitian symmetric spaces of rank two are $\mathbb{C}P^{k+1} \times \mathbb{C}P^{n-k}$ for $k \in \{0, \dots, n-1\}$, $G_2^+(\mathbb{R}^{n+3})$, $G_2(\mathbb{C}^{k+3})$ for $k \geq 3$, $SO(10)/U(5)$, $E_6/T \cdot Spin(10)$. This result was improved by Uchida [65] who classified all connected closed subgroups of $SU(n+1)$ acting on $\mathbb{C}P^n$ with cohomogeneity one, that is, whose principal orbits have codimension one. Uchida's approach to the classification problem is completely different and uses cohomological methods. In fact, Uchida classified all connected compact Lie groups acting with an orbit of codimension one on a simply connected smooth manifold whose rational cohomology ring is isomorphic to the one of a complex projective space. This includes, for instance, all odd-dimensional complex quadrics (which are real Grassmannians) $G_2^+(\mathbb{R}^{2n+1}) = SO(2n+1)/SO(2) \times SO(2n-1)$.

6.13. **Cohomogeneity one actions on quaternionic projective spaces.** For quaternionic projective spaces $\mathbb{H}P^n$ Iwata [35] used a method analogous to the one of Uchida and classified all connected compact Lie groups acting with an orbit of codimension one on a simply connected smooth manifold whose rational cohomology ring is isomorphic to the one of a quaternionic projective space. For instance, the symmetric space $G_2/SO(4)$ has the same rational cohomology as the quaternionic projective plane $\mathbb{H}P^2$. For the special case of $\mathbb{H}P^n$ Iwata's classification yields: Every cohomogeneity one action on $\mathbb{H}P^n$ is orbit equivalent to the action of $Sp(k+1) \times Sp(n-k)$ for some $k \in \{0, \dots, n-1\}$ or to the action of $SU(n+1)$.

A different proof, following the lines of Takagi, has been given by D'Atri [26]. The two singular orbits of the action of $Sp(k+1) \times Sp(n-k)$ on $\mathbb{H}P^n$ are totally geodesic $\mathbb{H}P^k$ and $\mathbb{H}P^{n-k-1}$. The two singular orbits of the action of $SU(n+1)$ on $\mathbb{H}P^n$ are a totally geodesic $\mathbb{C}P^n$ and the homogeneous space $SU(n+1)/SU(2) \times SU(n-1)$, which is the standard circle bundle over the Hermitian symmetric space $G_2(\mathbb{C}^{n+1}) = SU(n+1)/S(U(2) \times U(n-1))$.

6.14. **Cohomogeneity one actions on the Cayley projective plane.** For the Cayley projective plane $\mathbb{O}P^2$ Iwata [36] could also apply his cohomological methods and obtained: Every cohomogeneity one action on $\mathbb{O}P^2$ is orbit equivalent to the action of $Spin(9)$ or of $Sp(3) \times SU(2)$.

The Lie group $Spin(9)$ is the isotropy group of the isometry group F_4 of $\mathbb{O}P^2$. The two singular orbits of the action of $Spin(9)$ are a point and the corresponding polar, which is a Cayley projective line $\mathbb{O}P^1 = S^8$. The Lie group $Sp(3) \times SU(2)$ is a maximal subgroup of maximal rank of F_4, and the singular orbits of its action on $\mathbb{O}P^2$ are a totally geodesic quaternionic projective plane $\mathbb{H}P^2$ and an 11-dimensional sphere $S^{11} = Sp(3)/Sp(2)$.

6.15. **Cohomogeneity one actions on Riemannian symmetric spaces of compact type.** The classification of cohomogeneity one actions on irreducible simply connected Riemannian symmetric spaces of compact type is part of the more general classification of hyperpolar actions (up to orbit equivalence) on these spaces due to Kollross [41]. We describe the idea for the classification by Kollross in the special case when the action is of cohomogeneity one and the symmetric space $M = G/K$ is of rank ≥ 2 and not of group type. Suppose H is a maximal closed subgroup of G. If H is not transitive on M, then its cohomogeneity is at least one. Since the cohomogeneity of the action of any closed subgroup of H is at least the cohomogeneity of the action of H, and we are interested only in classification up the orbit equivalence, it suffices to consider only maximal closed subgroups of G. But it may happen that H acts transitively on G/K. This happens precisely in four cases, where we write down $G/K = H/(H \cap K)$:

$$
\begin{aligned}
SO(2n)/U(n) &= SO(2n-1)/U(n-1) \ (n \geq 4) \ , \\
SU(2n)/Sp(n) &= SU(2n-1)/Sp(n-1) \ (n \geq 3) \ , \\
G_2^+(\mathbb{R}^7) = SO(7)/SO(2) \times SO(5) &= G_2/U(2) \ , \\
G_3^+(\mathbb{R}^8) = SO(8)/SO(3) \times SO(5) &= Spin(7)/SO(4) \ .
\end{aligned}
$$

In these cases one has to go one step further and consider maximal closed subgroups of H which then never happen to act also transitively. Thus it is sufficient to consider maximal closed subgroups of G, with the few exceptions just mentioned. In order that a closed subgroup H acts with cohomogeneity one it obviously must satisfy $\dim H \geq \dim M - 1$. This rules already out a lot of possibilities. For the remaining maximal closed subgroups one has to calculate case by case the cohomogeneity. One way to do this is to calculate the codimension of the slice representation, this is the action of the isotropy group $H \cap K$ on the normal space at the corresponding point of the orbit through that point. This procedure eventually leads to the classification of all cohomogeneity one actions up to orbit equivalence, and hence to the classification of homogeneous hypersurfaces, on $M = G/K$. It turns out that with five exceptions all homogeneous hypersurfaces arise via the construction of Hermann. The exceptions come from the following actions:

1. The action of $G_2 \subset SO(7)$ on $SO(7)/U(3) = SO(8)/U(4) = G_2^+(\mathbb{R}^8)$.
2. The action of $G_2 \subset SO(7)$ on $SO(7)/SO(3) \times SO(4) = G_3^+(\mathbb{R}^7)$.
3. The action of $Spin(9) \subset SO(16)$ on $SO(16)/SO(2) \times SO(14) = G_2^+(\mathbb{R}^{16})$.
4. The action of $Sp(n)Sp(1) \subset SO(4n)$ on $SO(4n)/SO(2) \times SO(4n-2) = G_2^+(\mathbb{R}^{4n})$.
5. The action of $SU(3) \subset G_2$ on $G_2/SO(4)$.

All other homogeneous hypersurfaces can be obtained via the construction of Hermann. We refer to [41] for an explicit list of all Hermann actions of cohomogeneity one.

6.16. **Cohomogeneity one actions on Hadamard manifolds.** We assume from now on that M is a Hadamard manifold, that is, a connected, simply connected, complete Riemannian manifold of nonpositive curvature. For any point $p \in M$ the isotropy group G_p of G at p is a closed subgroup of G. As $I(M)_p$ is a compact subgroup of $I(M)$ and G is a closed subgroup of $I(M)$ it follows that $G_p = G \cap I(M)_p$ is a compact subgroup of G. Let K be a maximal compact subgroup of G. Then K is also compact in $I(M)$, and it follows from Cartan's fixed point theorem that K has a fixed point in M, say q. As $K \subset G_q$ and G_q is compact in G it follows that $K = G_q$. Thus there exists an orbit of the action of G such that the isotropy group at any of its points is a maximal compact subgroup of G. If M/G is homeomorphic to $\mathbb{R}$ this means that G_p is a maximal compact subgroup of G for any $p \in M$. Otherwise, the orbit $F := G \cdot q$ is a singular orbit of the action of G with the property that G_p is a maximal compact subgroup of G for all $p \in F$. Suppose there exists another singular orbit F' and let $q' \in F'$. As $G_{q'}$ is a compact subgroup of G, there exists some $g \in G$ so that $G_{q'} \subset gKg^{-1} = gG_qg^{-1} = G_{g(q)}$. There exists a unique geodesic γ in M connecting q' and $g(q)$. Any isometry in $G_{q'}$ fixes γ pointwise, which implies that $G_{q'}$ is contained in G_p for all p on γ. But as γ intersects principal orbits this gives a contradiction. Thus the action of G cannot have two singular orbits. We conclude that M/G is homeomorphic to $\mathbb{R}$ or $[0,\infty[$.

Let F be the singular orbit in the case M/G is homeomorphic to $[0,\infty[$ or any principal orbit in the case M/G is homeomorphic to $\mathbb{R}$. Denote be k the dimension of F, by exp the exponential map of M, by νF the normal bundle of F, and by $\tau : \nu F \to F$ the canonical projection. For any point $q \in M$ there exists exactly one normal vector $\xi \in \nu F$ so that $q = \exp(\xi)$, which implies that the map $f := \exp|\nu F : \nu F \to M$ is bijective. Moreover, f is smooth and the differential of f at any point must be an isomorphism since F has no focal points in M. Therefore f is a diffeomorphism onto M. Then $\tau \circ f^{-1} : M \to F$ is a fiber bundle with typical fiber $\mathbb{R}^{n-k}$, since at each point $p \in F$ the fiber is the normal space ν_pF of F at p. As M is diffeomorphic to $\mathbb{R}^n$, it now follows that F is diffeomorphic to $\mathbb{R}^k$. If $\dim F = n-1$ and F is a singular orbit, then $\tau \circ f^{-1}$ restricted to a principal orbit is a nontrivial covering map, which contradicts $\pi_1(F) = 0$. Thus we have $k = n-1$ if and only if M/G is homeomorphic to $\mathbb{R}$. If F is singular, then $\tau \circ f^{-1}$ restricted to a principal orbit is a fiber bundle with typical fiber S^{n-k-1}. As F is contractible it follows that any regular orbit is diffeomorphic to $\mathbb{R}^k \times S^{n-k-1}$. Eventually, as F is diffeomorphic to $\mathbb{R}^k$, there exists a solvable connected closed subgroup of G acting simply transitively on F. In the case $\dim F = n-1$ this group then acts simply transitively on any orbit of G. We summarize the previous discussion:

Let G be a connected closed subgroup of the isometry group of an n-dimensional Hadamard manifold M acting on M with cohomogeneity one. Then one of the following two possibilities holds:

(1) All orbits are principal and the isotropy group at any point is a maximal compact subgroup of G. Any orbit is diffeomorphic to $\mathbb{R}^{n-1}$ and there exists a solvable connected closed subgroup of G acting simply transitively on each orbit.

(2) There exists exactly one singular orbit F and the isotropy group at any point in F is a maximal compact subgroup of G. The singular orbit is diffeomorphic to $\mathbb{R}^k$ for some $k \in \{0,\dots,n-2\}$ and there exists a solvable connected closed subgroup of G acting simply transitively on F. Any principal orbit is diffeomorphic to $\mathbb{R}^k \times S^{n-k-1}$.

6.17. **Cohomogeneity one actions on Euclidean spaces.** Every principal orbit of a cohomogeneity one action has constant principal curvatures. The hypersurfaces with constant principal curvatures in $\mathbb{R}^n$ have been classified by Levi Civita [45] for $n = 3$ and by Segre [61] in general. This leads to the following classification (recall that the isometry group $I(\mathbb{R}^n)$ is the semidirect product $\mathbb{R}^n \times_\tau O(n)$): Every cohomogeneity one action on $\mathbb{R}^n$ is orbit equivalent to one of the following cohomogeneity one actions:

1. the action of $\{0\} \times_\tau SO(n)$: the singular orbit is a point and the principal orbits are spheres;
2. the action of $\mathbb{R}^k \times_\tau SO(n-k)$ for some $k \in \{1,\dots,n-2\}$: the singular orbit is a totally geodesic $\mathbb{R}^k \subset \mathbb{R}^n$ and the principal orbits are the tubes around it;

(3) the action of $\mathbb{R}^{n-1} \times_\tau \{e_{SO(n)}\}$; all orbits are principal and totally geodesic hyperplanes.

6.18. **Cohomogeneity one actions on real hyperbolic spaces.** Every principal orbit of a cohomogeneity one action has constant principal curvatures. The hypersurfaces with constant principal curvatures in a real hyperbolic space $\mathbb{R}H^n$ have been classified by E. Cartan [19]. His result leads to the following classification: Every cohomogeneity one action on $\mathbb{R}H^n$ is orbit equivalent to one of the following cohomogeneity one actions:

(1) the action of $SO^o(1,n)$: the singular orbit is a point and the principal orbits are geodesic hyperspheres;
(2) the action of $SO^o(1,k) \times SO(n-k)$ for some $k \in \{1,\ldots,n-2\}$: the singular orbit is a totally geodesic $\mathbb{R}H^k \subset \mathbb{R}H^n$ and the principal orbits are the tubes around it;
(3) the action of $SO^o(1,n-1)$; all orbits are principal, one orbit is a totally geodesic hyperplane $\mathbb{R}H^{n-1}$, and the other orbits are the equidistant hypersurfaces to it;
(4) the action of the nilpotent subgroup in an Iwasawa decomposition of $SO^o(1,n)$: all orbits are principal and are horospheres in $\mathbb{R}H^n$. The resulting foliation is the well-known horosphere foliation on $\mathbb{R}H^n$.

The method of Cartan does not work for the hyperbolic spaces $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$. The reason is that the Gauss-Codazzi equations become too complicated.

6.19. **Cohomogeneity one actions on Riemannian symmetric spaces of noncompact type.** Let $M = G/K$ be a Riemannian symmetric space of noncompact type with $G = I^o(M)$ and K the isotropy group of G at a point $o \in M$. Let n be the dimension of M and r the rank of M. We denote by $\mathfrak{M}$ the moduli space of all isometric cohomogeneity one actions on M modulo orbit equivalence.

The orbit space of an isometric cohomogeneity one action on a connected complete Riemannian manifold is homeomorphic to $\mathbb{R}$ or $[0,\infty)$. Geometrically this means that either all orbits are principal and form a Riemannian foliation on M or there exists exactly one singular orbit with codimension ≥ 2 and the principal orbits are tubes around the singular orbit. This induces a disjoint union $\mathfrak{M} = \mathfrak{M}_F \cup \mathfrak{M}_S$, where $\mathfrak{M}_F$ is the set of all homogeneous codimension one foliations on M modulo isometric congruence and $\mathfrak{M}_S$ is the set of all connected normal homogeneous submanifolds with codimension ≥ 2 in M modulo isometric congruence. Here a submanifold of M is called *normal homogeneous* if it is an orbit of a connected closed subgroup of $I^o(M)$ and the slice representation at a point acts transitively on the unit sphere in the normal space at that point.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of G and K, respectively, and B the Cartan-Killing form of $\mathfrak{g}$. If $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a Cartan decomposition of $\mathfrak{g}$. If $\theta : \mathfrak{g} \to \mathfrak{g}$ is the corresponding Cartan involution, we get a positive definite inner product on $\mathfrak{g}$ by $\langle X, Y\rangle = -B(X, \theta Y)$ for all $X, Y \in \mathfrak{g}$. We normalize the Riemannian metric on M such that its restriction to $T_oM \times T_oM$ coincides with $\langle\cdot,\cdot\rangle$, where we identify $\mathfrak{m}$ and T_oM in the usual manner.

Let $\mathfrak{a}$ be a maximal Abelian subspace in $\mathfrak{m}$ and denote by $\mathfrak{a}^*$ the dual vector space of $\mathfrak{a}$. Moreover, let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$$

be the restricted root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. The root system Σ is either reduced and then of type $A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2$ or nonreduced and then of type BC_r. For each $\lambda \in \mathfrak{a}^*$ let $H_\lambda \in \mathfrak{a}$ be the dual vector in $\mathfrak{a}$ with respect to the Cartan-Killing form, that is, $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{a}$. Then we get an inner product on $\mathfrak{a}^*$, which we also denote by $\langle \cdot, \cdot \rangle$, by means of $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$ for all $\lambda, \mu \in \mathfrak{a}^*$. We choose a set $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots in Σ and denote the resulting set of positive restricted roots by Σ^+.

By $\mathrm{Aut}(DD)$ we denote the group of symmetries of the Dynkin diagram associated to Λ. There are just three possibilities, namely

$$\mathrm{Aut}(DD) = \begin{cases} \mathfrak{S}_3 & , \text{if } \Sigma = D_4 \ , \\ \mathbb{Z}_2 & , \text{if } \Sigma \in \{A_r \ (r \geq 2),\ D_r \ (r \geq 2, r \neq 4),\ E_6\} \ , \\ \mathrm{id} & , \text{otherwise} \ . \end{cases}$$

where $\mathfrak{S}_3$ is the group of permutations of a set of three elements. The first two cases correspond to triality and duality principles on the symmetric space which were discovered by E. Cartan. The symmetric spaces with a triality principle are $SO(8, \mathbb{C})/SO(8)$ and the hyperbolic Grassmannian $G_4^*(\mathbb{R}^8)$. Each symmetry $P \in \mathrm{Aut}(DD)$ can be linearly extended to a linear isometry of $\mathfrak{a}^*$, which we also denote by P. Denote by Φ the linear isometry from $\mathfrak{a}^*$ to $\mathfrak{a}$ defined by $\Phi(\lambda) = H_\lambda$ for all $\lambda \in \mathfrak{a}^*$. Then $\widetilde{P} = \Phi \circ P \circ \Phi^{-1}$ is a linear isometry of $\mathfrak{a}$ with $\widetilde{P}(H_\lambda) = H_\mu$ if and only if $P(\lambda) = \mu$, $\lambda, \mu \in \mathfrak{a}^*$. Since P is an orthogonal transformation, $\widetilde{P}$ is just the dual map of $P^{-1} : \mathfrak{a}^* \to \mathfrak{a}^*$. In this way each symmetry $P \in \mathrm{Aut}(DD)$ induces linear isometries of $\mathfrak{a}^*$ and $\mathfrak{a}$, both of which we will denote by P, since it will always be clear from the context which of these two we are using.

We now define a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ by

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda \ ,$$

which then induces an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$. Then $\mathfrak{a} + \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$ with $[\mathfrak{a} + \mathfrak{n}, \mathfrak{a} + \mathfrak{n}] = \mathfrak{n}$. The connected subgroups A, N, AN of G with Lie algebras $\mathfrak{a}, \mathfrak{n}, \mathfrak{a} + \mathfrak{n}$, respectively, are simply connected and AN acts simply transitively on M. The symmetric space M is isometric to the connected, simply connected, solvable Lie group AN equipped with the left-invariant Riemannian metric which is induced from the inner product $\langle \cdot, \cdot \rangle$.

Let ℓ be a linear line in $\mathfrak{a}$. Since ℓ lies in the orthogonal complement of the derived subalgebra of $\mathfrak{a} + \mathfrak{n}$, the orthogonal complement $\mathfrak{s}_\ell = (\mathfrak{a} + \mathfrak{n}) \ominus \ell$ of ℓ in $\mathfrak{a} + \mathfrak{n}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$ of codimension one. Let S_ℓ be the connected Lie subgroup of AN with Lie algebra $\mathfrak{s}_\ell$. Then the orbits of the action of S_ℓ on M form a Riemannian foliation $\mathfrak{F}_\ell$ on M whose leaves are homogeneous hypersurfaces. If M has rank one then $\mathfrak{a}$ is one-dimensional and hence there exists only one such foliation, namely the one given by $S_\ell = S_{\mathfrak{a}} = N$. This is precisely the horosphere foliation on M all of whose leaves are isometrically congruent to each other. One can show that also for higher rank all leaves of $\mathfrak{F}_\ell$ are isometrically congruent to each other. Using structure theory of semisimple and solvable Lie algebras one can show that two foliations $\mathfrak{F}_\ell$ and $\mathfrak{F}_{\ell'}$ are isometrically congruent to each other if and only if there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\ell) = \ell'$. It follows that the set of all congruence classes of such foliations is parametrized my $\mathbb{R}P^{r-1}/\mathrm{Aut}(DD)$. Here, $\mathbb{R}P^{r-1}$ is the projective space of all linear lines ℓ in $\mathfrak{a}$, and the action of $\mathrm{Aut}(DD)$ on $\mathbb{R}P^{r-1}$ is the induced one from the linear action of $\mathrm{Aut}(DD)$ on $\mathfrak{a}$.

Let $\alpha_i \in \Lambda$, $i \in \{1,\dots,r\}$, be a simple root. For each unit vector $\xi \in \mathfrak{g}_{\alpha_i}$ the subspace $\mathfrak{s}_\xi = \mathfrak{a} + (\mathfrak{n} \ominus \mathbb{R}\xi)$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. Let S_ξ be the connected Lie subgroup of AN with Lie algebra $\mathfrak{s}_\xi$. Then the orbits of the action of S_ξ on M form a Riemannian foliation $\mathfrak{F}_\xi$ on M whose leaves are homogeneous hypersurfaces. If $\eta \in \mathfrak{g}_{\alpha_i}$ is another unit vector the induced foliation $\mathfrak{F}_\eta$ is congruent to $\mathfrak{F}_\xi$ under an isometry in the centralizer of $\mathfrak{a}$ in K. Thus for each simple root $\alpha_i \in \Lambda$ we obtain a congruence class of homogeneous foliations of codimension one on M. We denote by $\mathfrak{F}_i$ a representative of this congruence class. By investigating the geometry of these foliations one can prove that $\mathfrak{F}_i$ and $\mathfrak{F}_j$ are isometrically congruent if and only if there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\alpha_i) = \alpha_j$. Thus the set of all congruence classes of such foliations is parametrized by $\{1,\dots,r\}/\mathrm{Aut}(DD)$, where the action of $\mathrm{Aut}(DD)$ on $\{1,\dots,r\}$ is given by identifying $\{1,\dots,r\}$ with the vertices of the Dynkin diagram. The geometry of these foliations is quite fascinating. Among all leaves there exists exactly one which is minimal. All leaves together form a *homogeneous* isoparametric system on M, and if the rank of M is ≥ 3 there exist among these systems some which are noncongruent but have the same principal curvatures with the same multiplicities. Such a feature had already been discovered by Ferus, Karcher and Münzner [28] for *inhomogeneous* isoparametric systems on spheres.

Using structure theory of semisimple and solvable Lie algebras it was proved by the author and Tamaru in [9] that every homogeneous codimension one foliation on M is isometrically congruent to one of the above: Let M be a connected irreducible Riemannian symmetric space of noncompact type and with rank r. The moduli space $\mathfrak{M}_F$ of all noncongruent homogeneous codimension one foliations on M is isomorphic to the orbit space of the action of $\mathrm{Aut}(DD)$ on $\mathbb{R}P^{r-1} \cup \{1,\dots,r\}$:

$$\mathfrak{M}_F \cong (\mathbb{R}P^{r-1} \cup \{1,\dots,r\})/\mathrm{Aut}(DD)\ .$$

It is very surprising and remarkable that $\mathfrak{M}_F$ depends only on the rank and on possible duality or triality principles on the symmetric space. For instance, for the symmetric spaces $SO(17,\mathbb{C})/SO(17)$, $Sp(8,\mathbb{R})/U(8)$, $Sp(8,\mathbb{C})/Sp(8)$, $SO(16,\mathbb{H})/U(16)$, $SO(17,\mathbb{H})/U(17)$, $E_8^8/SO(16)$, $E_8^{\mathbb{C}}/E_8$ and for the hyperbolic Grassmannians $G_8^*(\mathbb{R}^{n+16})$ ($n \geq 1$), $G_8^*(\mathbb{C}^{n+16})$ ($n \geq 0$), $G_8^*(\mathbb{H}^{n+16})$ ($n \geq 0$) the moduli space $\mathfrak{M}_F$ of all noncongruent homogeneous codimension one foliations is isomorphic to $\mathbb{R}P^7 \cup \{1,\dots,8\}$.

We now discuss the case when the rank r is one, that is, M is a hyperbolic space over one of the normed real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. From the above theorem we see that there are exactly two congruence classes of homogeneous codimension one foliations on M. The first one, coming from the 0-dimensional real projective space, is the well-known horosphere foliation. The second foliation is not so well-known except for the real hyperbolic case. In case of $\mathbb{R}H^n$ we get the foliation whose leaves are a totally geodesic $\mathbb{R}H^{n-1} \subset \mathbb{R}H^n$ and its equidistant hypersurfaces. Comparing this with Cartan's classification for $\mathbb{R}H^n$ we see that we got indeed all homogeneous codimension one foliations on $\mathbb{R}H^n$. In case of $\mathbb{C}H^n$ the minimal orbit of the second foliation is a minimal ruled real hypersurface and can be constructed in the following way. Consider a horocycle in a totally geodesic and totally real $\mathbb{R}H^2 \subset \mathbb{C}H^n$. At each point of the horocycle we attach a totally geodesic $\mathbb{C}H^{n-1}$ orthogonal to the complex hyperbolic line determined by the tangent vector of the horocycle at that point. By varying with the points on the horocycle we get the minimal ruled real hypersurface in $\mathbb{C}H^n$.

In order to complete the classification of homogeneous hypersurfaces in connected irreducible Riemannian symmetric spaces of noncompact type it now remains to determine the moduli space $\mathfrak{M}_S$. In the case of $\mathbb{R}H^n$ we already know that $\mathfrak{M}_S$ consists of $n-1$ elements given by the real hyperbolic subspace $\mathbb{R}H^k \subset \mathbb{R}H^n$, $k \in \{0, \ldots, n-2\}$, and the tubes around $\mathbb{R}H^k$. An obvious consequence from Cartan's classification is the nonobvious fact that a singular orbit of a cohomogeneity one action of $\mathbb{R}H^n$ is totally geodesic.

In [10] the subset $\mathfrak{M}_s^{tg}$ of $\mathfrak{M}_S$ consisting of all cohomogeneity one actions for which the singular orbit is totally geodesic has been determined. In this special situation one can use duality between symmetric spaces of compact type and noncompact type to derive the classification. An explicit list of all totally geodesic singular orbits can be found in [10], which can be summarized as follows. The set $\mathfrak{M}_s^{tg}$ is empty for the exceptional symmetric spaces of $E_7^{\mathbb{C}}$ and $E_8^{\mathbb{C}}$ and all their noncompact real forms, and of $E_6^{\mathbb{C}}$ and its split real form. For all other symmetric spaces, and this includes all classical symmetric spaces, $\mathfrak{M}_S^{tg}$ is nonempty and finite. It is $\#\,\mathfrak{M}_S^{tg} = n > 3$ only for the hyperbolic spaces $\mathbb{R}H^{n+1}$, $\mathbb{C}H^{n-1}$ and $\mathbb{H}H^{n-1}$. For the symmetric spaces $\mathbb{R}H^4$, $\mathbb{C}H^2$, $\mathbb{H}H^2$, $\mathbb{O}H^2$, $G_3^*(\mathbb{R}^7)$, $G_2^*(\mathbb{R}^{2n})$ $(n \geq 3)$ and $G_2^*(\mathbb{C}^{2n})$ $(n \geq 3)$ we have $\#\,\mathfrak{M}_S^{tg} = 3$. For the symmetric spaces $\mathbb{R}H^3$, $G_k^*(\mathbb{R}^n)$ $(1 < k < n-k, (k,n) \neq (3,7), (2,2m), m > 2)$, $G_3^*(\mathbb{R}^6)$, $G_k^*(\mathbb{C}^n)$ $(1 < k < n-k, (k,n) \neq (2,2m), m > 2)$, $G_k^*(\mathbb{H}^n)$ $(1 < k < n-k)$, $SL(3,\mathbb{H})/Sp(3)$, $SL(3,\mathbb{C})/SU(3)$, $SL(4,\mathbb{C})/SU(4) = SO(6,\mathbb{C})/SO(6)$, $SO(7,\mathbb{C})/SO(7)$, $G_2^2/SO(4)$ and E_6^{-24}/F_4 we have $\#\,\mathfrak{M}_S^{tg} = 2$. In all remaining cases we have $\#\,\mathfrak{M}_S^{tg} = 1$.

Of course, the natural question now is whether a singular orbit of a cohomogeneity one action on M is totally geodesic. As we already know the answer is yes for $\mathbb{R}H^n$. In [8], the author and Brück investigated this question for the other hyperbolic spaces $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$. The surprising outcome of their investigations is that in all these spaces there exist cohomogeneity one actions with non-totally geodesic singular orbits. In the following we describe the construction of these actions. Let M be one of these hyperbolic spaces and consider an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ of the Lie algebra of the isometry group of M. The restricted root system Σ associated to M is of type BC_1 and hence nonreduced. The nilpotent Lie algebra $\mathfrak{n}$ decomposes into root spaces $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$, where α is a simple root in Σ. The root space $\mathfrak{g}_{2\alpha}$ is the center of $\mathfrak{n}$. The Lie algebra $\mathfrak{n}$ is a Heisenberg algebra in case of $\mathbb{C}H^n$, a generalized Heisenberg algebra with 3-dimensional center in case of $\mathbb{H}H^n$, and a generalized Heisenberg algebra with 7-dimensional center in case of $\mathbb{O}H^2$.

We first consider the case of $\mathbb{C}H^n$, $n \geq 3$, in which case $\mathfrak{g}_\alpha$ is a complex vector space of complex dimension ≥ 2. Denote by J its complex structure. We choose a linear subspace $\mathfrak{v}$ of $\mathfrak{g}_\alpha$ such that its orthogonal complement $\mathfrak{v}^\perp$ in $\mathfrak{g}_\alpha$ has constant Kähler angle, that is, there exists a real number $\varphi \in [0, \pi/2]$ such that the angle between $J(\mathbb{R}v)$ and $\mathfrak{v}^\perp$ is φ for all nonzero vectors $v \in \mathfrak{v}^\perp$. If $\varphi = 0$ then $\mathfrak{v}$ is a complex subspace. It is easy to classify all subspaces with constant Kähler angle in a complex vector space. In particular there exist such subspaces for each given angle φ. It is clear that $\mathfrak{s} = \mathfrak{a}+\mathfrak{v}+\mathfrak{g}_{2\alpha}$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$. Let S be the connected closed subgroup of AN with Lie algebra S and $N_K^o(S)$ the identity component of the normalizer of S in $K = S(U(1) \times U(n))$. Then $N_K^o(S)S \subset KAN = G$ acts on $\mathbb{C}H^n$ with cohomogeneity one and singular orbit $S \subset AN = G/K = \mathbb{C}H^n$. If $\varphi \neq 0$ then S is not totally geodesic.

A similar construction works in the quaternionic hyperbolic space $\mathbb{H}H^n$, $n \geq 3$. In this case the root space $\mathfrak{g}_\alpha$ is a quaternionic vector space of quaternionic dimension $n-1$ and for $\mathfrak{v}$ one has to choose linear subspaces for which the orthogonal complement $\mathfrak{v}^\perp$ of $\mathfrak{v}$ in $\mathfrak{g}_\alpha$ has constant quaternionic Kähler angle. If $n = 2$ we may choose any linear subspace $\mathfrak{v}$ of $\mathfrak{g}_\alpha$ of real dimension one or two.

Finally, in case of the Cayley hyperbolic plane $\mathbb{O}H^2$, the root space $\mathfrak{g}_\alpha$ is isomorphic to the Cayley algebra $\mathbb{O}$. Let $\mathfrak{v}$ be a linear subspace of $\mathfrak{g}_\alpha$ of real dimension 1, 2, 4, 5 or 6. Let S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s} = \mathfrak{a} + \mathfrak{v} + \mathfrak{g}_{2\alpha}$ and $N_K^o(S)$ the identity component of the normalizer of S in $K = Spin(9)$. For instance, if $\dim \mathfrak{v} = 1$ then $N_K^o(S)$ is isomorphic to the exceptional Lie group G_2. The action of G_2 on the 7-dimensional normal space $\mathfrak{v}^\perp$ is equivalent to the standard 7-dimensional representation of G_2. Since this is transitive on the 6-dimensional sphere it follows that $G_2S \subset KAN = G = F_4$ acts on $\mathbb{O}H^2$ with cohomogeneity one and with S as a non-totally geodesic singular orbit. For the dimensions 2, 4, 5 and 6 the corresponding normalizer is isomorphic to $U(4)$, $SO(4)$, $SO(3)$ and $SO(2)$ respectively, and one also gets cohomogeneity one actions on $\mathbb{O}H^2$ with a non-totally geodesic singular orbit. Surprisingly, if $\mathfrak{v}$ is 3-dimensional, this method does not yield such a cohomogeneity one action.

It is an open problem whether for $\mathbb{C}H^n$, $\mathbb{H}H^n$ or $\mathbb{O}H^2$ the moduli space $\mathfrak{M}_S$ contains more elements than described above. Also, for higher rank the explicit structure of $\mathfrak{M}_S$ is still unknown.

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