Dispersive estimates for the Navier-Stokes equations with the Coriolis force

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Introduction

The Navier-Stokes equations with the Coriolis force

$$
\text{(NSC)} \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \Omega e_3 \times u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^3, \end{cases}
$$

where

$u = (u_1(t,x), u_2(t,x), u_3(t,x))$: velocity filed

$p = p(t,x)$: pressure

$u_0 = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$: (given) initial velocity field

$\Omega \in \mathbb{R}$: the Coriolis parameter

$e_3 := (0, 0, 1)$

Known Results

Global existence of solutions for large data

- Chemin–Desjardins–Gallagher–Grenier ('02, '06)
  $\forall u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$, $\exists \Omega_0 = \Omega_0(u_0) > 0$ *s.t.*
  $|\Omega| \geqslant \Omega_0 \Rightarrow \exists! u \in C([0,\infty); \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$: *global sol.*
- Iwabuchi–T. ('12)
  $\forall u_0 \in \dot{H}^s(\mathbb{R}^3)$ $(1/2 < s < 3/4)$:

$$\Omega_0 = \Omega_0(\|u_0\|_{\dot{H}^s}) = C\|u_0\|_{\dot{H}^s}^{\frac{2}{s-1/2}}$$

Uniform global existence of solutions

- Giga–Inui–Mahalov–Saal ('08)
  $\exists \delta > 0$: *independent of* $\Omega \in \mathbb{R}$ *s.t.* $\forall u_0 \in FM_0^{-1}(\mathbb{R}^3)$
  with $\|u_0\|_{FM_0^{-1}} \leqslant \delta, \exists! u \in C([0,\infty); FM_0^{-1}(\mathbb{R}^3))$: *global sol.*
- Hieber–Shibata ('10) : $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$

Semigroup associated with (NSC)

$\mathbb{P} := \left(\delta_{jk} + R_j R_k\right)_{1\leqslant j,k\leqslant 3}$: the Helmholtz projection
$L_\Omega u := -\Delta u + \mathbb{P}\Omega e_3 \times u$

$$\text{(NSC}'\text{)} \begin{cases} \dfrac{\partial u}{\partial t} + L_\Omega u + \mathbb{P}(u\cdot\nabla)u = 0, \quad \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^3, \end{cases}$$

$$e^{-tL_\Omega} f = \frac{1}{2} e^{i\Omega t \frac{D_3}{|D|}} \left[e^{t\Delta}(I + \mathscr{R}) f\right] + \frac{1}{2} e^{-i\Omega t \frac{D_3}{|D|}} \left[e^{t\Delta}(I - \mathscr{R}) f\right],$$

where

$$e^{\pm i\Omega t \frac{D_3}{|D|}} f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t \frac{\xi_3}{|\xi|}} \widehat{f}(\xi) d\xi \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

$$\mathscr{R} := \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix} \qquad R_j : \text{the Riesz transform}$$

Dispersive Estimates

Dispersive Estimates for $e^{\pm i\Omega t\frac{D_3}{|D|}}$

Since the phase $\xi_3/|\xi|$ is homogeneous of degree 0, by the Littlewood–Paley theory, it suffices to consider

$$\mathcal{G}_\Omega(t)f(x) = \mathcal{G}_\Omega^\pm(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t\frac{\xi_3}{|\xi|}}\varphi(\xi)\widehat{f}(\xi)d\xi,$$

where $\varphi \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\varphi \subset \left\{\xi \in \mathbb{R}^3 \,\middle|\, 1/2 \leqslant |\xi| \leqslant 2\right\}$.

- Dutrifoy ('05)

$$\|\mathcal{G}_\Omega(t)f\|_{L^\infty} \lesssim \frac{\log(e+|\Omega|t)}{(1+|\Omega|t)^{\frac{1}{2}}}\|f\|_{L^1}$$

- Iwabuchi–T. ('11)

$$\|\mathcal{G}_\Omega(t)f\|_{L^\infty} \lesssim \frac{\{\log(e+|\Omega|t)\}^{\frac{1}{2}}}{(1+|\Omega|t)^{\frac{1}{2}}}\|f\|_{L^1}$$

Main Results

Theorem (Dispersive Estimates)

$$\|\mathscr{G}_\Omega(t)f\|_{L^\infty} \lesssim \frac{1}{1+|\Omega|t}\|f\|_{L^1}$$

From the above dispersive estimates and the abstract theory by Keel–Tao ('98), we can prove the following:

Theorem (Strichartz Estimates)

Let $2 \leqslant q, r \leqslant \infty$ *with* $(q,r) \neq (2,\infty)$. *Then,*

$$\|\mathscr{G}_\Omega(t)f\|_{L^q_t L^r_x} \lesssim |\Omega|^{-\frac{1}{q}}\|f\|_{L^2}$$

if and only if

$$\frac{1}{q}+\frac{1}{r} \leqslant \frac{1}{2}$$

Main Results

Corollary (Strichartz Estimates)

Let $2 \leqslant q \leqslant \infty$ *and* $2 \leqslant r < \infty$ *satisfy*

$$\frac{1}{q} + \frac{1}{r} \leqslant \frac{1}{2}.$$

Then,

$$\left\| e^{\pm i\Omega t \frac{D_3}{|D|}} f \right\|_{L_t^q L_x^r} \lesssim |\Omega|^{-\frac{1}{q}} \|f\|_{\dot{H}^{\frac{3}{2}-\frac{3}{r}}}$$

Remarks

The loss of the derivative $s = 3/2 - 3/r$ is *sharp* by the scaling. Furthermore, by the Sobolev embedding

$$\dot{H}^{\frac{3}{2}-\frac{3}{r}}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3),$$

i.e, $e^{\pm i\Omega t \frac{D_3}{|D|}}$ have no smoothing effect w.r.t. space variables.

Proof of Dispersive Estimates

Lemma (Littman ('63))

Let $d\mu$ be a surface measure on a smooth surface $S \subset \mathbb{R}^d$, and let $\phi \in C_0^\infty(\mathbb{R}^d)$. Suppose that for all $x \in S$, at least k of the principal curvatures are not zero. Then,

$$\left|\widehat{\phi d\mu}(\xi)\right| \lesssim |\xi|^{-\frac{k}{2}}$$

Let $\Phi(\xi) := \xi_3/|\xi|$ for $\xi \in \mathbb{R}^3 \setminus \{0\}$ and consider the surface

$$\Sigma_0 := \left\{(\xi,\rho) \in \mathbb{R}^{3+1} \,\middle|\, \rho = \Phi(\xi),\ \frac{1}{2} \leqslant |\xi| \leqslant 2\right\}.$$

For the surface Σ_0, the number of non–vanishing principal curvatures is equal to the number of non–zero eigenvalues, that is, the rank of the Hessian matrix $H\Phi$.

Proof of Dispersive Estimates

A direct calculation gives that

$$H\Phi(\xi) = \frac{1}{|\xi|^5}\begin{pmatrix} \xi_3(3\xi_1^2 - |\xi|^2) & 3\xi_1\xi_2\xi_3 & \xi_1(3\xi_3^2 - |\xi|^2) \\ 3\xi_1\xi_2\xi_3 & \xi_3(3\xi_2^2 - |\xi|^2) & \xi_2(3\xi_3^2 - |\xi|^2) \\ \xi_1(3\xi_3^2 - |\xi|^2) & \xi_2(3\xi_3^2 - |\xi|^2) & -3\xi_3(\xi_1^2 + \xi_2^2) \end{pmatrix}$$

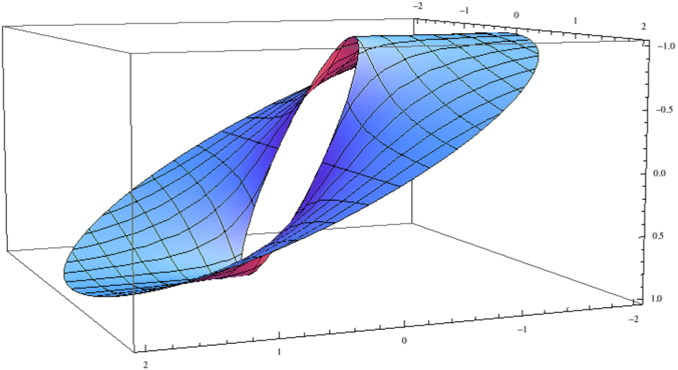
and

$$\det H\Phi(\xi) = \frac{(\xi_1^2 + \xi_2^2)\xi_3}{|\xi|^9}.$$

The surface Σ_0 has 3 non–vanishing principal curvatures (non–vanishing Gaussian curvature) unless

$$(\xi_1, \xi_2) = 0 \quad \text{or} \quad \xi_3 = 0.$$

Proof of Dispersive Estimates



$$\Sigma_0^{3D} := \left\{ (\xi, \rho) = (\xi_1, \xi_2, \rho) \in \mathbb{R}^{2+1} \;\middle|\; \rho = \frac{\xi_2}{|\xi|}, \quad \frac{1}{2} \leqslant |\xi| \leqslant 2 \right\}$$

$(0, \xi_2, \pm 1)$: **degenerate critical points** $\quad \left(\frac{1}{2} \leqslant |\xi_2| \leqslant 2 \right)$

Proof of Dispersive Estimates

$\chi \in C_0^\infty([0,\infty))$ s.t. $\chi(r) = 1$ on $[0, 1/2)$ & $\chi(r) = 0$ on $[1, \infty)$
$0 < c \ll 1$: small constant, $\quad \xi_h := (\xi_1, \xi_2)$

$$\begin{aligned}\mathcal{G}_\Omega(t) f(x) &= \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t \frac{\xi_3}{|\xi|}} \varphi(\xi) \widehat{f}(\xi) d\xi \\ &=: \mathcal{G}_\Omega^1(t) f(x) + \mathcal{G}_\Omega^2(t) f(x) + \mathcal{G}_\Omega^3(t) f(x)\end{aligned}$$

where

$$\mathcal{G}_\Omega^1(t) f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t \frac{\xi_3}{|\xi|}} \left\{1 - \chi\left(\frac{|\xi_h|}{c}\right)\right\} \left\{1 - \chi\left(\frac{|\xi_3|}{c}\right)\right\} \varphi(\xi) \widehat{f}(\xi) d\xi$$

$$\mathcal{G}_\Omega^2(t) f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t \frac{\xi_3}{|\xi|}} \chi\left(\frac{|\xi_h|}{c}\right) \varphi(\xi) \widehat{f}(\xi) d\xi$$

$$\mathcal{G}_\Omega^3(t) f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t \frac{\xi_3}{|\xi|}} \chi\left(\frac{|\xi_3|}{c}\right) \varphi(\xi) \widehat{f}(\xi) d\xi$$

Proof of Dispersive Estimates

Estimate for $\mathscr{G}^1_\Omega(t)f(x)$

$$\mathscr{G}^1_\Omega(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t\frac{\xi_3}{|\xi|}} \left\{1-\chi\left(\frac{|\xi_h|}{c}\right)\right\}\left\{1-\chi\left(\frac{|\xi_3|}{c}\right)\right\}\varphi(\xi)\widehat{f}(\xi)d\xi$$

Note that

$$\operatorname{supp}\left[\left\{1-\chi\left(\frac{|\xi_h|}{c}\right)\right\}\left\{1-\chi\left(\frac{|\xi_3|}{c}\right)\right\}\varphi(\xi)\right]$$
$$\subset \left\{\xi=(\xi_h,\xi_3)\in\mathbb{R}^3 \;\middle|\; |\xi_h|\geqslant\frac{c}{2},\ |\xi_3|\geqslant\frac{c}{2},\ \frac{1}{2}\leqslant|\xi|\leqslant 2\right\}$$

Therefore

$$\det H\Phi(\xi) = \frac{(\xi_1^2+\xi_2^2)\xi_3}{|\xi|^9} \neq 0$$

We have by the standard stationary phase method

$$\left\|\mathscr{G}^1_\Omega(t)f\right\|_{L^\infty} \lesssim (1+|\Omega|t)^{-\frac{3}{2}}\|f\|_{L^1}$$

Proof of Dispersive Estimates

Estimate for $\mathcal{G}^2_\Omega(t)f(x)$

$$\mathcal{G}^2_\Omega(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t\frac{\xi_3}{|\xi|}} \chi\left(\frac{|\xi_h|}{c}\right)\varphi(\xi)\widehat{f}(\xi)d\xi$$

Note that

$$\operatorname{supp}\left[\chi\left(\frac{|\xi_h|}{c}\right)\varphi(\xi)\right] \subset \left\{\xi = (\xi_h, \xi_3) \in \mathbb{R}^3 \;\middle|\; |\xi_h| \leqslant c,\ \frac{1}{2} \leqslant |\xi| \leqslant 2\right\}$$

Since $0 < c \ll 1$, it suffices to consider the case $\xi_h = 0$:

$$H\Phi(0,0,\xi_3) = \frac{1}{|\xi_3|^5}\begin{pmatrix} -\xi_3^3 & 0 & 0 \\ 0 & -\xi_3^3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $|\xi| \sim 1$ and c is small, we have $|\xi_3| \sim 1$. Hence

$$\left\|\mathcal{G}^2_\Omega(t)f\right\|_{L^\infty} \lesssim (1+|\Omega|t)^{-1}\|f\|_{L^1}$$

Proof of Dispersive Estimates

Estimate for $\mathscr{G}^3_\Omega(t)f(x)$

$$\mathscr{G}^3_\Omega(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t\frac{\xi_3}{|\xi|}} \chi\left(\frac{|\xi_3|}{c}\right)\varphi(\xi)\widehat{f}(\xi)d\xi$$

Note that

$$\operatorname{supp}\left[\chi\left(\frac{|\xi_3|}{c}\right)\varphi(\xi)\right] \subset \left\{\xi = (\xi_h, \xi_3) \in \mathbb{R}^3 \;\middle|\; |\xi_3| \leqslant c,\ \frac{1}{2} \leqslant |\xi| \leqslant 2\right\}$$

Since $0 < c \ll 1$, it suffices to consider the case $\xi_3 = 0$:

$$H\Phi(\xi_1, \xi_2, 0) = \frac{1}{|\xi_h|^5}\begin{pmatrix} 0 & 0 & -\xi_1|\xi_h|^2 \\ 0 & 0 & -\xi_2|\xi_h|^2 \\ -\xi_1|\xi_h|^2 & -\xi_2|\xi_h|^2 & 0 \end{pmatrix}$$

By a direct calculation,

$$\det\left(\lambda I - H\Phi(\xi_1, \xi_2, 0)\right) = \lambda\left(\lambda - |\xi_h|^{-2}\right)\left(\lambda + |\xi_h|^{-2}\right)$$

Proof of Dispersive Estimates

$$\det\left(\lambda I - H\Phi(\xi_1, \xi_2, 0)\right) = \lambda\left(\lambda - |\xi_h|^{-2}\right)\left(\lambda + |\xi_h|^{-2}\right)$$

Since $|\xi| \sim 1$, $|\xi_3| \leqslant c$ and c is small, we have $|\xi_h| \sim 1$. Thus,

$$\left\|\mathscr{G}^3_\Omega(t) f\right\|_{L^\infty} \lesssim (1 + |\Omega| t)^{-1} \|f\|_{L^1}$$

Lemma

$$\left\|\mathscr{G}^1_\Omega(t) f\right\|_{L^\infty} \lesssim (1 + |\Omega| t)^{-\frac{3}{2}} \|f\|_{L^1}$$
$$\left\|\mathscr{G}^2_\Omega(t) f\right\|_{L^\infty} \lesssim (1 + |\Omega| t)^{-1} \|f\|_{L^1}$$
$$\left\|\mathscr{G}^3_\Omega(t) f\right\|_{L^\infty} \lesssim (1 + |\Omega| t)^{-1} \|f\|_{L^1}$$

Thank you very much for your kind attention!

Proof of Dispersive Estimates

$$\det(\lambda I - H\Phi(\xi_1,\xi_2,0)) = \lambda\left(\lambda - |\xi_h|^{-2}\right)\left(\lambda + |\xi_h|^{-2}\right)$$

Since $|\xi| \sim 1$, $|\xi_3| \leqslant c$ and c is small, we have $|\xi_h| \sim 1$. Thus,

$$\left\|\mathscr{G}^3_\Omega(t) f\right\|_{L^\infty} \lesssim (1+|\Omega|t)^{-1}\|f\|_{L^1}$$

Lemma

$$\left\|\mathscr{G}^1_\Omega(t) f\right\|_{L^\infty} \lesssim (1+|\Omega|t)^{-\frac{3}{2}}\|f\|_{L^1}$$
$$\left\|\mathscr{G}^2_\Omega(t) f\right\|_{L^\infty} \lesssim (1+|\Omega|t)^{-1}\|f\|_{L^1}$$
$$\left\|\mathscr{G}^3_\Omega(t) f\right\|_{L^\infty} \lesssim (1+|\Omega|t)^{-1}\|f\|_{L^1}$$

Thank you very much for your kind attention!