Steady Rotational Water Waves

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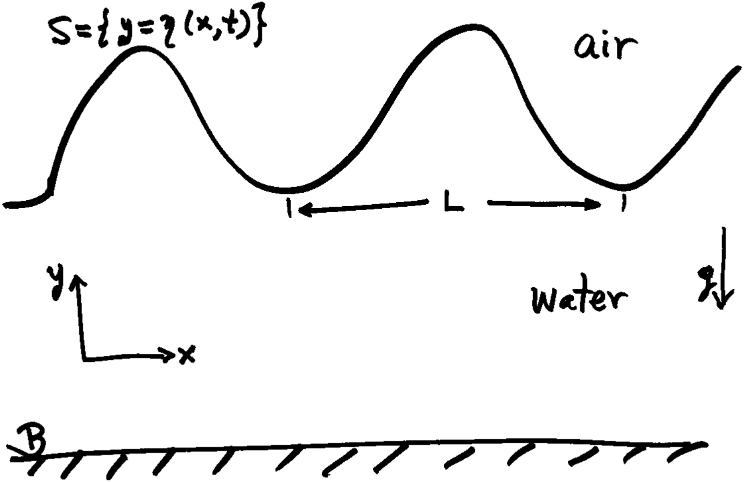
Naturally, island nations have been particularly active in the mathematical theory of fluids and water waves. See, for instance, the influential book of Okamoto & Shoji and the papers of T. Nishida, B. Benjamin, J. Toland,....


S={y=η(x,t)}
air
L
y
x
water
g
B

Joint work with:

Adrian Constantin

Joy Ko

Miles Wheeler

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We consider water in, say, a lake with a flat bottom B and a free surface S under gravity g.
Above is air with atmospheric pressure P_{atm}.

The water: 2D, incompressible, inviscid, no surface tension.

Not irrotational!

Not an approximation!

OUTLINE

- The equations
- History
- Periodic waves: existence theorem
- Sketch of proof
- Solitary waves
- Open problems

2D Euler Equations

Velocity $\vec{u} = [u, v] = [u(x, y, t), v(x, y, t)]$, pressure$= P(x, y, t)$, density$= 1$. Inside the fluid [Euler ∼1750]:

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \nabla P = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

On the surface $S = \{y = \eta(x, t)\}$:

$$P = P_{atm}, \quad v = \eta_t + u\,\eta_x$$

On the bottom $B = \{y = -d\}$: $\quad v = 0.$

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Define the stream function ψ: $\quad \psi_x = -v, \;\; \psi_y = u$

and the 2D vorticity ω: $\quad \omega = v_x - u_y = -\Delta\psi.$

Traveling (i.e. Steady) Waves

We consider waves of speed c:
velocity $\vec{u}(x - ct, y)$, surface $\eta(x - ct)$.
and of period L.
Therefore we can change to a moving frame: $x - ct \to x$.
Then the Euler equations imply that ω and $\psi - cy$ are functionally dependent, so that we get the nonlinear elliptic PDE

$$-\Delta\psi(x, y) = \gamma[\psi(x, y) - cy] = \omega(x, y)$$

where $\Delta\psi = \psi_{xx} + \psi_{yy}$. The vorticity function γ is a completely arbitrary function of one variable; we assume it's single-valued.

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There are two constants:
1. The flux $p_0 = \int_{-d}^{\eta(x)} [u(x, y) - c]\, dy$ is independent of x.
2. The relative stream function $\psi(x, y) - cy$, restricted to the free surface S, is a constant.

We will first consider waves periodic in x.

Irrotational Case (no vorticity):

Cauchy (1815), Poisson, Airy, Stokes (1847), ...

Existence:
Nekrasov, Levi-Civita, Struik (1920's),
Keady & Norbury, Amick & Toland & Fraenkel ($\sim$1980), ...

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Theorem (2010). Within the fluid:
(i) The pressure decreases horizontally away from the crest line, and increases with depth.
(ii) If $u < c$, all the particle trajectories are *non-closed* looping orbits unless there is a background current.

All of this work strongly uses the fact that ψ is a harmonic function inside the fluid.

Steady Waves with Vorticity

Rotational effects (i.e. due to nontrivial vorticity) are significant for

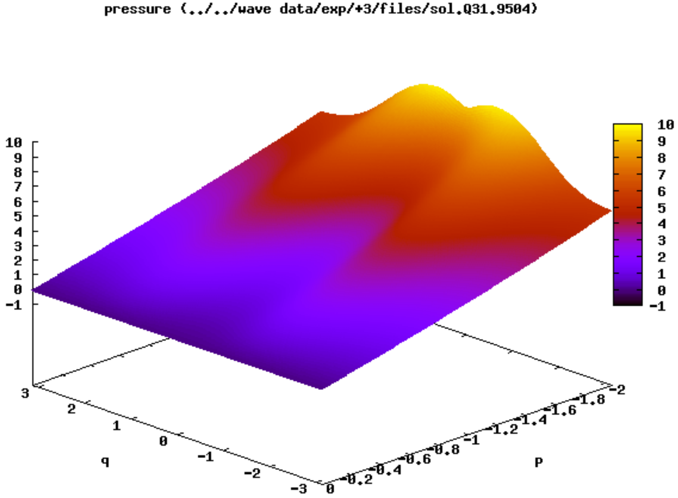
- wind-driven waves
- waves riding upon a sheared current
- waves near a ship
- tsunamis approaching a shore

Important example: Gerstner (1802)

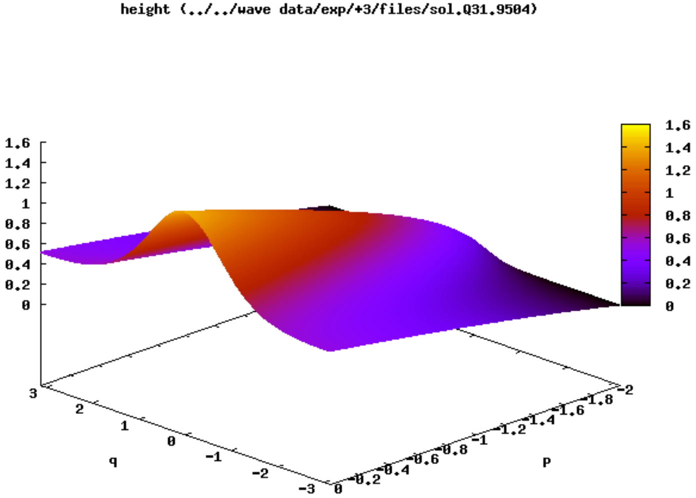
Existence:
Dubreil-Jacotin (1934), Constantin & S (2004)

This situation requires looking inside the fluid because ψ is no longer a harmonic function.

If ∃ vorticity, the pressure is not necessarily monotone:



The height of the same wave:



Existence of Traveling Waves

Theorem [Constantin & S, 2004-2011]

Let arbitrary constants $c > 0, L > 0, p_0 < 0$
and an arbitrary C^∞ function $\gamma(\cdot)$ be given
subject to: either $\gamma \leq 0$ or $|p_0|$ not too big or L is not too small.
Then there exists a continuum $\mathcal{C}$ (a connected set) of C^∞
symmetric traveling waves with $u(x, y) < c$, each one with a
single crest and trough per period.

- $\mathcal{C}$ contains a trivial laminar flow with S flat,
- as well as waves for which $\max u \nearrow c$ ("stagnation").

More generally, if, for some $0 < \alpha < 1$, the vorticity is merely C^α
except for a finite number of jumps, then
$h \in C^{1+\alpha}, u, v \in C^\alpha, \eta \in C^{1+\alpha}$. □

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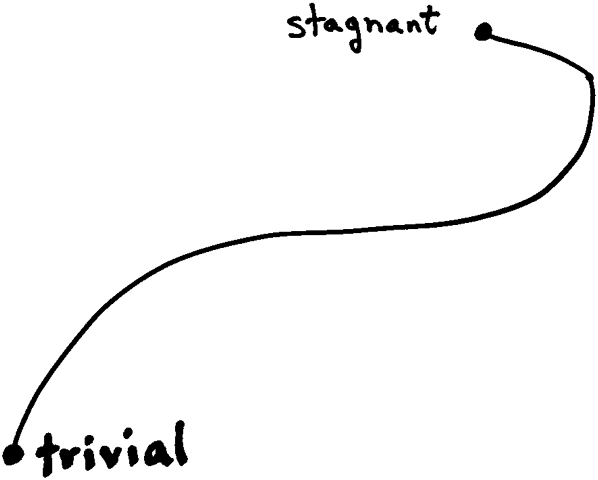
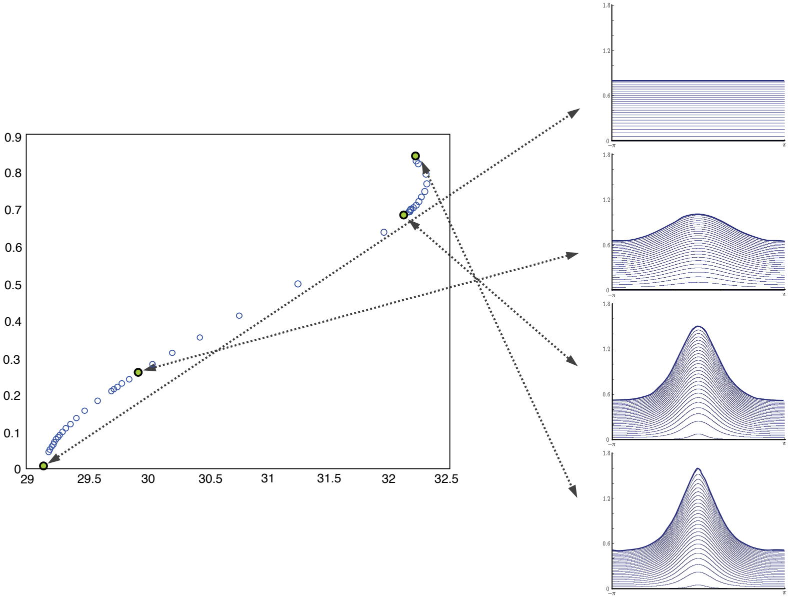
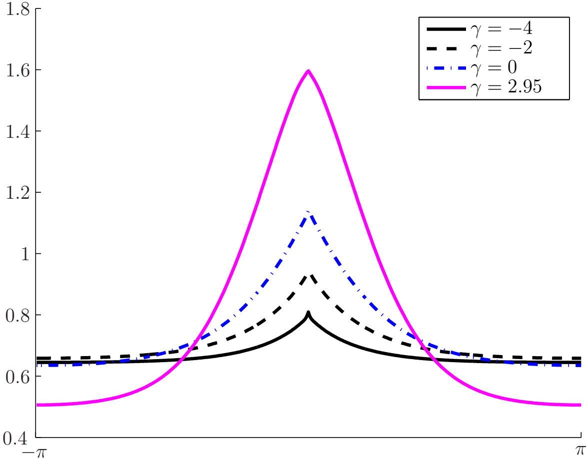
Computations [Joy Ko & S.]. The computations below show
that the vorticity can have a big qualitative effect on the
solutions! (We take $L = 2\pi, \quad p_0 = -2, \quad g = 9.8$.)


stagnant
trivial

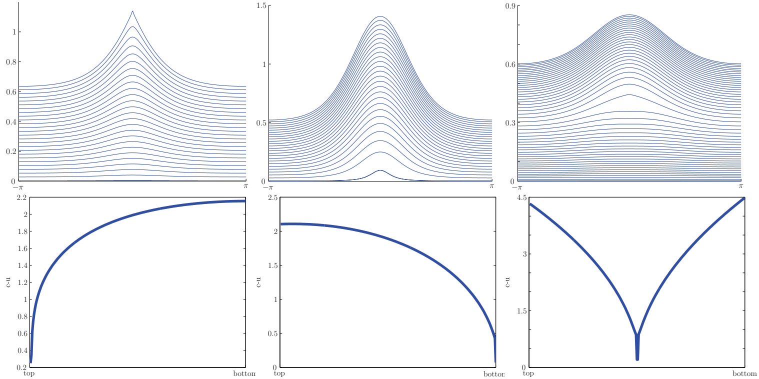
The continuum: amplitude vs. energy for $\gamma \equiv 2.95$



Profiles of waves near stagnation for $\gamma \equiv -4,\ -2,\ 0,\ 2.95$



Extreme waves with top, bottom and internal stagnation



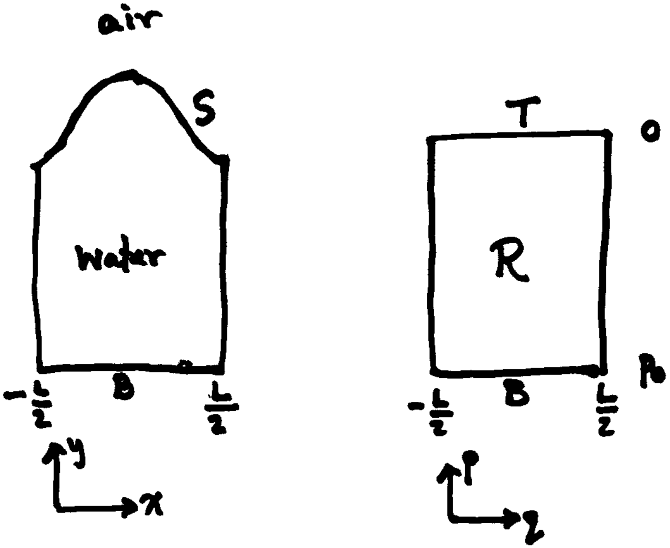
Change of coordinates

First step in the proof: fix the free boundary by introducing a new independent variable $p = cy - \psi(x, y)$.

It would be very nice for the other coordinate q to be its harmonic conjugate but that is possible only if ψ is harmonic, that is, if there is no vorticity.

So Dubreil-Jacotin in 1933 simply put $q = x$. This permits the vorticity to be arbitrary.

D-J Transformation:



The D-J transformation

$$p = cy - \psi(x, y), \quad q = x$$

leads to the following system for the height

$$h(q, p) = y + d$$

in the rectangle $R = \{0 < q < L,\ p_0 < p < 0\}$:

$$\left\{ -\frac{1 + h_q^2}{2h_p^2} + \Gamma(p) \right\}_p + \left\{ \frac{h_q}{h_p} \right\}_q = 0 \quad \text{in} \quad R,$$

$$-\frac{1 + h_q^2}{2h_p^2} - gh + \frac{Q}{2} = 0 \quad \text{on the top} \quad T$$

$$h = 0 \qquad \text{on the bottom } B$$

period L in the q variable.

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Notice that the PDE is elliptic so long as $h_p > 0$.
Because $h_p = 1/(c - u)$, this means we will require $u < c$.
Notice that the only free parameters are the head Q, the dimensions of R, and the vorticity function $\gamma(\cdot)$.

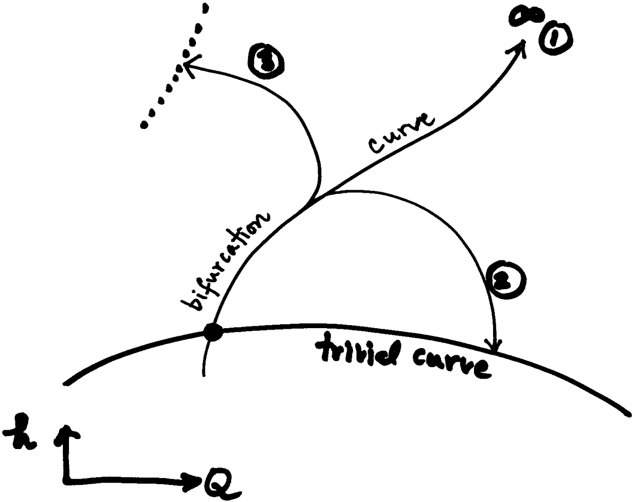
Proof of Theorem

First, local bifurcation.
We treat Q as a bifurcation parameter.
Suppose $\gamma(\cdot)$ is an arbitrary smooth function.

There is a curve of trivial (flat surface) solutions $Q(\lambda), H(p; \lambda)$ in the (Q, h) space $\mathbb{R} \times C^{3,\alpha}$, where $0 < \alpha < 1$.

We look along this curve for a bifurcation point $Q(\lambda^*), H(p, \lambda^*)$. At such a point the linearized operator must have a kernel.

Even though this is not a standard eigenvalue problem, it is still possible to apply the Crandall-Rabinowitz bifurcation - from - a - simple - eigenvalue theorem (1971) (or the Liapunov-Schmidt method) to get a unique local curve of non-trivial solutions. Their amplitudes are small.



∞
①
③
②
curve
bifurcation
trivial curve
h
Q

Global Continuation

We want to continue the bifurcation curve in $\mathbb{R} \times C^{3,\alpha}$.
Our tools are:

- Degree theory, Leray & Schauder (1930's), but specifically the degree of Healey & Simpson (1998).
- Global bifurcation method of Rabinowitz (1971).
- Schauder estimates for elliptic operators.

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We obtain one of the following possibilities:

1. Unboundedness: $\mathcal{C}$ becomes unbounded.
2. Self-intersection: $\mathcal{C}$ intersects the trivial curve at another point.
3. Degeneracy: The PDE becomes non-elliptic or the BC becomes non-oblique.

- Eliminate self-intersection by using the nodal properties (nonvanishing derivative between crest and trough).
  Tool: the maximum principles of Hopf and Serrin (1971).

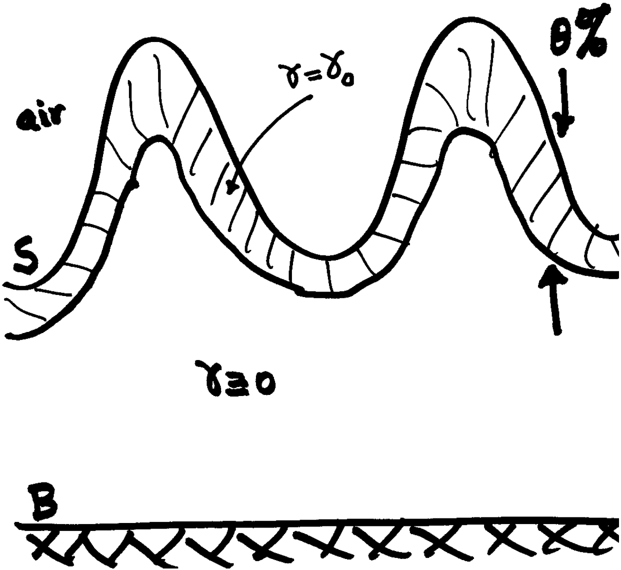
- Reduce $\|h\|_{C^{3,\alpha}} \to \infty$ (unboundedness of h) to the much weaker statement that $\|h_p\|_{L^\infty} \to \infty$ along $\mathcal{C}$.
  Tool: the Lieberman-Trudinger regularity estimates (1986).

- Eventually reduce all possibilities to

$$\sup u \nearrow c \qquad \text{along } \mathcal{C}.$$

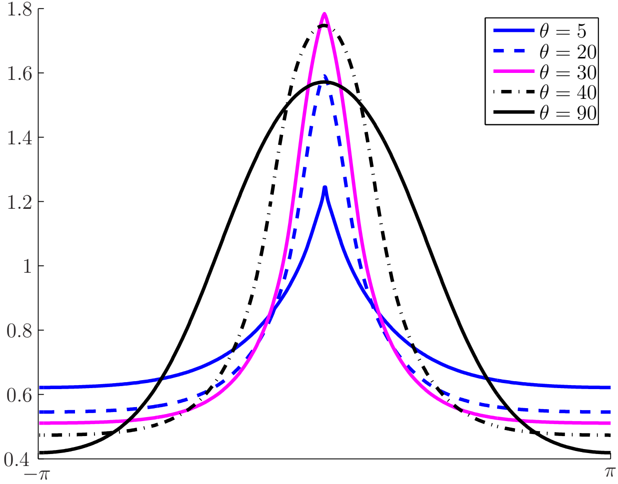
  Recall that $h_p = 1/(c - u)$.

  Tools: (1) A lower bound on u at the crest.
  (2) A lower bound on the pressure P.

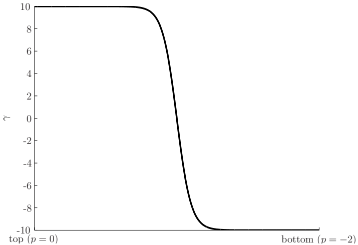


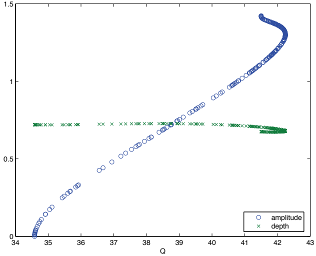
air
$\gamma=\gamma_0$
θ%
S
$\gamma\equiv 0$
B

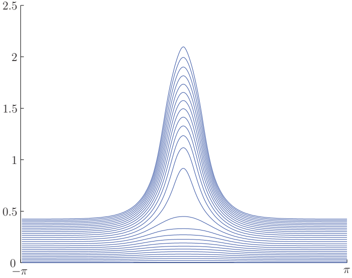
Profiles of waves near stagnation for 5 thicknesses with $\gamma_0 = 10$

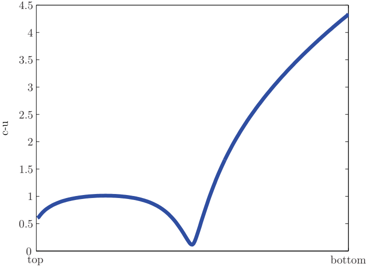


Continuous vorticity with internal stagnation









Solitary Waves

These are non-periodic waves which approach a constant depth as $x \to \pm\infty$, the same at both ends. I will consider only symmetric waves with a single crest or trough.

Let $u = U(y),\ v = 0$ denote the trivial (flat) flow which is approached as $x \to \pm\infty$. Define F and $U^*(y)$ by

$$\frac{1}{F^2} = g \int_0^d \frac{dy}{(c - U(y))^2}, \quad U(y) = c - FU^*(y).$$

F is non-dimensional and is called the *generalized Froude number*. It is used as a measure of the drag of a ship. There is vorticity provided $U^*(y)$, the normalized flow at ∞, is not a constant.

Constructions of Solitary Waves

In the irrotational case there are several constructions of solitary waves of small amplitude going back to the 1950's. A big advance was made by Amick and Toland (1981), who constructed a global continuum of solitary waves with $1 < F < 2$.

In the rotational case, first Hur (2008) and then Groves & Wahlén (2008) constructed a local curve $\mathcal{C}_{loc}$ of small-amplitude solitary waves with F slightly bigger than 1 (supercritical flow). The local construction is *much* more complicated than the periodic case because the linearized operator is not Fredholm; it requires a KdV type of scaling.

The first global construction of rotational solitary waves has just been accomplished by Miles Wheeler.

Theorem [Wheeler(2012)]

Let the speed c and the average depth d be given, as well as an arbitrary $C^{2,\frac{1}{2}}$ trivial flow $U^*(y) > 0$.
Then there exists a continuum $\mathcal{C} \supset \mathcal{C}_{loc}$ of solitary traveling waves with $C^{1,\frac{1}{2}}$ surfaces and $F > 1$.
Each wave is a symmetric monotone wave of elevation with a single crest and with $u(x, y) < c$.
The continuum $\mathcal{C}$ contains waves that
either approach stagnation or $F \to \infty$ or $F \searrow 1$. □

In the irrotational case [Amick & Toland], only the first alternative can occur and the stagnation occurs at the crest.

Method of proof: Use a continuation argument starting from a point on $\mathcal{C}_{loc}$. This requires detailed knowledge of the local waves. In order to use degree theory, a compactness property is needed which requires the use of a weight function as $x \to \infty$. Therefore one must work in weighted Hölder spaces.

Additional results on steady waves with vorticity

- Infinite depth: large-amplitude waves. [Vera Hur]
- Surface tension: There can be multiple bifurcating curves even from the lowest eigenvalue. These curves are global. [Erik Wahlén, Sam Walsh]
- Stratified fluid: large-amplitude waves. [Sam Walsh]
- Stagnation: If the vorticity is nowhere positive, there is an extreme wave (like Stokes') with its stagnation point at the crest. [Eugen Varvaruca]
- Critical layers: These are waves with layers where $u = c$. [Wahlén, Varvaruca & Constantin]
- Vortex sheets. [Ambrose et al]

250 years after Euler, still many fundamental open problems

- Overhanging steady waves
- 3D steady waves [Iooss and Plotnikov]
- Time-dependent flows;
  • Short-time existence in Sobolev spaces by Sijue Wu et al.
  • Long-time existence for small data in 2D by Sijue Wu.
  • Infinite-time existence for small data in 3D by Germain, Masmoudi, Shatah and by Wu.
- Unstable? Stable? [Zhiwu Lin: instability of some large amplitude waves]
- Instability and turbulent flows!