Sharp Morawetz Estimates*

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Contents

1. Morawetz estimates (original form, applications, history, etc.)

2. Relation to other important notion

3. Previous results

4. Main results

5. Method of proof

Basic Literature

W.A. Strauss, Nonlinear Wave Equations, AMS1989.
V. Georgiev, Semilinear Hyperbolic Equations, MSJ 2000.
T. Cazenave, Semilinear Schrödinger Equations, AMS 2003.
T. Tao, Nonlinear Dispersive Equations, AMS 2006.

Nonlinear equations of classical fields

$$\partial_t^2 u - \Delta u + f(u) = 0$$ **Nonlinear Wave Equation (NLW)**

$$u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$ **wavefunction (neutral, massless field)**

$$u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$$ **wavefunction (charged, massless field)**

$$\partial_t^2 u - \Delta u + u + f(u) = 0$$ **Nonlinear Klein-Gordon Equation (NLKG)**

$$u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$ **wavefunction (neutral, massive field)**

$$u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$$ **wavefunction (charged, massive field)**

$$i\partial_t u - \Delta u + f(u) = 0$$ **Nonlinear Schrödinger Equation (NLS)**

$$u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$$ **wavefunction (scalar field)**

$$f : \mathbb{C} \to \mathbb{C} \ (\text{or } \mathbb{R} \to \mathbb{R})$$ **nonlinearity (self-interaction)**

Variational Formulation (Lagrange Formulation)

$$\exists V : \mathbb{C} \to \mathbb{R} \text{ (or } \mathbb{R} \to \mathbb{R}) : f(u) = \partial V / \partial \bar{u} \text{ (or } f(u) = V'(u))$$

NLW $$\mathfrak{L}(u) = -\frac{1}{2}\partial_t u \, \overline{\partial_t u} + \frac{1}{2}\nabla u \cdot \overline{\nabla u} + V(u)$$

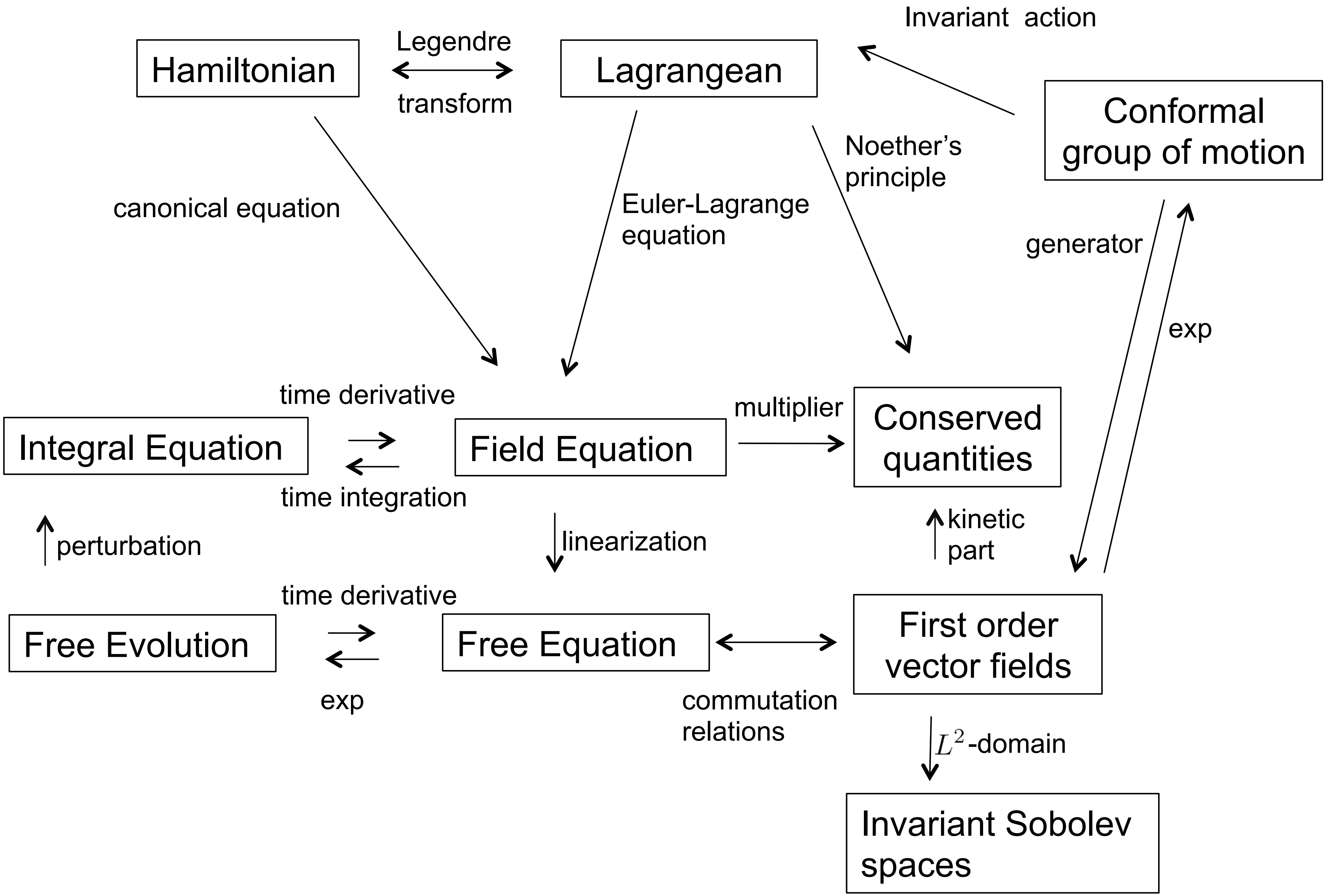
$$\left(\text{ or } \mathfrak{L}(u) = -\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}(\nabla u)^2 + V(u)\right)$$

NLKG $$\mathfrak{L}(u) = -\frac{1}{2}\partial_t u \, \overline{\partial_t u} + \frac{1}{2}\nabla u \cdot \overline{\nabla u} + \frac{1}{2}u\overline{u} + V(u)$$

$$\left(\text{ or } \mathfrak{L}(u) = -\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}(\nabla u)^2 + \frac{1}{2}u^2 + V(u)\right)$$

NLS $$\mathfrak{L}(u) = \frac{1}{4i}(u\overline{\partial_t u} - \overline{u}\partial_t u) + \frac{1}{2}\nabla u \cdot \overline{\nabla u} + V(u)$$

$$L_\Omega(u) = \int_\Omega \mathfrak{L}(u)\,dx\,dt, \quad \Omega \subset \mathbb{R} \times \mathbb{R}^n$$ **Lagrangean (action)**


Invariant action
Hamiltonian
Legendre
transform
Lagrangean
Conformal
group of motion
Noether's
principle
canonical equation
Euler-Lagrange
equation
generator
exp
time derivative
Integral Equation
Field Equation
multiplier
Conserved
quantities
time integration
perturbation
linearization
kinetic
part
time derivative
Free Evolution
Free Equation
First order
vector fields
exp
commutation
relations
L^2-domain
Invariant Sobolev
spaces

NLW [NLKG]

Conformal group of motion	generator	conserved quantity	invariant Sobolev space
time-translations	time-derivative	energy	$H^1(\mathbb{R}^n)$
space-translations	space-derivatives	momentum	
space-rotations	angular-derivatives	angular-momentum	
Lorentz transforms	Lorentz derivatives	Lorentz angular momentum	
space-dilations	radial-derivative	Morawetz' identity	
(space-time dilations	generator of space-time dilations	dilation identity	$H^1 \cap \mathcal{F}(H^1)$)
inversions	generator of conformal transform	conformal identity	$H^1 \cap \mathcal{F}(H^1)$

NLS

Conformal group of motion	**generator**	**conserved quantity**	**invariant Sobolev space**
gauge-transform	Imaginary unit	charge	$L^2(\mathbb{R}^n)$
time-translations	time-derivative	energy	$H^1(\mathbb{R}^n)$
space translations	space-derivatives	momentum	
space-rotations	angular-derivatives	angular-momentum	
Galilei-transform	generator of Galilei-transform	Galilei-momentum	
space-dilations	radial-derivative	Morawetz’ identity	
space-time dilations	generator of space-time dilations	dilation identity	$H^1 \cap \mathcal{F}(H^1)$
pseudo-inversions	generator of pseudo-conformal transform	pseudo-conformal identity	$H^1 \cap \mathcal{F}(H^1)$

Morawetz' identity for NLW

multiplier : symmetric part of the radial derivative $\partial_r = \frac{x}{|x|} \cdot \nabla$:

$$M = \frac{1}{2}\left(\frac{x}{|x|}\cdot\nabla + \nabla\cdot\frac{x}{|x|}\right) = \frac{x}{|x|}\cdot\nabla + \frac{n-1}{2|x|}.$$

$n \geq 4$ $\quad \frac{d}{dt}\,\mathrm{Re}\int \partial_t u \cdot \overline{Mu}$

$$= -\int (|\nabla u|^2 - |\partial_r u|^2)\frac{1}{|x|}dx - \frac{(n-1)(n-3)}{4}\int \frac{|u|^2}{|x|^3}dx - \frac{n-1}{2}\int (\mathrm{Re}\ (\overline{u}f(u)) - 2V(u))\frac{1}{|x|}dx$$

$n = 3$ $\quad \frac{d}{dt}\,\mathrm{Re}\int \partial_t u \cdot \overline{Mu}$

$$= -\int (|\nabla u|^2 - |\partial_r u|^2)\frac{1}{|x|}dx - 2\pi|u(t,0)|^2 - \int (\mathrm{Re}\ (\overline{u}f(u)) - 2V(u))\frac{1}{|x|}dx$$

Applications : $\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \frac{|u(t,x)|^{p+1}}{|x|}dxdt \leq C\|u(0); H^1\|^2$

Asymptotic completeness of wave operators in the energy space $H^1(\mathbb{R}^n)$ for large data (Morawetz, Strauss, Lin, Ginibre, Velo, Nakanishi, Colliander, Keel, Staffilani, Takaoka, Tao,・・・)

Linear Case : smoothing estimates

$$||Au; L^2(\mathbb{R}\times\mathbb{R}^n)\| \le C\|u(0); L^2(\mathbb{R}^n)\|,\ \ A = |x|^{-s}.$$

$$\text{LW}:\quad \int_{\mathbb{R}}\int_{\mathbb{R}^n} |u(t,x)|^2 \frac{1}{|x|^3} dx dt \le C(\|\nabla u(0); L^2\|^2 + \|\partial_t u(0); L^2\|^2),\ \ n \ge 4.$$

Kato smoothing : Tosio Kato, Math. Annalen (1996), Studia Math. (1968).

$\mathcal{H}$: Hilbert space, H : selfadjoint operator in $\mathcal{H}$, $R(z) = (z-H)^{-1}$,

A : densely defined closed operator in $\mathcal{H}$ with $D(H) \subset D(A)$.

$$\sup_{\phi\in\mathcal{H}\setminus\{0\}} \frac{1}{2\pi}\int_{-\infty}^{\infty} \|Ae^{-itH}\phi\|^2 dt/\|\phi\|^2$$

$$= \sup_{\substack{\phi\in\mathcal{H}\setminus\{0\}\\ \varepsilon>0}} \frac{1}{4\pi^2}\int_{-\infty}^{\infty} (\|AR(\lambda+i\varepsilon)\phi\|^2 + \|AR(\lambda-i\varepsilon))\phi\|^2) d\lambda/\|\phi\|^2$$

$$= \sup_{\substack{\phi\in\mathcal{H}\setminus\{0\}\\ \varepsilon>0}} \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \|A(R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon))\phi\|^2) d\lambda/\|\phi\|^2$$

$$= \sup_{I\subset\mathbb{R}} \|AE_H(I)\|^2/|I|$$

Kato smoothing in the free Schrödinger case

$$\mathcal{H} = L^2(\mathbb{R}^n), \;\; H = -\Delta,$$

$$A = (1+|x|^2)^{-s/2}(-\Delta)^{1/4} \qquad (s > 1/2)$$ Ben-Artzi & Klainerman

$$A = |x|^{s-1}(-\Delta)^{s/2} \qquad (1 - n/2 < s < 1/2)$$ Kato & Yajima, Sugimoto

$$A = (1+|x|^2)^{-1/2}(1-\Delta)^{1/4} \qquad (n \geq 3)$$ Kato & Yajima

Method of Proof depends on one of the following estimates :

$$\int_{|\xi|=\rho} |\mathcal{F}(A^*\phi)(\xi)|^2 dS \leq C\rho \|\phi; L^2\|^2$$ Restriction theorem (trace lemma)

$$\sup_{\mathrm{Im} z > 0} |(R(z)A^*\phi, A^*\phi)| \leq C\|\phi; L^2\|^2$$ Resolvent estimate (Limiting absorption pinciple)

In fact : $\mathrm{Im}(R(\rho^2 + i0)\phi, \phi) = \dfrac{1}{4(2\pi)^{n-1}\rho} \displaystyle\int_{|\xi|=\rho} |\mathcal{F}\phi|^2 dS$

(Hoshiro, Sugimoto, Ruzhansky, ･･･)

Sharp Morawetz estimates for free solutions

- Schrödinger : $i\partial_t u - \Delta u = 0$ (Simon, JFA (1992))

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |u(t,x)|^2 \frac{1}{|x|^2} dxdt \leq \frac{\pi}{n-2}\|u(0); L^2\|^2, \quad n \geq 3$$

- Wave : $\partial_t^2 u - \Delta u = 0$ (Vega & Visciglia, JFA(2008))

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |u(t,x)|^2 \frac{1}{|x|^3} dxdt \leq \frac{4}{(n-1)(n-3)}(\|\nabla u(0); L^2\|^2 + \|\partial_t u(0); L^2\|^2), \quad n \geq 4$$

Goal

Sharp Morawetz estimates for LS, LW, LKG in fractional setting
Characterization of the maximisers
Elementary proof (without restriction theorem and resolvent estimates)

Main Theorems (Related works by Bez & Sugimoto)

$$C(n,s) := \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n-1}{2}-s)}{\Gamma(\frac{n-1}{2}+s)}\frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})}\ , \quad 0 < s < \frac{n-1}{2}$$

Theorem 1. $\partial_t^2 u - \Delta u = 0$ in $\mathbb{R}\times\mathbb{R}^n$, $n \geq 2$, $0 < s < \frac{n-1}{2}$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |u(t,x)|^2 \frac{1}{|x|^{1+2s}} dxdt \leq C(n,s)(\|u(0);\dot{H}^s\|^2 + \|\partial_t u(0);\dot{H}^{s-1}\|^2).$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radial.

Corollary 1. $\partial_t^2 u - \Delta u = 0$ in $\mathbb{R}\times\mathbb{R}^n$, $n \geq 4$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |u(t,x)|^2 \frac{1}{|x|^3} dxdt \leq \frac{4}{(n-1)(n-3)}(\|\nabla u(0);L^2\|^2 + \|\partial_t u(0);L^2\|^2).$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radial.

$$C(n,s) := \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n-1}{2}-s)}{\Gamma(\frac{n-1}{2}+s)}\frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})}, \quad 0<s<\frac{n-1}{2}$$

Theorem 2. $\partial_t^2 u - \Delta u + u = 0 \ \text{in} \ \mathbb{R}\times\mathbb{R}^n, \ \ n \geq 2, \ -\frac{1}{2} < s < \frac{n-1}{2} - 1.$ Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|(1-\Delta)^{1/4} u(t,x)\right|^2 \frac{1}{|x|^{2+2s}} dx dt$$

$$\leq C\left(n, s+\tfrac{1}{2}\right)\left(\|u(0);\dot{H}^s\|^2 + \|\nabla u(0);\dot{H}^s\|^2 + \|\partial_t u(0);\dot{H}^s\|^2\right).$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radial.

Corollary 2. $\partial_t^2 u - \Delta u + u = 0 \ \text{in} \ \mathbb{R}\times\mathbb{R}^n, \ \ n \geq 3.$ Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|(1-\Delta)^{1/4} u(t,x)\right|^2 \frac{1}{|x|^{2}} dx dt$$

$$\leq \frac{\pi}{n-2}(\|u(0);L^2\|^2 + \|\nabla u(0);L^2\|^2 + \|\partial_t u(0);L^2\|^2).$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radial.

$$C(n,s) := \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n-1}{2}-s)}{\Gamma(\frac{n-1}{2}+s)}\frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})}, \quad 0<s<\frac{n-1}{2}$$

Theorem 3. $i\partial_t u - \Delta u = 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}^n, \quad n\geq 2, \quad -\frac{1}{2}<s<\frac{n-1}{2}-1.$ Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |u(t,x)|^2\frac{1}{|x|^{2+2s}}dxdt \leq C(n,s+\frac{1}{2})\|u(0);\dot{H}^s\|^2.$$

Equality is attained if and only if $u(0)$ is radial.

Corollary 3. $i\partial_t u - \Delta u = 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}^n, \quad n\geq 3.$ Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |u(t,x)|^2\frac{1}{|x|^{2}}dxdt \leq \frac{\pi}{n-2}\|u(0);L^2\|^2.$$

Equality is attained if and only if $u(0)$ is radial.

Proof of Theorems 1 and 3

$$i\partial_t u = (-\Delta)^{a/2}u,\ \ a>0,\ \ u(0,x)=f(x).$$

$$u(t,x) = (\exp(-it(-\Delta)^{a/2})f)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi - it|\xi|^a}\hat{f}(\xi)d\xi,$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx\ .$$

Lemma 1. $n\geq 2,\ a>0,\ \dfrac{1-a}{2}<s<\dfrac{n-a}{2}$

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |(-\Delta)^{-s/2}u(t,x)|^2\frac{1}{|x|^{a+2s}}dxdt \leq A(n,s,a)\int_0^\infty \left(\int_{S^{n-1}} |\hat{f}(r\omega)|^p d\omega\right)^{2/p} r^{n-1}dr,$$

where $p = \dfrac{2(n-1)}{n+a-2+2s}$,

$$A(n,s,a) = \frac{1}{a(2\pi)^{n-1}(4\pi)^{s+\frac{a}{2}}}\left(\frac{\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{\frac{2s+a-1}{n-1}}\frac{\Gamma(s+\frac{a-1}{2})}{\Gamma(s+\frac{n+a}{2}-1)}\frac{\Gamma(\frac{n-a}{2}-s)}{\Gamma(\frac{a}{2}+s)}$$

Equality is attained if and only if f satisfies

$$\hat{f}(r\omega) = \frac{h(r)}{(1-\zeta(r)\cdot\omega)^{\frac{n+a}{2}+s-1}}$$

with $h:\mathbb{R}_+\to\mathbb{C}$ and $\zeta:\mathbb{R}_+\to B^n=\{x\in\mathbb{R}^n; |x|<1\}.$

Proof of Lemma 1

$$(-\Delta)^{-s/2}u(t,x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi - it|\xi|^a}|\xi|^{-s}\hat{f}(\xi)d\xi$$

$$= (2\pi)^{-n}\int_0^\infty \int_{S^{n-1}} e^{ir\omega\cdot x - itr^a} r^{n-1-s}\hat{f}(r\omega)d\omega dr$$

$$= \frac{1}{a(2\pi)}\int_0^\infty \int_{S^{n-1}} e^{ir^{1/a}\omega\cdot x - itr} r^{\frac{n-a-s}{a}}\hat{f}(r^{1/a}\omega)d\omega dr,$$

$$\int_{-\infty}^\infty |(-\Delta)^{-s/2}u(t,x)|^2 dt = \frac{1}{a^2(2\pi)^{2n-1}}\int_0^\infty |\int_{S^{n-1}} e^{ir^{1/a}\omega\cdot x} r^{\frac{n-a-s}{a}}\hat{f}(r^{1/a}\omega)d\omega|^2 dr$$

$$= \frac{1}{a(2\pi)^{2n-1}}\int_0^\infty |\int_{S^{n-1}} e^{ir\omega\cdot x} r^{n-\frac{a+1}{2}-s}\hat{f}(r\omega)d\omega|^2 dr,$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |(-\Delta)^{-s/2}u(t,x)|^2 \frac{1}{|x|^{a+2s}}dxdt$$

$$= \frac{1}{a(2\pi)^{2n-1}}\int_{\mathbb{R}}\int_0^\infty\int_{S^{n-1}}\int_{S^{n-1}} r^{2n-a-1-2s}\hat{f}(r\omega)\overline{\hat{f}(r\sigma)}e^{ir(\omega-\sigma)\cdot x}d\omega d\sigma dr\frac{1}{|x|^{a+2s}}dx$$

$$= \frac{1}{a(2\pi)^{2n-1}}\frac{2^\alpha \pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\int_0^\infty\int_{S^{n-1}}\int_{S^{n-1}} \frac{\hat{f}(r\omega)\overline{\hat{f}(r\sigma)}}{|\omega-\sigma|^\alpha}d\omega d\sigma\ r^{n-1}dr, \quad \alpha = n-a-2s.$$

The lemma follows by the sharp Hardy-Littlewood-Sobolev inequality on the unit sphere due to Lieb :

$$\int_{S^{n-1}}\int_{S^{n-1}} \frac{g(\omega)h(\sigma)}{|\omega-\sigma|^{\alpha}} d\omega d\sigma$$

$$\leq \pi^{\frac{\alpha}{2}} \left(\frac{\Gamma(n-1)}{\Gamma(\frac{2-1}{2})}\right)^{1-\frac{\alpha}{n-1}} \frac{\Gamma(\frac{n-1-\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})} \|g; L^p(S^{n-1})\| \|h : L^p(S^{n-1})\|,$$

where $\frac{2}{p}+\frac{\alpha}{n-1}=2.$ Equality is attained if and only if

$$g(\omega)=\frac{c_1}{(1-\zeta\cdot\omega)^{n-1-\frac{\alpha}{2}}},\ h(\omega)=\frac{c_2}{(1-\zeta\cdot\omega)^{n-1\frac{\alpha}{2}}}$$

for some $c_1,\ c_2 \in \mathbb{C}$ and $|\xi|<1.$

Proof of Theorem 3

(1) $$\|\hat{f}(r\cdot); L^p(S^{n-1})\| \leq |S^{n-1}|^{\frac{1}{p}-\frac{1}{2}} \|\hat{f}(r\cdot); L^2(S^{n-1})\|,$$

(2) $$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |(-\Delta)^{-s/2} u(t,x)|^2 \frac{1}{|x|^{a+2s}} dx dt \leq B(n,s,a) \|f; L^2\|^2,$$

where $B(n,s,a) = (2\pi)^{n/2} A(n,s,a) |S^{n-1}|^{\frac{2}{p}-1}, \quad p = \dfrac{2(n-1)}{n+a-2+2s}.$

Taking $a=2$ implies $B(n,s,2) = C(n, s+\frac{1}{2})$.

Equality is attained in (1) and therefore (2) if and only if $\hat{f}(r\cdot)$ is constant almost everywhere on S^{n-1}. That is to say, equality is attained if and only if f is radial.

Proof of Theorem 1

$$u_\pm(t) = e^{\pm it(-\Delta)^{1/2}} f_\pm, \quad u(0) = f_+ + f_-, \;\; \partial_t u(0) = i(-\Delta)^{1/2}(f_+ - f_-).$$

$$(\mathcal{F}_t u_\pm)(\tau) = (2\pi)^{-n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{ix\cdot\xi \pm it|\xi|} \hat{f}_\pm(\xi) d\xi \; e^{-it\tau} dt$$

$$= (2\pi)^{1-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi \pm it|\xi|} \delta(\tau \mp |\xi|) \hat{f}_\pm(\xi) d\xi,$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |u(t,x)|^2 \frac{1}{|x|^{1+2s}} dx dt = \sum_\pm \int_{\mathbb{R}} \int_{\mathbb{R}^n} |u_\pm(t,x)|^2 \frac{1}{|x|^{1+2s}} dx dt$$

$$\leq B(n,s,1)(\|f_+; \dot{H}^2\|^2 + \|f_-; \dot{H}^s\|^2)$$

$$= \frac{1}{2} B(n,s,1)(\|u(0); \dot{H}^2\|^2 + \|\partial_t u(0); \dot{H}^{s-1}\|^2).$$