One-dimensional model equations for incompressible fluid motion

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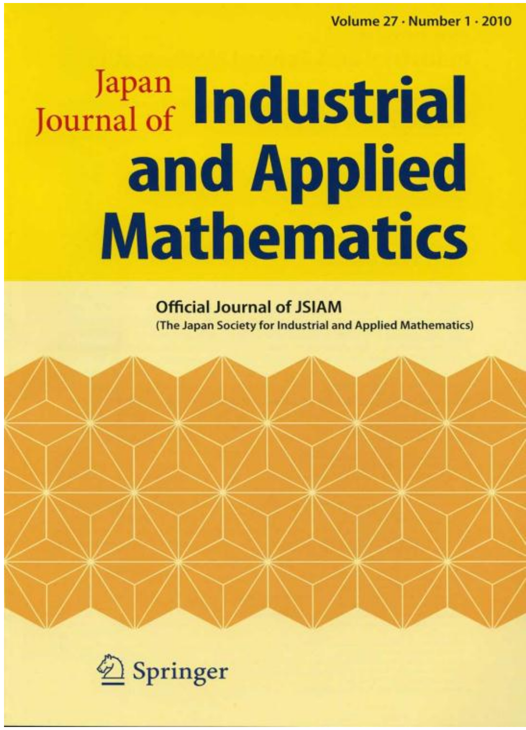
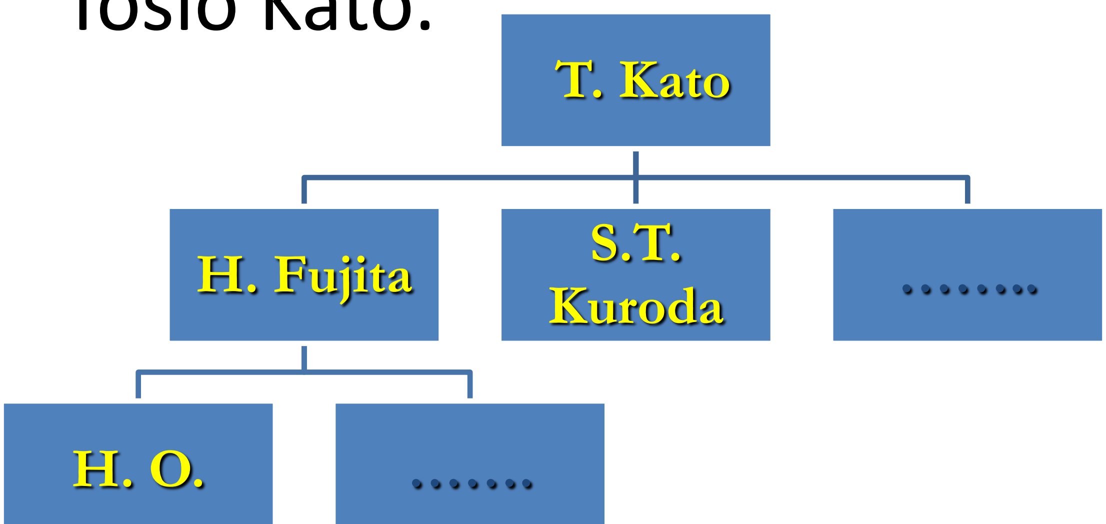
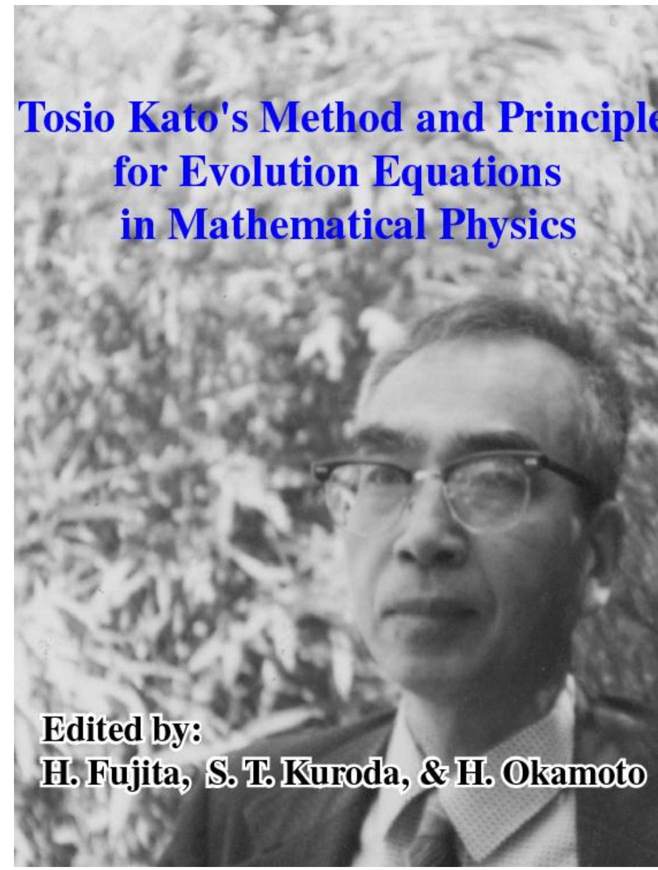


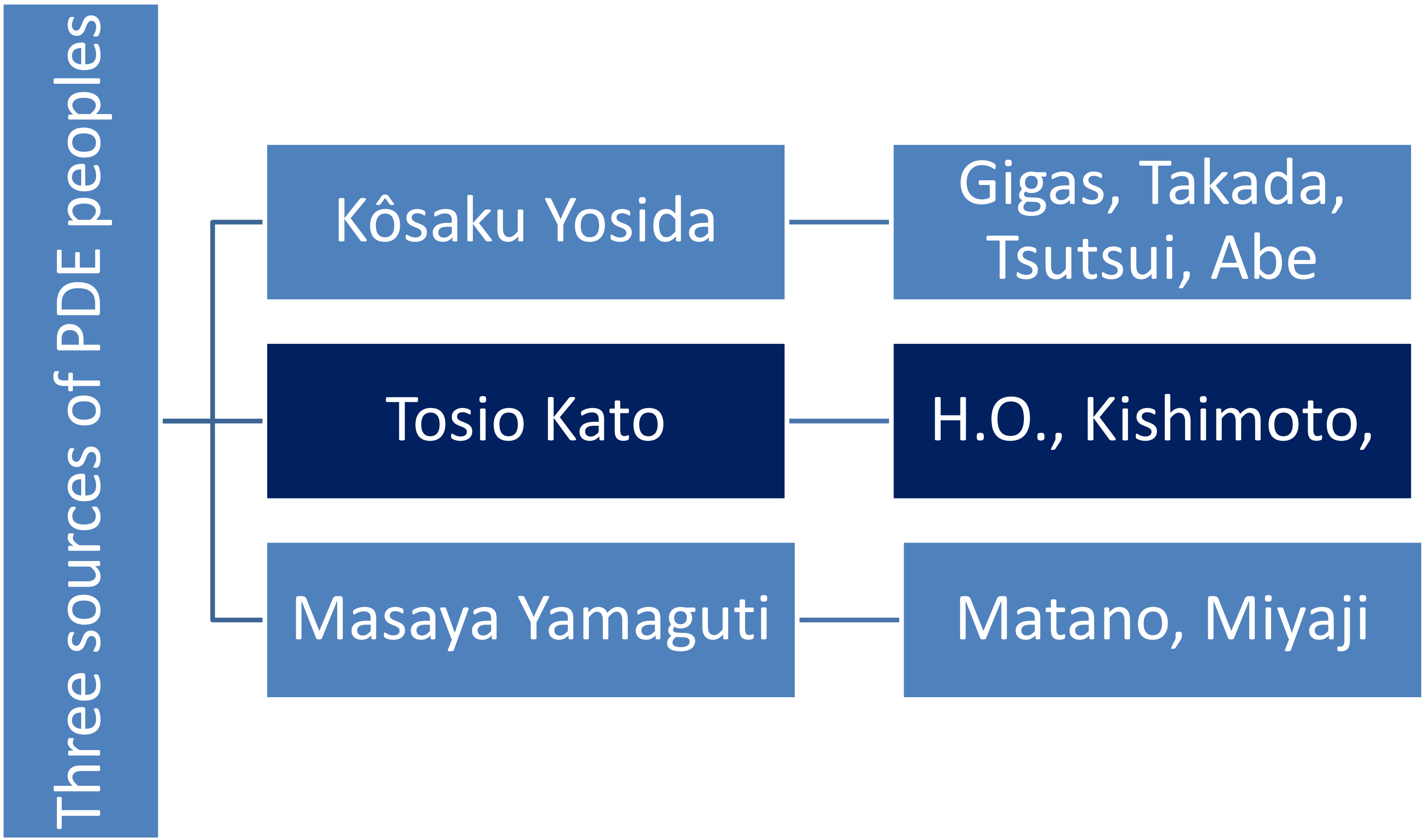
A short introduction of myself

- Educated in Univ. Tokyo---Hiroshi Fujita---Tosio Kato
- Tosio Kato died in 1999.
- His wife died in 2011 and left thousands of slides taken by Tosio Kato.





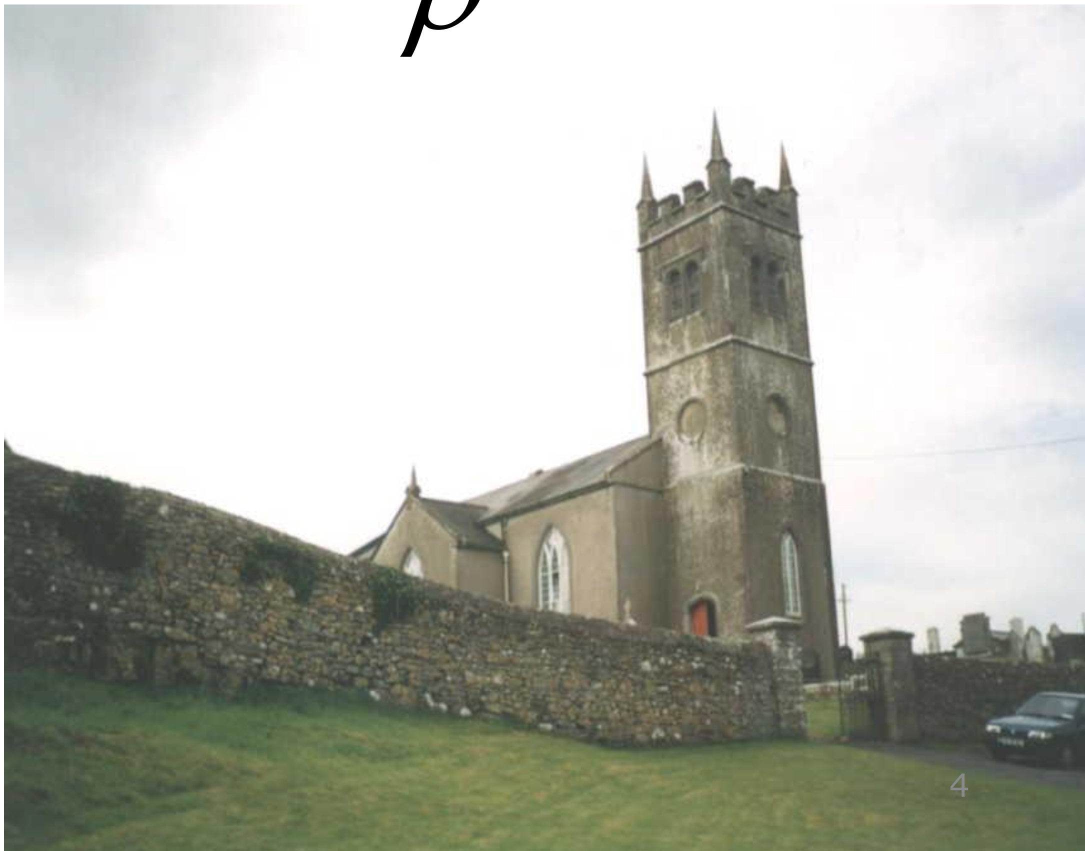
Three sources of PDE peoples



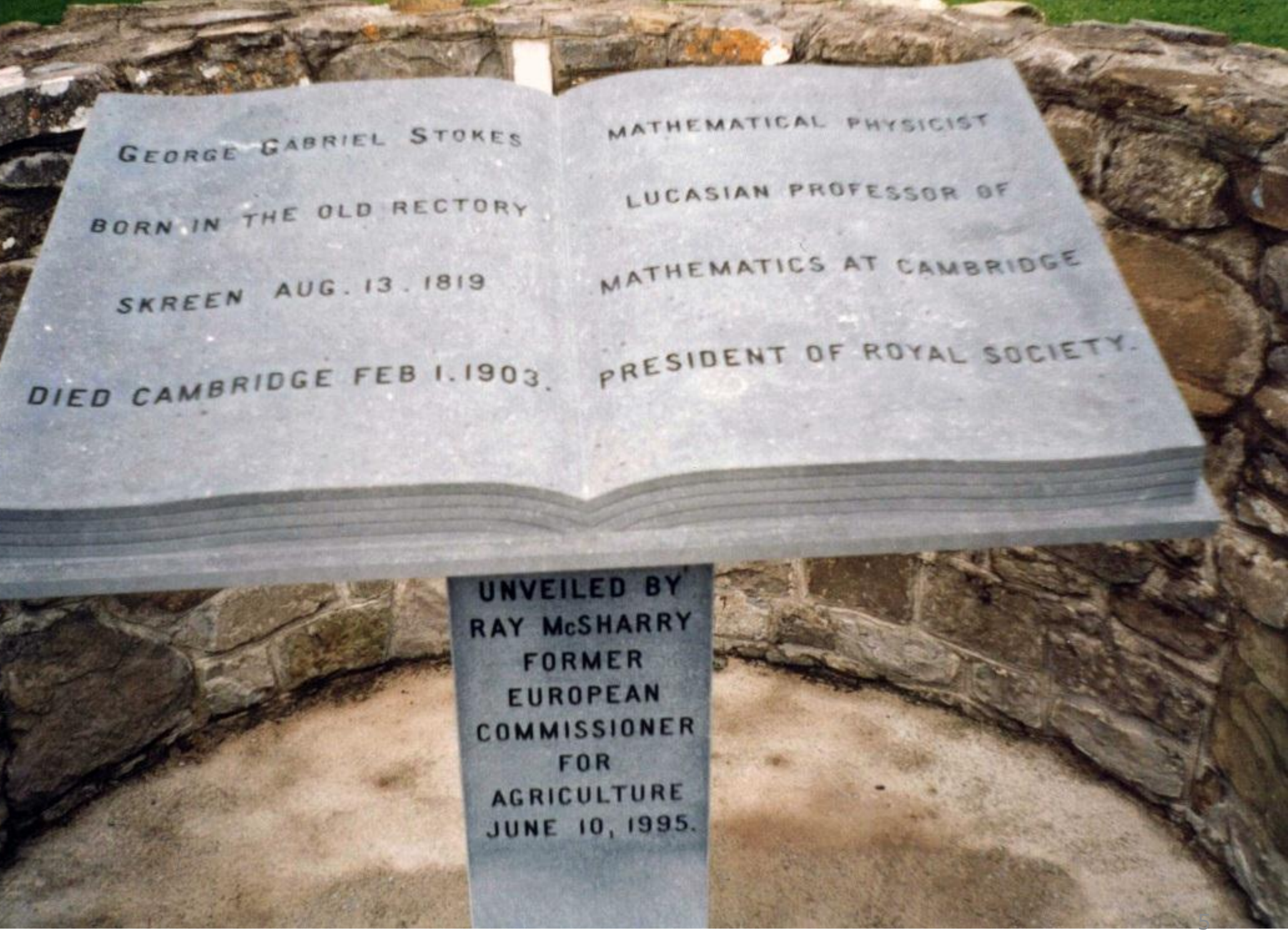
Navier-Stokes

$$\mathbf{u}_t + (\mathbf{u} \bullet \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \frac{1}{\rho} \nabla p$$

$$\operatorname{div} \mathbf{u} = 0$$



The parish where Stokes was born. His father was the parish minister.


GEORGE GABRIEL STOKES
BORN IN THE OLD RECTORY
SKREEN AUG. 13. 1819
DIED CAMBRIDGE FEB 1. 1903.
MATHEMATICAL PHYSICIST
LUCASIAN PROFESSOR OF
MATHEMATICS AT CAMBRIDGE
PRESIDENT OF ROYAL SOCIETY.
UNVEILED BY
RAY McSHARRY
FORMER
EUROPEAN
COMMISSIONER
FOR
AGRICULTURE
JUNE 10, 1995.

3D Navier-Stokes: A bad problem

Turbulence is a bad Problem!? How about the NS itself?

Try simpler models:

- **Burgers** **('15 Bateman, '39 Burgers)**
- **Proudman--Johnson eq.** **('62)**
- **Fujita's eq.** $u_t = \Delta u + u^p$ ('66)
- **De Gregorio** **('90)**
- **Strain-vorticity dynamics** **(unbounded sol.)**
- **Quasi-geostrophic eq.**
- **Many others.**

Navier-Stokes is nonlinear & nonlocal

- Navier-Stokes eqns. are *integro-differential* eqns. rather than differential eqns.

$$\boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = \nu \Delta \boldsymbol{\omega}$$

$$\mathbf{u} = (\mathrm{curl})^{-1}\boldsymbol{\omega}, \quad \mathrm{Biot-Savart}$$

$$\mathbf{u}(t,x) = \frac{-1}{4\pi}\iiint \frac{x - \xi}{|x - \xi|^3}\boldsymbol{\omega}(t,\xi)d\xi$$

$$\mathrm{nonlocal} \Leftrightarrow \nabla p \Leftrightarrow \mathrm{Helmoltz\ decom.}$$

Therefore models must be nonlinear & nonlocal.

model ①

The Proudman-Johnson equation. ‘62

- Derived from 2D Navier-Stokes

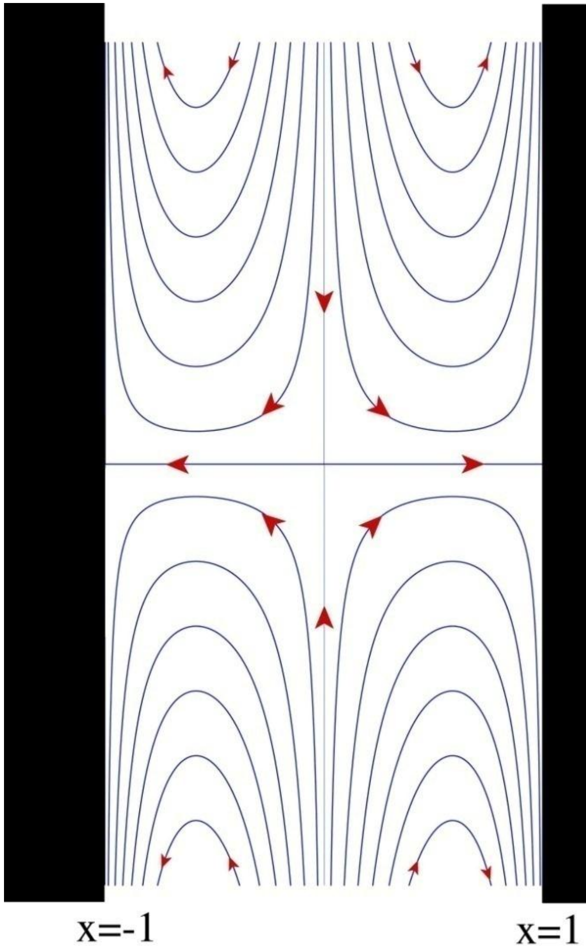
$$\mathbf{u} = \left(u(t,x), -yu_x(t,x)\right)$$

(unbounded solution of NS)

$$u_{txx} + uu_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$

$$(0 < t,\ 0 < x < 1)$$

periodic BC & $u_{xx}(0,x) = -\phi(x)$



Global existence or finite time blow-up?

$$u_{txx} + uu_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$

$$\omega = -u_{xx}$$

Order -2

$$\omega_t + u\omega_x - u_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \phi(x)$$

$$\boldsymbol{\omega}_t + (\mathbf{u}\bullet\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\bullet\nabla)\mathbf{u} = \nu\Delta\boldsymbol{\omega}$$

$$\mathbf{u} = (\mathrm{curl})^{-1}\boldsymbol{\omega}, \quad \mathrm{Biot-Savart}$$

Order -1

In 1989, a paper appeared in J. Fluid Mech.

- Finite time blow up was predicted by numerical computation.

Global existence was proved by X. Chen

Theorem. Assume that $\nu > 0$.

For any initial data $\omega(t=0)$ in $L^2(-1,1)$, a solution exists uniquely for all t and tends to zero as $t \to \infty$,

if homogeneous Dirichlet, Neumann, or the periodic boundary condition.

Xinfu Chen and O., Proc. Japan Acad., 2000.

Blow-up if non-homogeneous Dirichlet BC.????

Grundy & McLaughlin (1997).

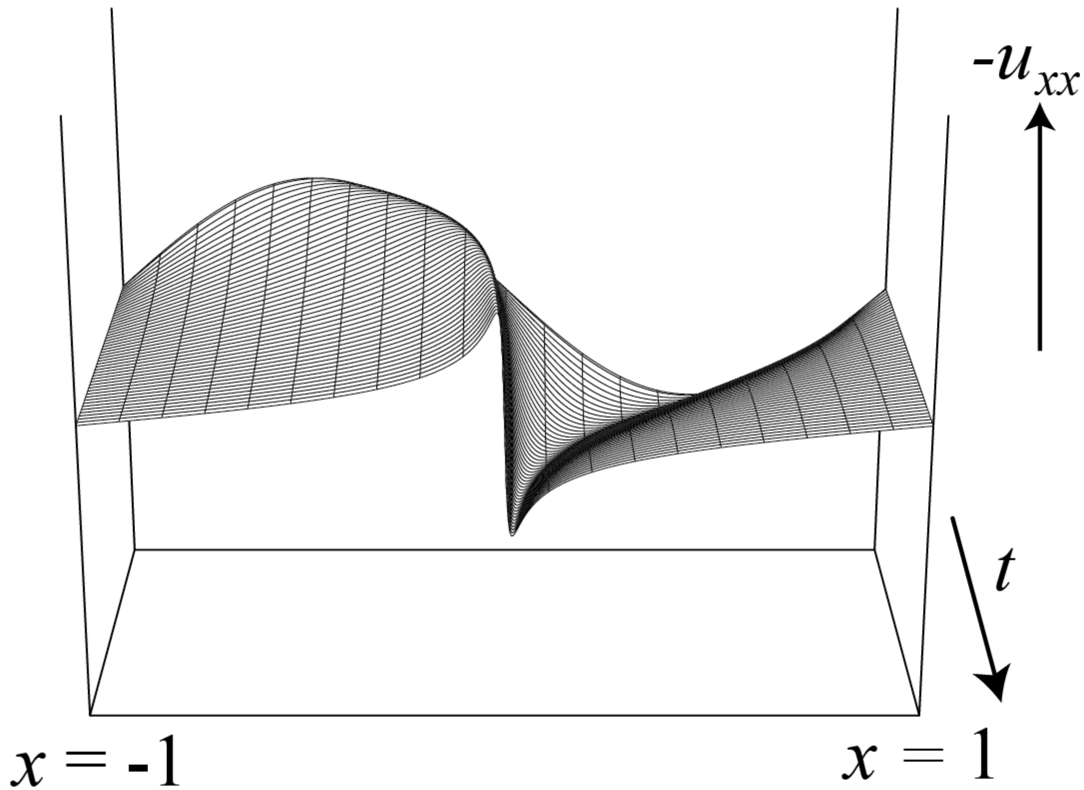
Be careful for numerical solution

- Somebody may say:

A remark on numerical experiments

- In the case of ν=0, numerical experiments are sometimes misleading.

Rigorous analysis is necessary



Prime suspect of the blow-up is the stretching term.

$$\boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = \nu \Delta \boldsymbol{\omega}$$

convection stretching diffusion

Conjecture: blow-up is caused by the stretching term.
The convection term is the by-stander.

Effect of convection term

$$\omega_t + u\omega_x - u_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

convection stretching difusion

$$u_{txx} - u_x u_{xx} = \nu u_{xxxx}$$

The convection term is NOT important in blow-up.

$$u_{tx} - \frac{1}{2}{u_x}^2 = \nu u_{xxx} + \text{constant}$$

$$U = \frac{1}{2}u_x, \quad U_t = \nu U_{xx} + U^2 - b(t)$$

Close to the Fujita eqn.

$$U_t = U_{xx} + U^2 - \frac{1}{2}\int_{-1}^{1} U(t,x)^2\,dx, \quad (0 < t, -1 < x < 1)$$

$$\int_{-1}^{1} U(t,x)\,dx = 0, \quad \text{periodic BC}$$

blow-up

$$U_t = U_{xx} + PU^2, \quad P : L^2 \to L^2 / \mathbf{R}$$

Fujita+Projection

$$\omega_{txx} + u\omega_x - u_x\omega = \nu\omega_{xx},$$

Global existence

$$\omega_{txx} \qquad - u_x\omega = \nu\omega_{xx}$$

Blow-up

A proper convection term prevents solutions from blowing-up.

(O. & J. Zhu, Taiwanese J. Math., 2000)

Budd, Dold & Stuart ('93), Zhu &O. ('00)

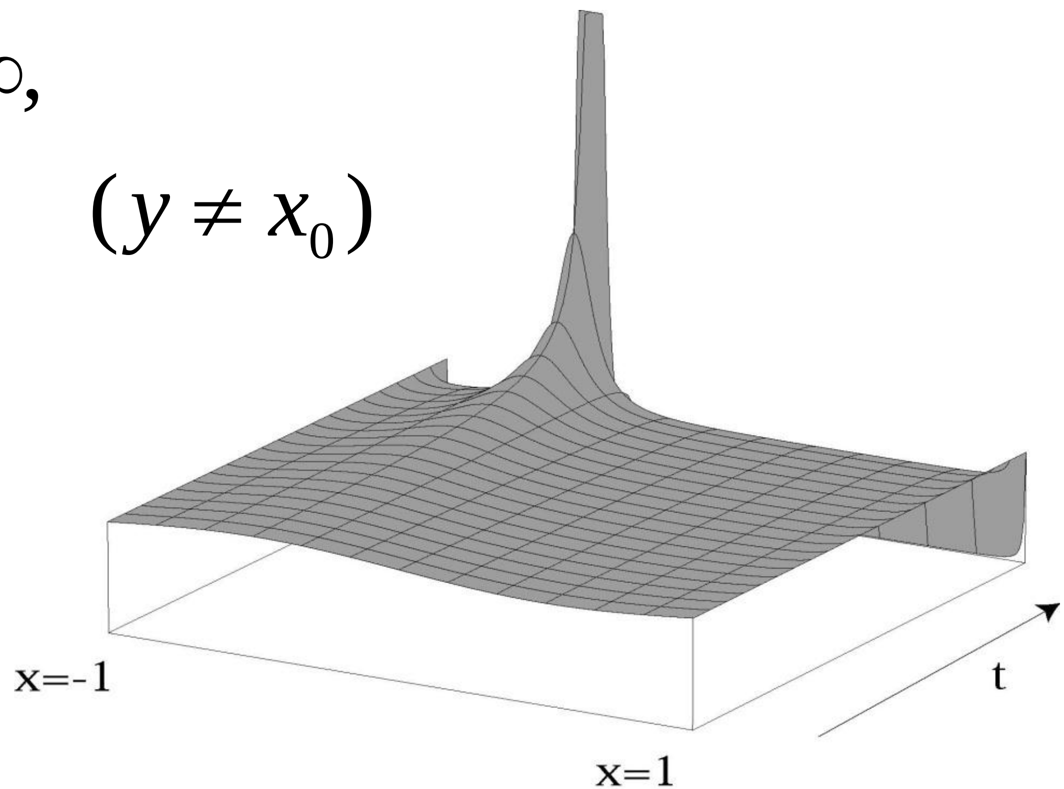
- ∃ x_0

$$u_t = \nu u_{xx} + u^2 - \int_0^1 u(t,x)^2\,dx.$$

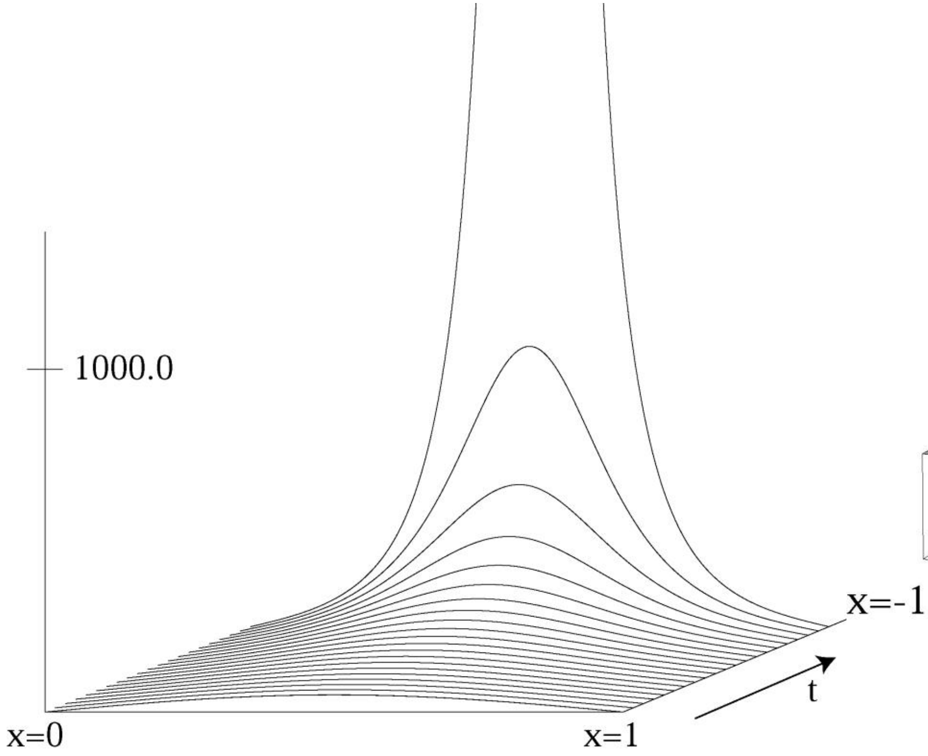
$$\lim_{t\to T} u(t,x_0) = +\infty,$$

$$\lim_{t\to T} u(t,y) = -\infty \quad (y \neq x_0)$$

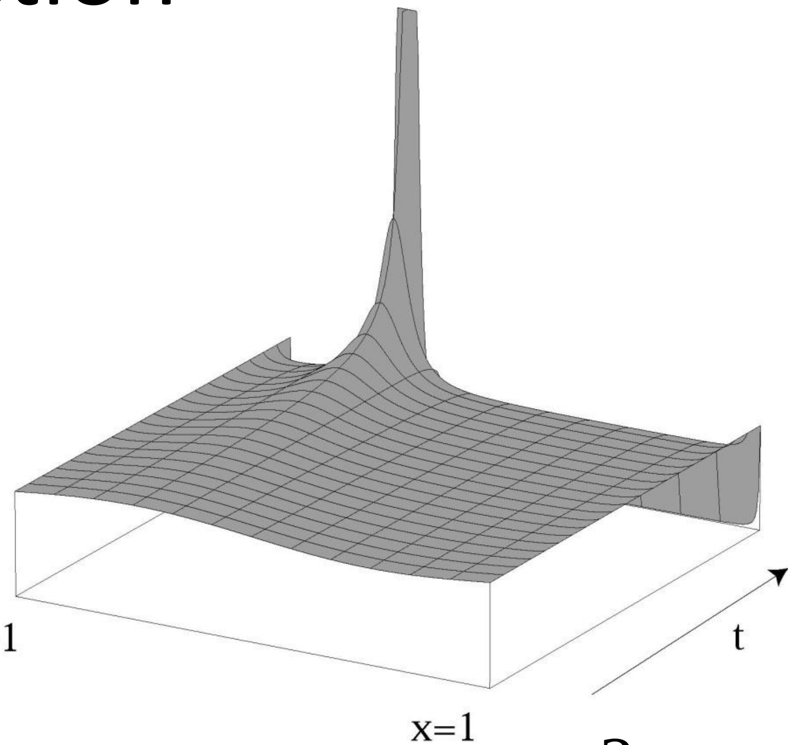
$$\lim_{t\to T} \frac{u(t,y)}{u(t,x_0)} = 0$$



Blow-up with or without the projection



$$u_t = u_{xx} + u^2$$



$$u_t = u_{xx} + Pu^2$$

Everywhere blow-up is likely
Proof?

model ② Generalized Proudman-Johnson equation

- A model:

$$\omega_{txx} + u\omega_x - a u_x \omega = \nu \omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \phi(x)$$

❶ a = -(m-3)/(m-1), axisymmetric exact solutions of the Navier-Stokes eqns in $\boldsymbol{R}^m$.

❷ a = 1 (m=2) Proudman-Johnson eqn

❸ a = -2, ν=0. Hunter-Saxton equation ('91)

❹ a = -3 the Burgers equation ('46)

$$\frac{d^2}{dx^2} \quad u_t + uu_x = \nu u_{xx} \quad \Rightarrow \quad u_{txx} + uu_{xxx} + 3u_x u_{xx} = \nu u_{xxxx}$$

Prime suspect of the blow-up is the stretching term.

$$\boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = \nu \Delta \boldsymbol{\omega}$$

convection stretching diffusion

$$\omega_{txx} + u\omega_x - au_x\omega = \nu\omega_{xx}$$

Conjecture: blow-up for large $|a|$
global existence for small $|a|$.

Xinfu Chen's proof of global existence

- X. Chen and O., Proc. Japan Acad., vol. 78 (2002),
- periodic boundary condition.

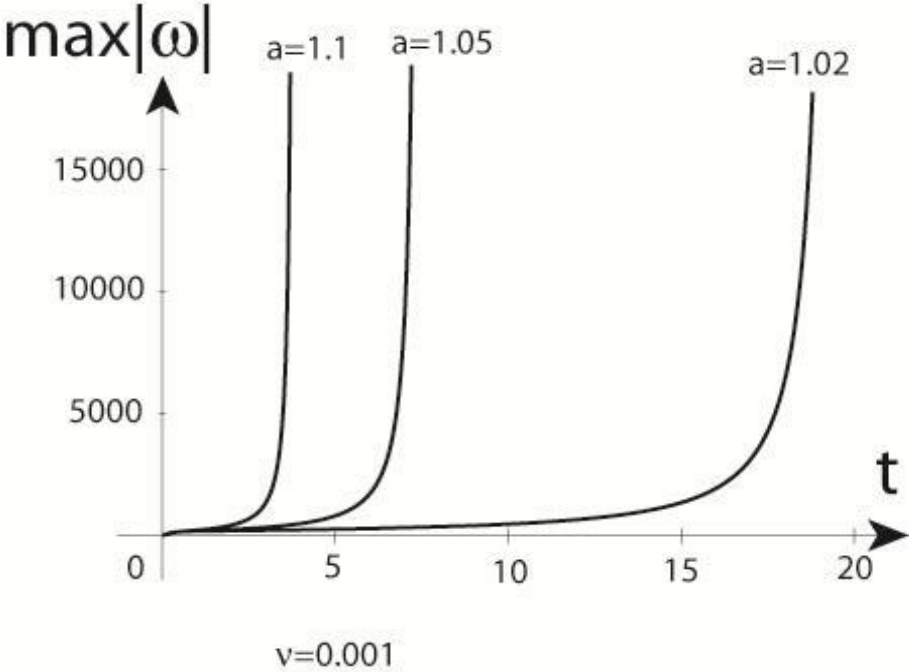
Burgers

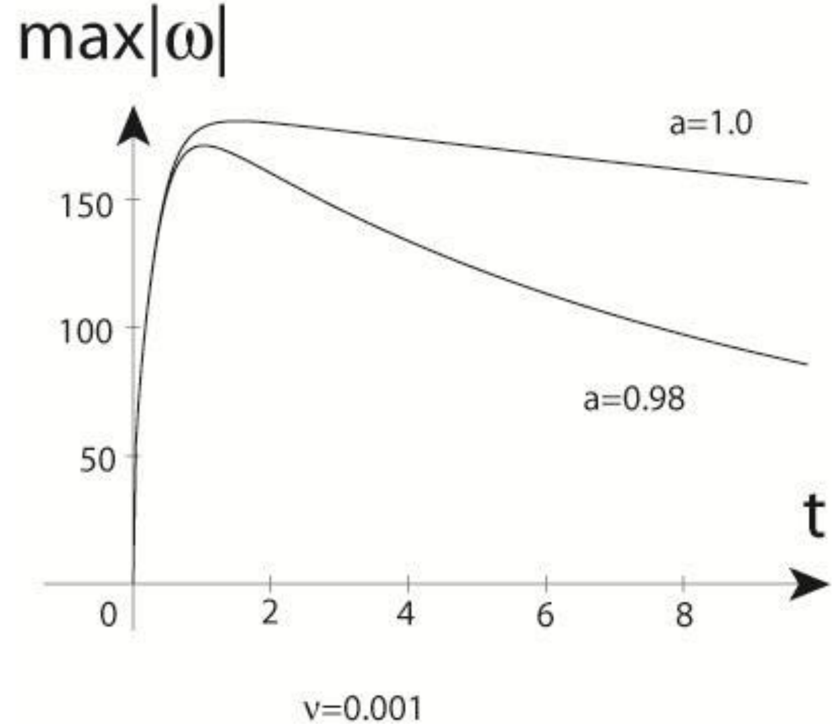
Proudman-Johnson

- **THEOREM.** If $0 < \nu$ & $-3 \leqq a \leqq 1$, the solutions exist globally in time for all initial data.

If $a < -3$, or $1 < a$, then …

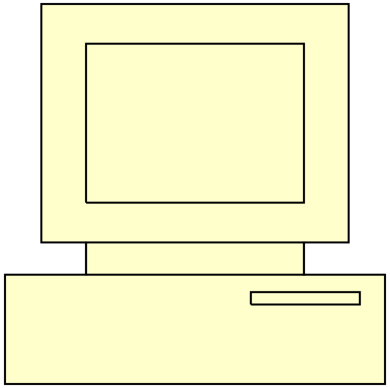
- Global existence for small initial data. Blow-up for large initial data --- numerical evidence but no proof.



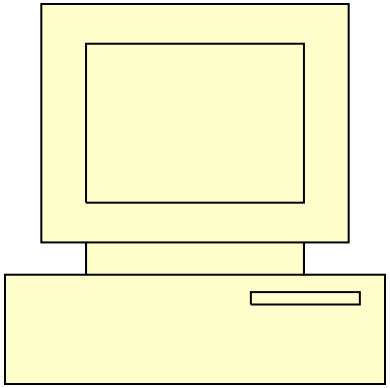


$a = 1$ is a threshold.

Numerical experiments

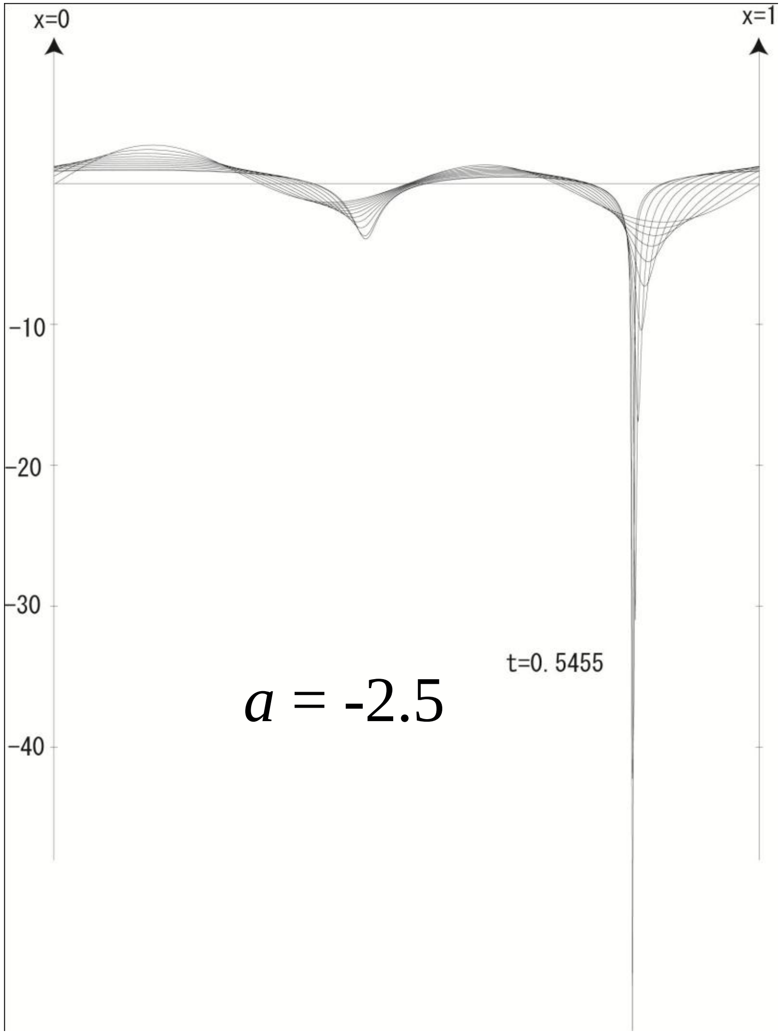


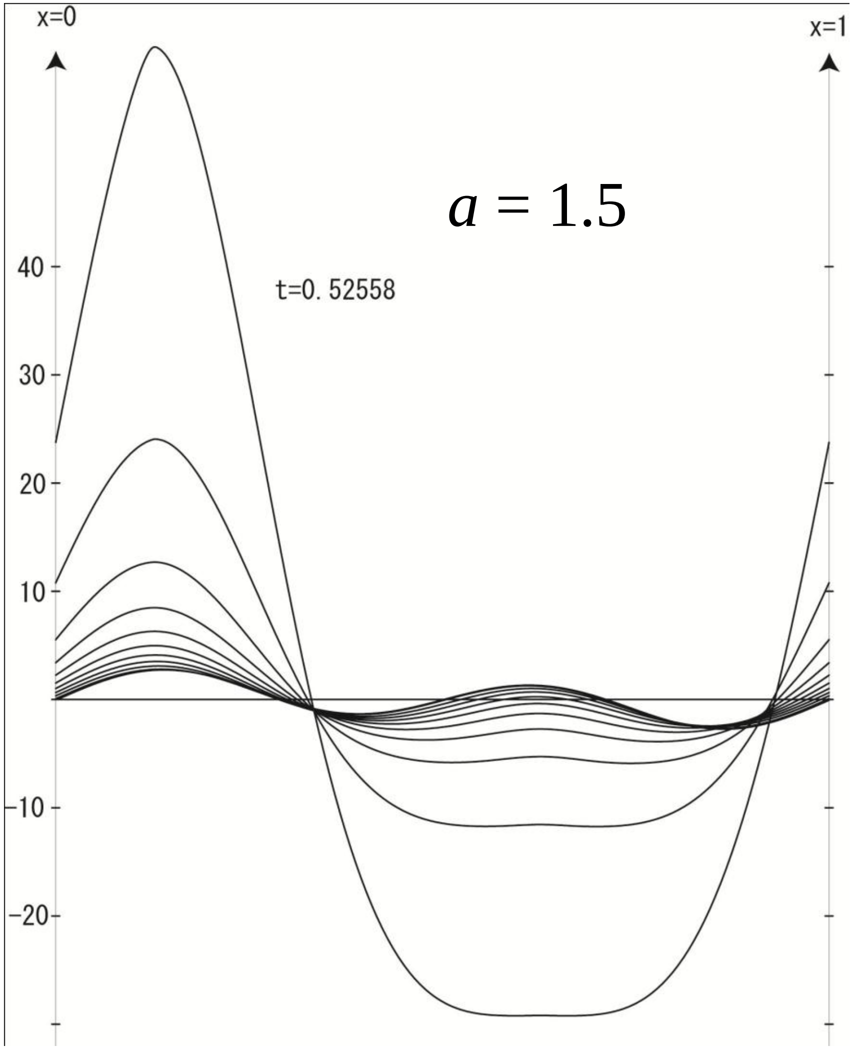
$$\omega_0 = 30\sin(2\pi x)$$
$$a = 10$$



$$\omega_0 = \sin(2\pi x)$$
$$a = 10$$
$$\omega / \max|\omega|$$

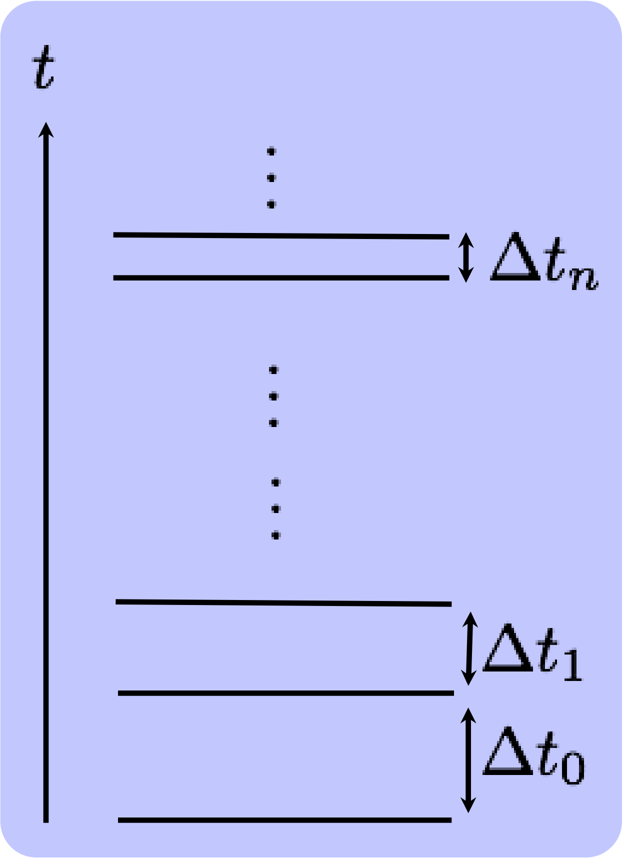
If $1 < a$, we expect blow-up occurs even for smooth initial data.





- Nakagawa's method(1976)
adaptive Δt_n

- W. Ren & X.-P. Wang's
iterative grid redistribution
method(2000)
adaptive Δx_n



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Initial data

$$
\begin{aligned}
u_0(x) &= 300\sin(2\pi x)\\
u_0(x) &= 200\sin(2\pi x) + 400\cos(2\pi x)\\
u_0(x) &= 300\sin(2\pi x) - 200\cos(4\pi x)\\
u_0(x) &= 250\sin(4\pi x) + 100\cos(2\pi x)\\
u_0(x) &= 300\sin(2\pi x) - 200\cos(2\pi x) + 100\sin(4\pi x)\\
u_0(x) &= 200\sin(2\pi x) - 100\cos(2\pi x)\\
&\qquad +50\sin(4\pi x) + 75\cos(4\pi x)
\end{aligned}
$$

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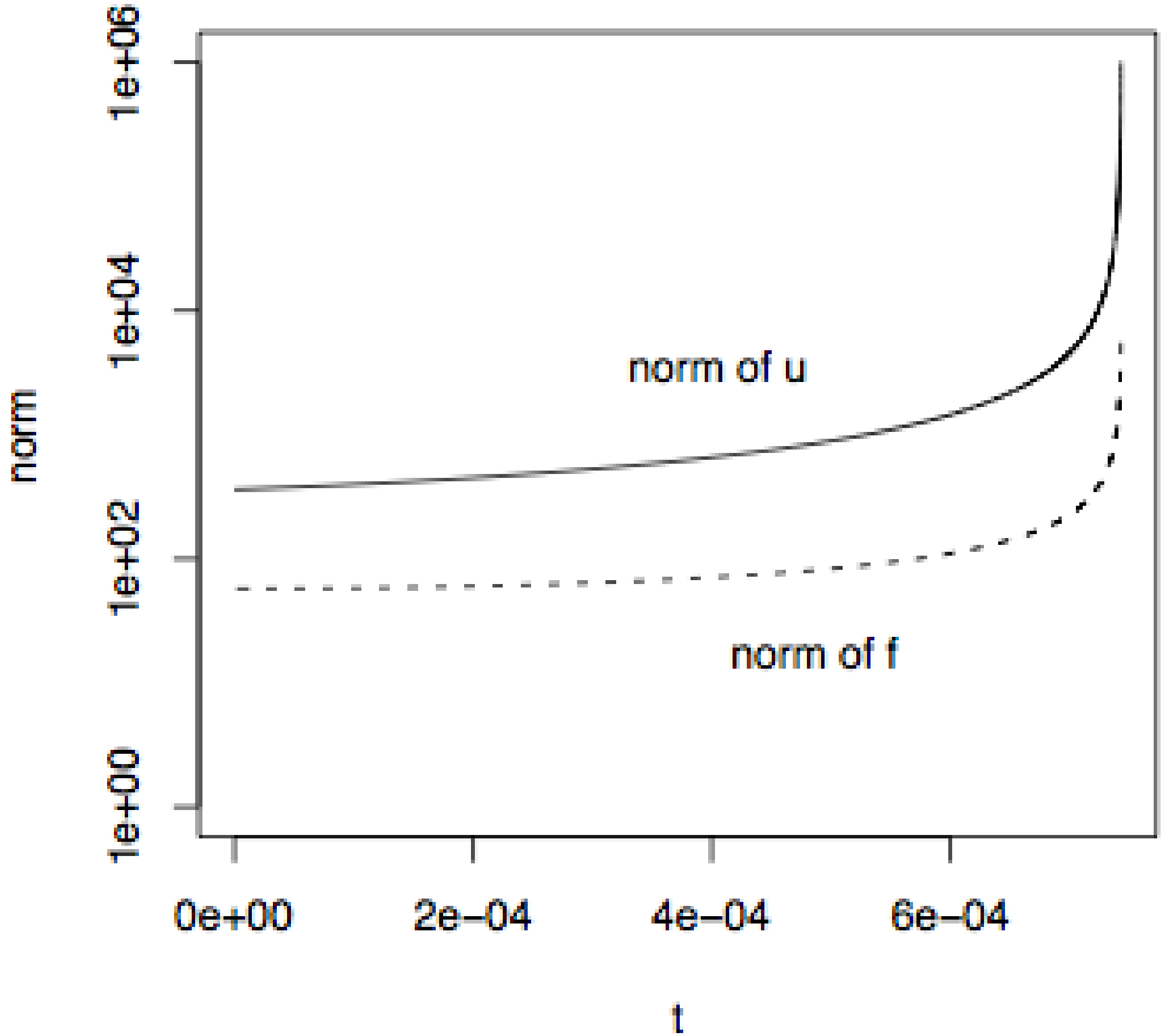
Max norm of u & u_x



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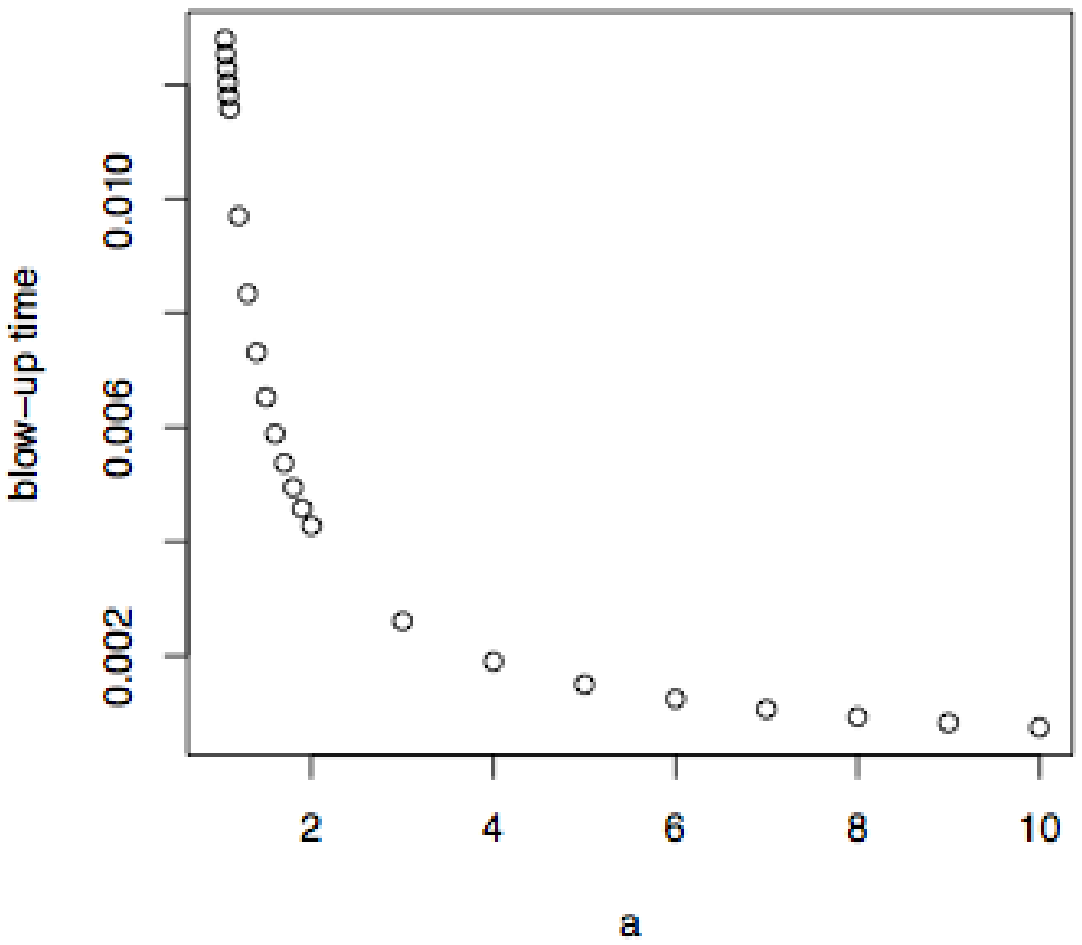
Blow-up time versus a



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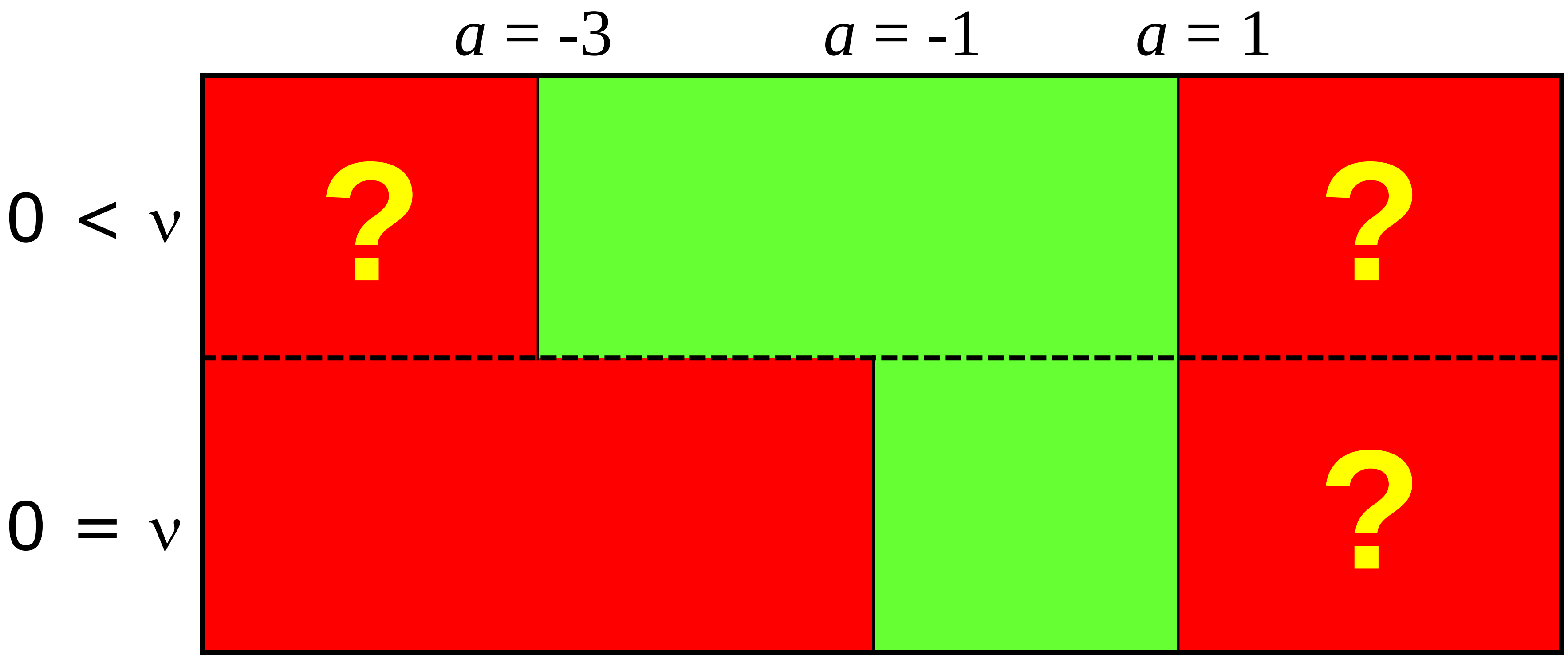
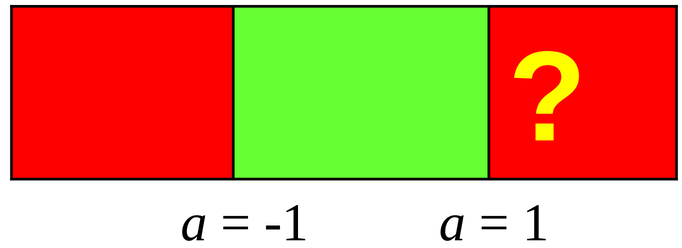
Current Status



O. J. Math. Fluid Mech. 2009

Summary for $\nu = 0$



O. J. Math. Fluid Mech. 2009

- Blow-up for $-\infty < a < -1$. (Remember that the solutions can exist globally in this region if $\nu > 0$. Viscosity helps global existence.)

- Global existence if $-1 \leqq a < 1$ & if smooth initial data.

- Self-similar, non-smooth blow-up solutions exist for $-1 < a < \infty$.

- So far, I have no conclusion in the case of $1 < a$.

Weak sol. of the generalized PJ

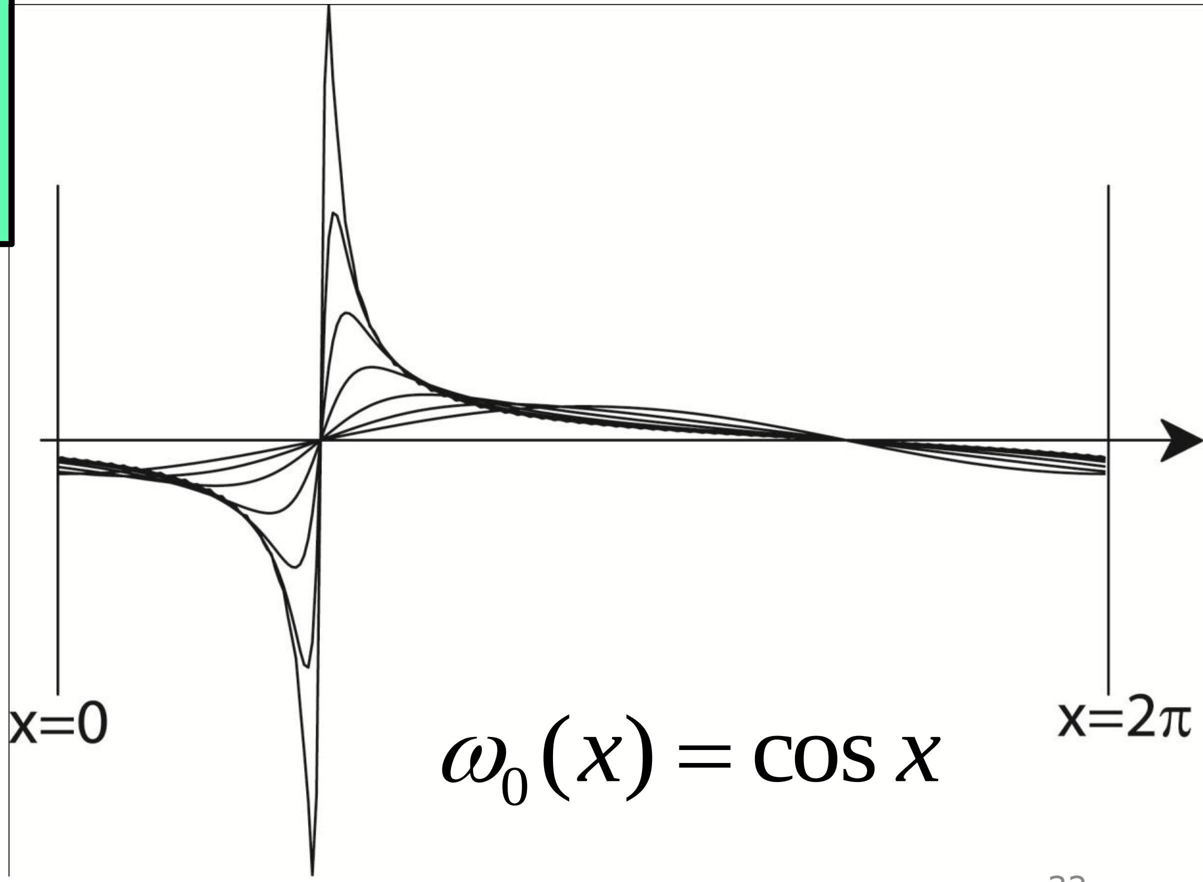
Cho & Wunsch, (2010), $a = -(n+2)/(n+1)$

model ③ Constantin-Lax-Majda

$$\omega_t - \omega u_x = 0$$

$$u_x = H\omega$$

A necessary and sufficient condition is known (Constantin, Lax, & Majda 1985).



$$\omega_0(x) = \cos x$$

De Gregorio '90

$$\omega_t + u\omega_x - \omega u_x = 0$$

$$u_x = H\omega$$

Global existence???

Does the convection term delete the blow-up?

$$\omega_t + au\omega_x - \omega u_x = 0$$

O, Sakajo & Wunsch '08

$$u_x = H\omega, \quad a \in \mathbf{R}$$

$-\infty < a \leqq 0.$ Blow-up

Castro & Cordoba '09

Constantin-Lax-Majda & De Gregorio & Proudman-Johnson can be unified.

$$\omega_{txx} + u\omega_x - au_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \phi(x)$$

$$\omega_{txx} + u\omega_x - au_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-\beta/2}\omega$$

$$\omega(0,x) = \phi(x)$$

$\beta = 1$ & $a = \infty$ ⇒ Blow-up Constantin-Lax-Majda `85

$\beta = 1$ & $a = 1$ ⇒ ??? De Gregorio's `90

$\beta = 1$ & $-\infty < a < 0$ ⇒ Blow-up Castro & Cordoba '09

Unified equation & b-equation

$$\omega_{txx} + u\omega_x + bu_x\omega = \nu\omega_{xx}, \quad u = \left(m^2 - \frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \phi(x)$$

Holm & Hone 2005

Escher & Seiler 2010

The generalized P-J with ν=0.

$$u_{txx} + uu_{xxx} - au_x u_{xx} = 0$$

$$(0 < t,\ 0 < x < 1)$$

periodicBC

$$u_{xx}(0, x) = -\phi(x)$$

- 3D axisymmetric Euler for $a = 0$.
- Hunter-Saxton model for nematic liquid crystal for a = -2.
- Burgers for a = -3.

Starting point: local existence theorem

- With a help of Kato & Lai's theorem (J. Func. Anal. '84),

$$\omega = -u_{xx}, \quad \omega_t + u\omega_x - au_x\omega = 0$$

- Locally well-posed if $\omega(0,\bullet) \in L^2(0,1)/\mathbf{R}$,

- Global existence if $\omega(0,\bullet) \in L^2(0,1)/\mathbf{R}$,

Different methods were needed for global existence/blow-up in

$$-\infty < a < -2, \quad -2 \leqq a < -1, \quad -1 \leqq a < 0, \quad 0 \leqq a < 1$$

- The case of $-\infty < a < -2$ is settled in Zhu & O., Taiwanese J. Math. (2000).

$$\phi(t) \equiv \int_0^1 u_x(t,x)^2 dx$$

$$\frac{d^2}{dt^2}\phi(t) \geq b\phi(t)^3$$

$-2 \leqq a < -1$. Follows the recipe of Hunter & Saxton (’91)

- Use the Lagrangian coordinates

$$X_t = u(t, X(t,\xi)), \quad X(0,\xi) = \xi, \quad (0 \le \xi \le 1)$$

- Define $V(t,\xi) = X_\xi(t,\xi).$

$$VV_{tt} = (V_t)^2 - I(t)V, \quad I(t) = \int_0^1 \frac{V_t^2}{V}\, d\xi$$

- V tends to $-\infty$.
- Global weak solution in the case of $a = -2$ (Bressan & Constantin ‘05).

Blow-up occurs both in $-\infty < a < -2$ and in $-2 \leqq a < -1$, but

- Asymptotic behavior is quite different.
- $\|u_x(t)\|_{L^2}$ blow up. $(-\infty < a < -2)$
- $\|u_x(t)\|_{L^2}$ is bounded. $\|u_x(t)\|_{L^\infty}$ blows up.

$(-2 \leqq a < -1)$

-1 ≦ $a < 0$. Follows the recipe of Chen & O. Proc. Japan Acad., (2002)

- Define $$\Phi(s) = |s|^{-1/a}$$
- Invariance

$$\frac{d}{dt}\int_0^1 \Phi(u_{xx}(t,x))dx = \int_0^1 \Phi'(u_{xx})[-uu_{xxx} + au_x u_{xx}]dx$$

$$= \int_0^1 [\Phi(u_{xx}) + au_{xx}\Phi'(u_{xx})]u_x dx = 0.$$

- Boundedness of $\int_0^1 \left|u_{xx}(t,x)\right|^{-1/a} dx, \quad \int_0^1 \left|u_{xx}(t,x)\right| dx$

-1 $\leqq a < 0$. Continued.

- $\|u_x(t)\|_\infty \le c$

- $u_{txx} + uu_{xxx} - au_x u_{xx} = \nu\, u_{xxxx}$ gives us

$$\frac{d}{dt}\int_0^1 u_{xx}(t,x)^2\,dx = (2a+1)\int_0^1 u_x {u_{xx}}^2\,dx$$

$$\frac{d}{dt}\int_0^1 u_{xx}(t,x)^2\,dx \le c(2a+1)\int_0^1 u_{xx}(t,x)^2\,dx$$

$0 \leqq a < 1$. Follows the recipe of Chen &O. Proc. Japan Acad., (2002)

- Define

$$\Phi(s) = \begin{cases} |s|^{1/(1-a)} & (s < 0) \\ 0 & (0 < s) \end{cases}$$

- Then $$\frac{d}{dt}\int_0^1 \Phi(u_{xxx})dx = a\int_0^1 u_{xx}^2 \Phi'(u_{xxx})dx \leq 0$$

- $\int_0^1 \left|u_{xxx}(t,x)\right| dx$ is bounded.

Non-smooth, self-similar blow-up solutions when $-1 < a < +\infty$

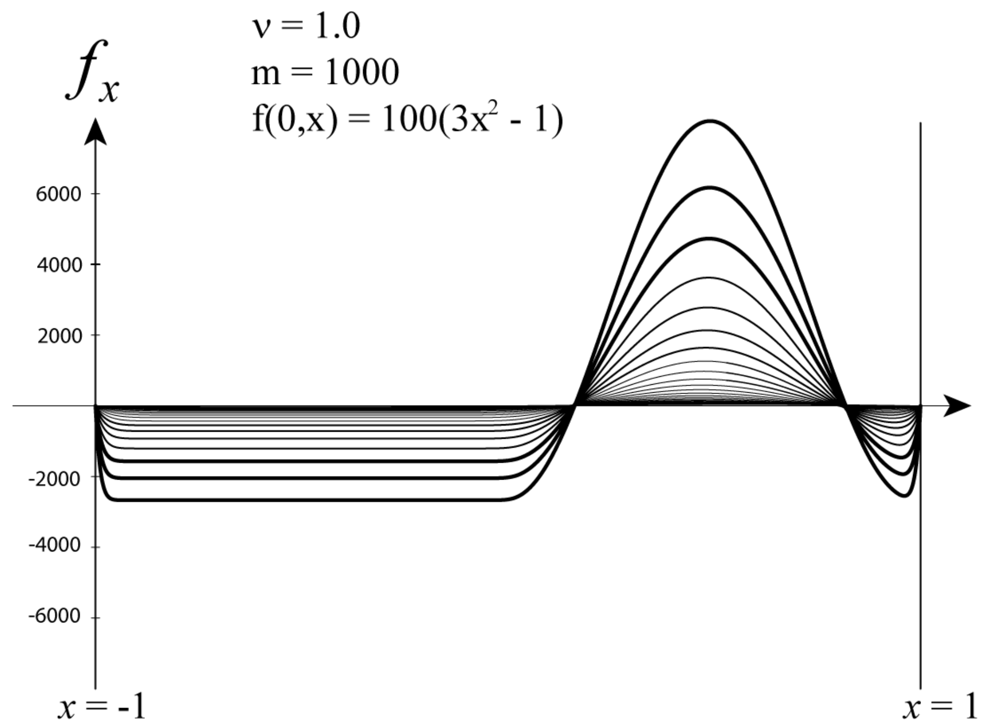
- $$u(t,x) = \frac{F(x)}{T-t}$$

$$F'' + FF''' - aF'F'' = 0.$$

- Nontrivial solution exists for all $-1 < a < +\infty$.

Another

- 3D Navier-Stokes exact sol.



$$f_{txx} + (f - Sf)f_{xxx} - (f_x - (Sf)_x)f_{xx} = \nu f_{xxxx}$$

$$Sf(t,x) = f(t,-x)$$

- Nagayama and O., '02 numerical experiment.
- Proof ???

2D Example (with K. Ohkitani)

J. Phys. Soc. Japan, vol. 74 (2005), 2737--2742

- 2D Euler

$$\omega_t + \mathbf{u} \bullet \nabla \omega = 0$$

$$\omega = \mathrm{curl}\, \mathbf{u}$$

$$\boldsymbol{\chi} = (\omega_y, -\omega_x) = -\Delta \mathbf{u}$$

$$\boldsymbol{\chi}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\chi} - (\boldsymbol{\chi} \bullet \nabla)\mathbf{u} = 0$$

The convection term is now deleted.

$$\boldsymbol{\chi} = (\omega_y, -\omega_x) = -\Delta \mathbf{u}$$

$$\boldsymbol{\chi}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\chi} - (\boldsymbol{\chi} \bullet \nabla)\mathbf{u} = 0$$

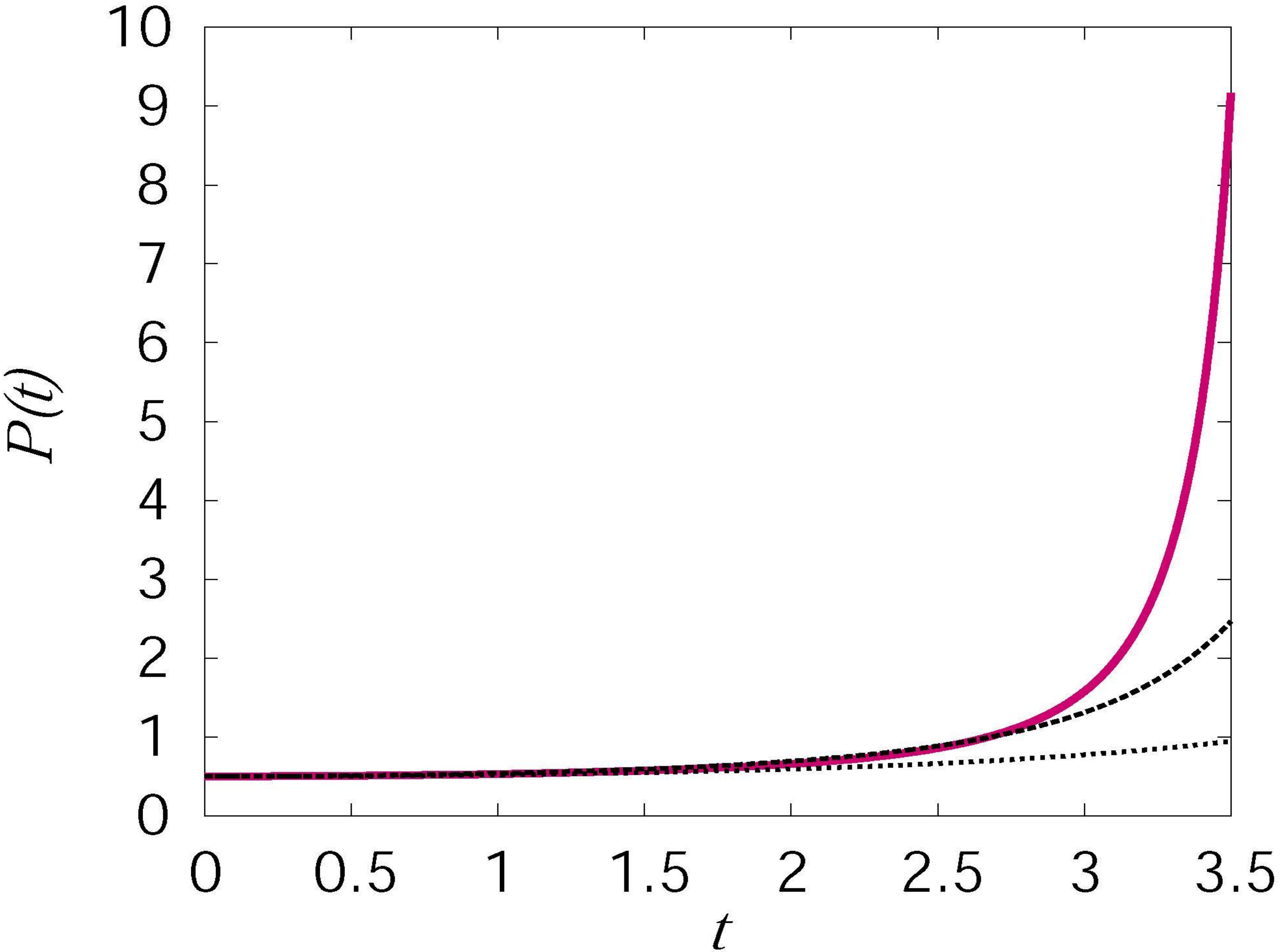
$$\boldsymbol{\chi}_t \qquad - (\boldsymbol{\chi} \bullet \nabla)\mathbf{u} = 0$$

$$\mathbf{u} = (-\Delta)^{-1} \boldsymbol{\chi}$$

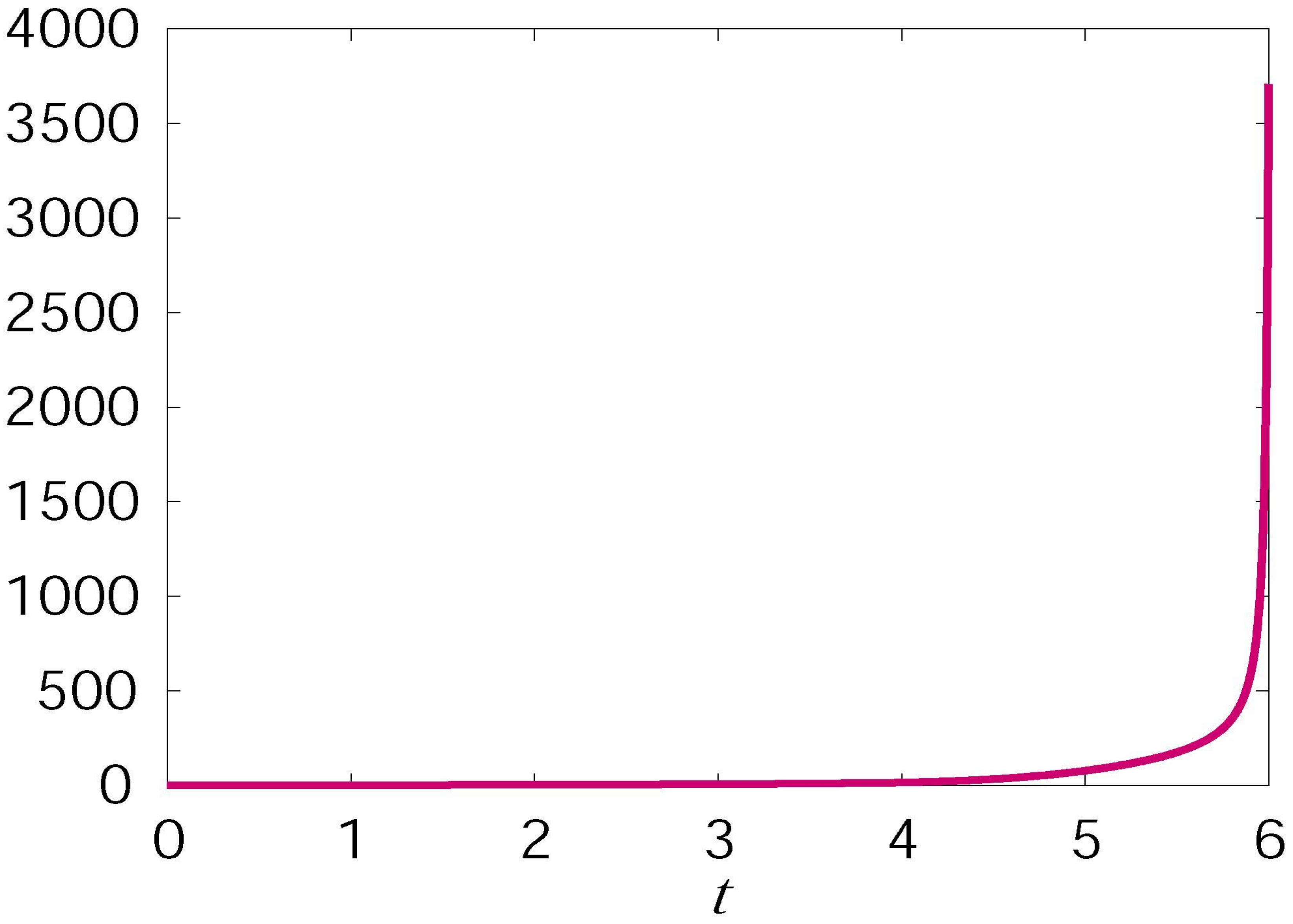
$$\boldsymbol{\chi}_t - (\boldsymbol{\chi} \bullet \nabla)\mathbf{u} = 0$$

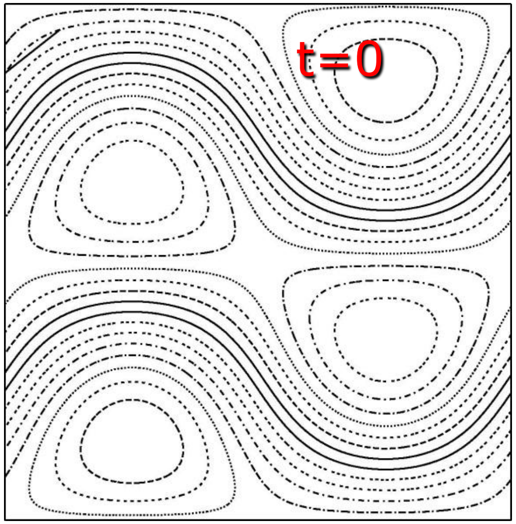
$$\mathbf{u} = P(-\Delta)^{-1} \boldsymbol{\chi}$$

L^2-norm of χ

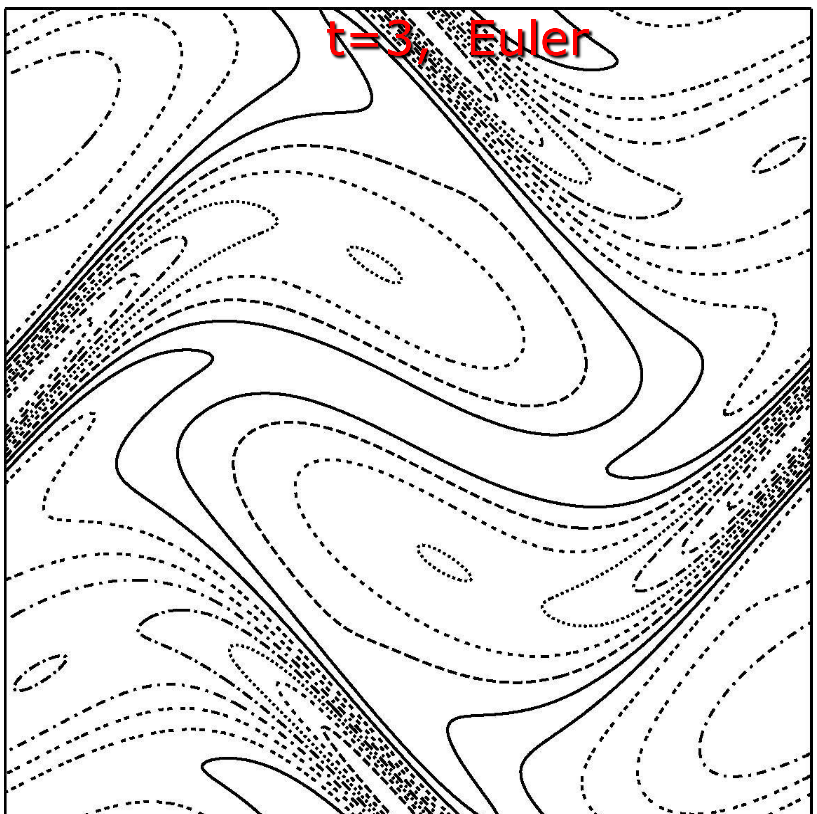


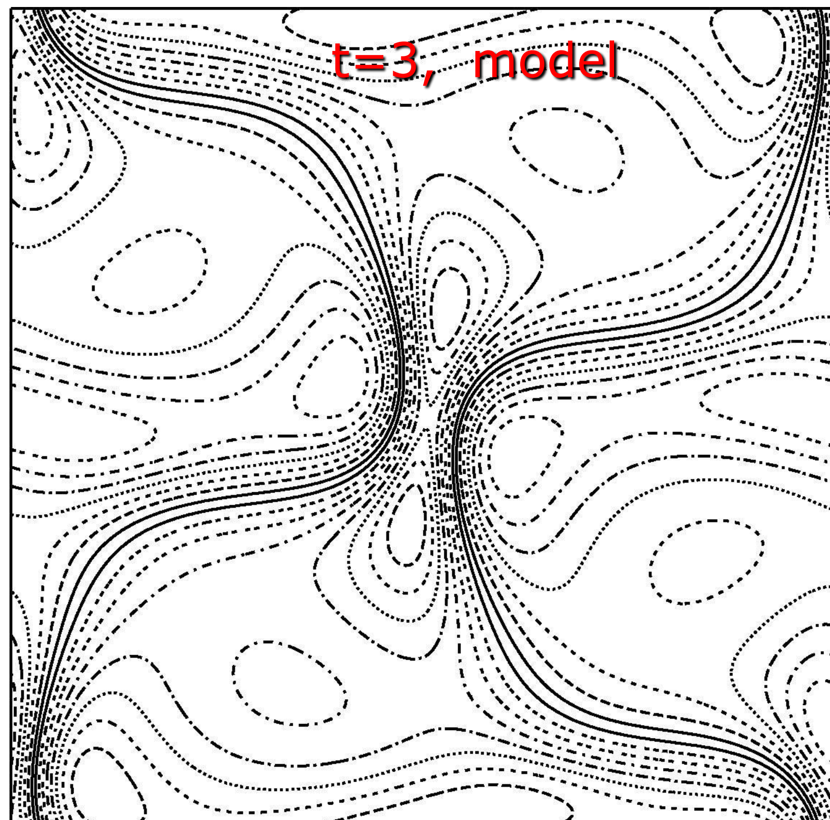
$\int_0^t |\chi(s)|_\infty dx$





$$(-\Delta)^{1/2}\omega \sim |\chi|$$





Conclusions

- Similarity solutions of the Navier-Stokes eqns can blow up in finite time: necessity of the energy inequality.
- A proper convection term prevent the solution from blowing-up. Or, at least, rapid growth is slowed down by a convection term.
- There are some cases where proof is needed.
- Blow-up behavior is very different from a nonlinear heat eqn: *the yoke of non-locality*.

Thank you very much.