# Front propagation in spatially ergodic media


Hiroshi Matano

(University of Tokyo)



UK-Japan Winter School

*"Nonlinear Analysis"*

*Royal Academy of Engineering, London,  Jan 7-11, 2013*

# Outline

1. Introduction Formulation of the problem
2. Basic concepts Ergodicity and traveling waves
3. Main results
4. Outline of the proof
5. The case of Random media

Joint work with B. Lou and K.-I. Nakamura

and partly with J. Nolen

# 1. Introduction

## Formulation of the problem

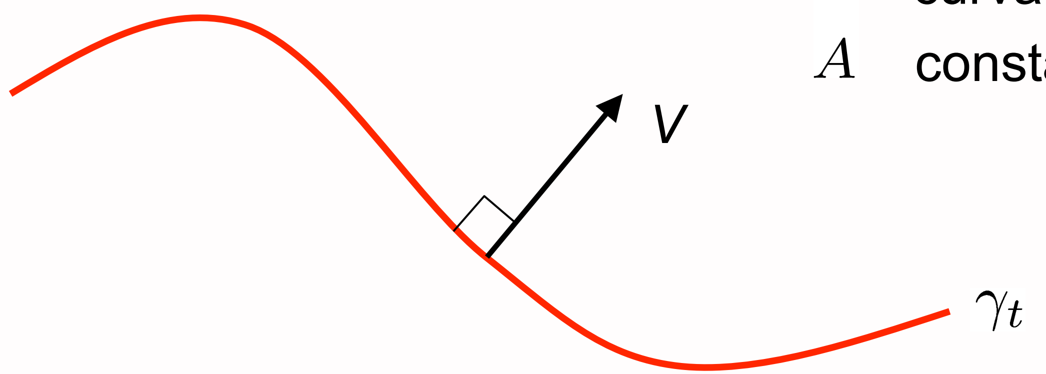
# Curvature-dependent motion of a plane curve

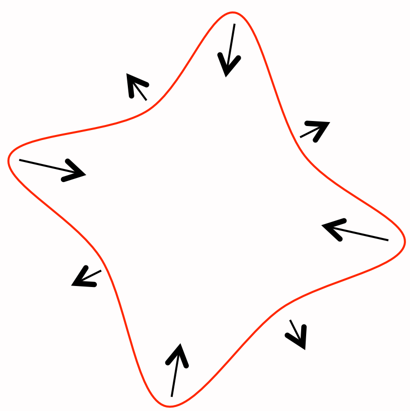
$$V = \kappa + A$$

$V$ normal velocity

$\kappa$ curvature

$A$ constant $> 0$



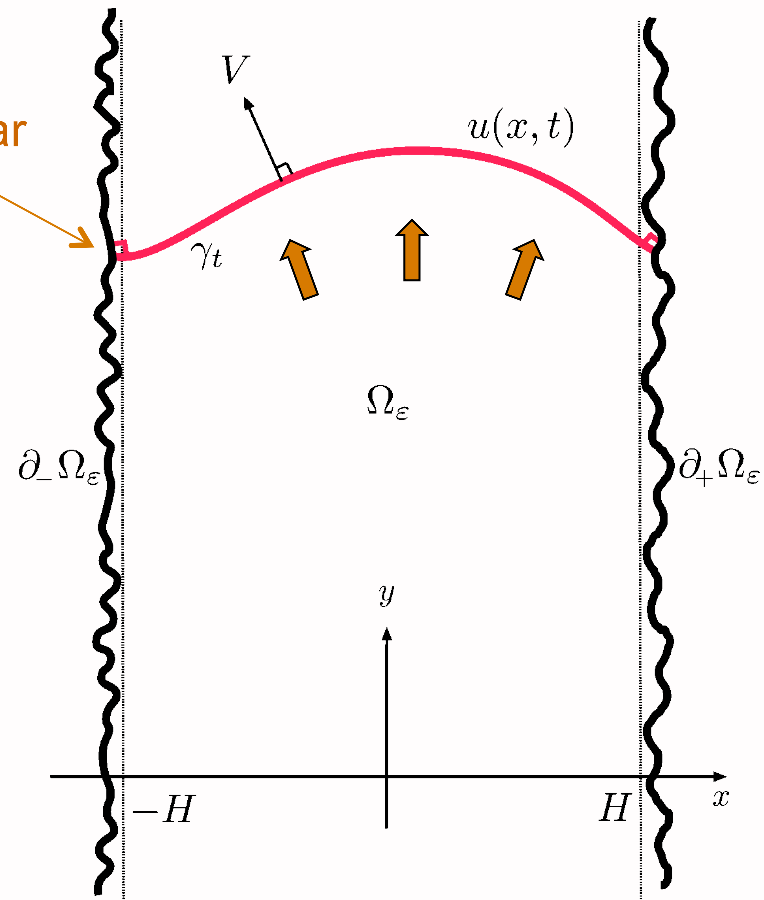



- Evolution of a phase boundary.
- Singular limit of the Allen-Cahn equation

$$\partial_t u = \Delta u + f(u)$$

$$V = \kappa + A$$

| | | |
|---|---|---|
| $V$ | normal velocity | |
| $\kappa$ | curvature | ← surface tension |
| $A$ | constant $> 0$ | ← pressure from behind |

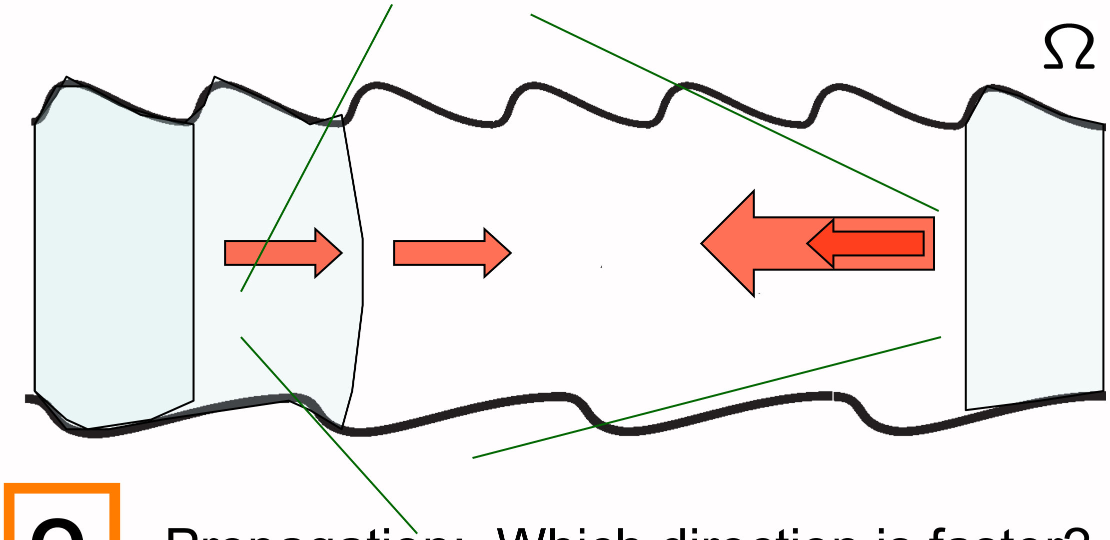



How does the front move in the presence of rugged boundary?

## Motivations

1. To study the effect of geometry on the speed of propagation.
2. Generalized notion of traveling waves in heterogeneous environments.
3. A good example of homogenization under non-periodic heterogeneity.

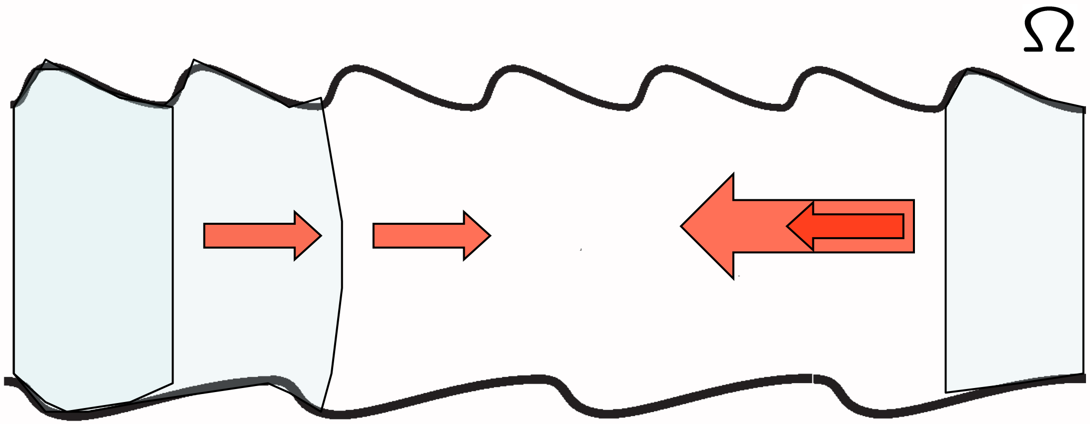
# Effect of geometry on front propagation



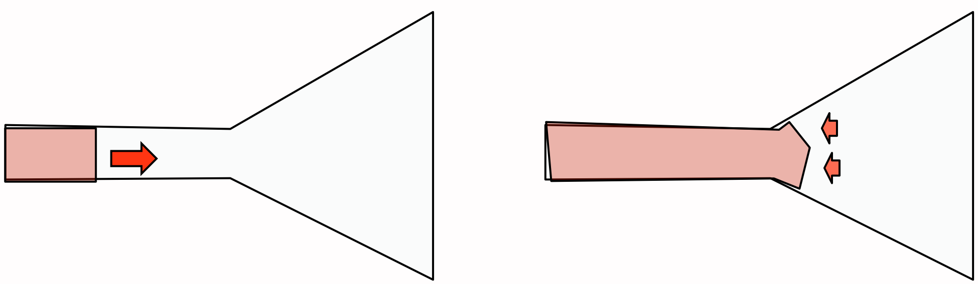

The bigger the opening angles, the slower the speed

**Q** Propagation: Which direction is faster?

# Effect of geometry on front propagation



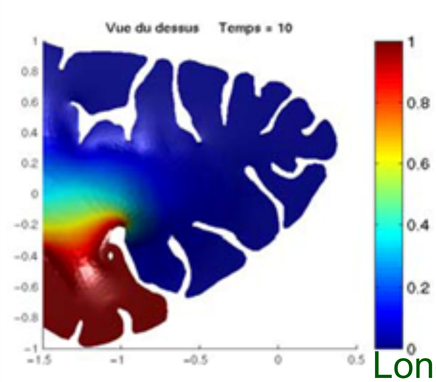

The bigger the opening angles, the slower the speed



Too rapid widening blocks propagation.

M. (1979), M.-Mimura (1983), Jimbo (1988)

The same fact was used, e.g., in the study of spreading depression (SD) by Dronne *et al* (2004).




London 2013

$$V = \kappa + A$$

$V$ normal velocity

$\kappa$ curvature

$A$ constant $> 0$

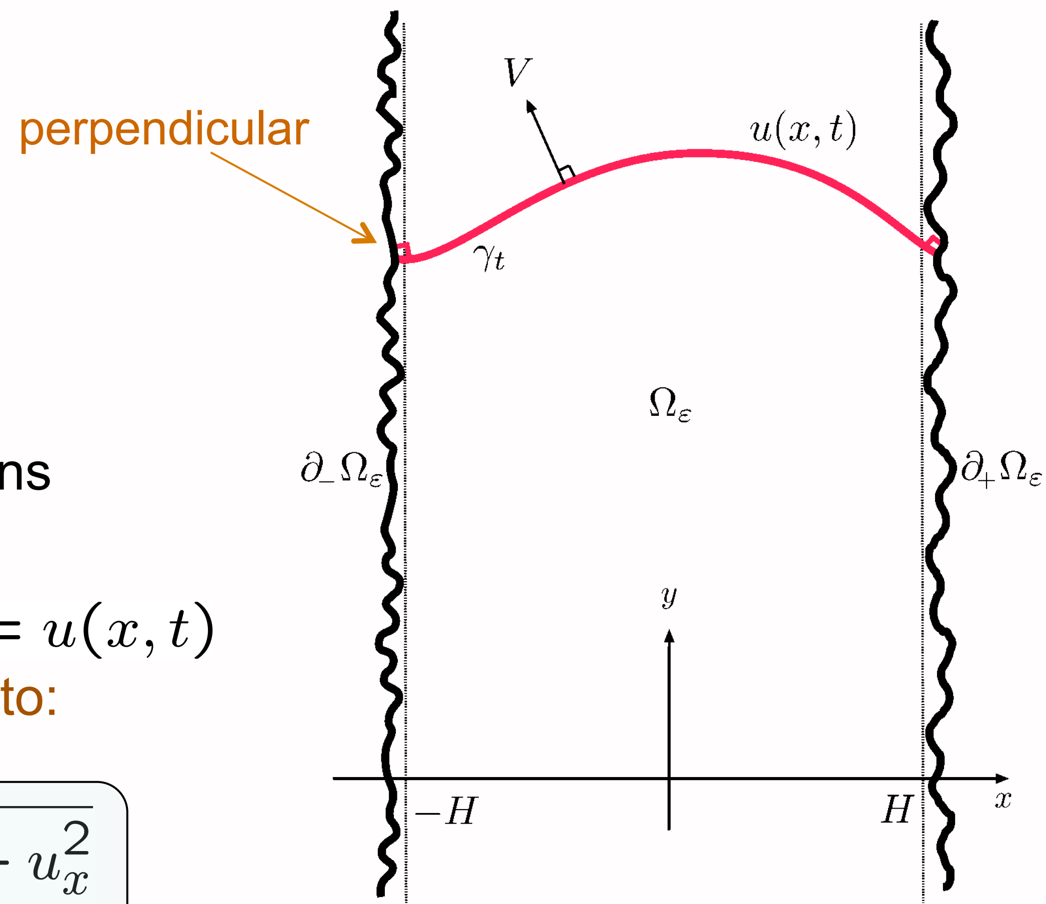
$$g_\pm^\varepsilon(y) = \varepsilon g_\pm(y/\varepsilon)$$

$g_\pm(y)$ recurrent functions

If the curve is a graph: $y = u(x,t)$
then the equation reduces to:

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}$$

$$\Omega_\varepsilon = \{-H - g_-^\varepsilon(y) < x < H + g_+^\varepsilon(y),\ y \in \mathbf{R}\}$$



## GOAL

① Existence and non-existence of traveling waves

② Average speed

③ Homogenization limit as $\varepsilon \to 0$.

<u>Earlier results</u> For $g_+(y) = g_-(y)$ : periodic
( Lou-M.-Nakamura 2006)

<u>What if *g* is non-periodic ?</u>

Features specific to non-periodic environments:

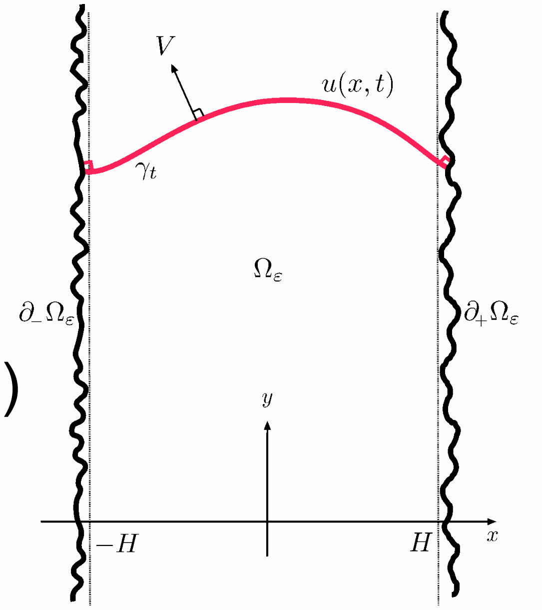
- virtual pinning ( related to topic ② )
- slower convergence rate ( related to topic ③ )

Another topic

The case when the undulation is random (i.e. non-deterministic).

④ Existence of average speed (almost surely)

⑤ Central limit theorem for observed data.




(joint work with J. Nolen)

# Homogenization

$$g_{\pm}^{\varepsilon}(y) = \varepsilon g_{\pm}(y/\varepsilon)$$

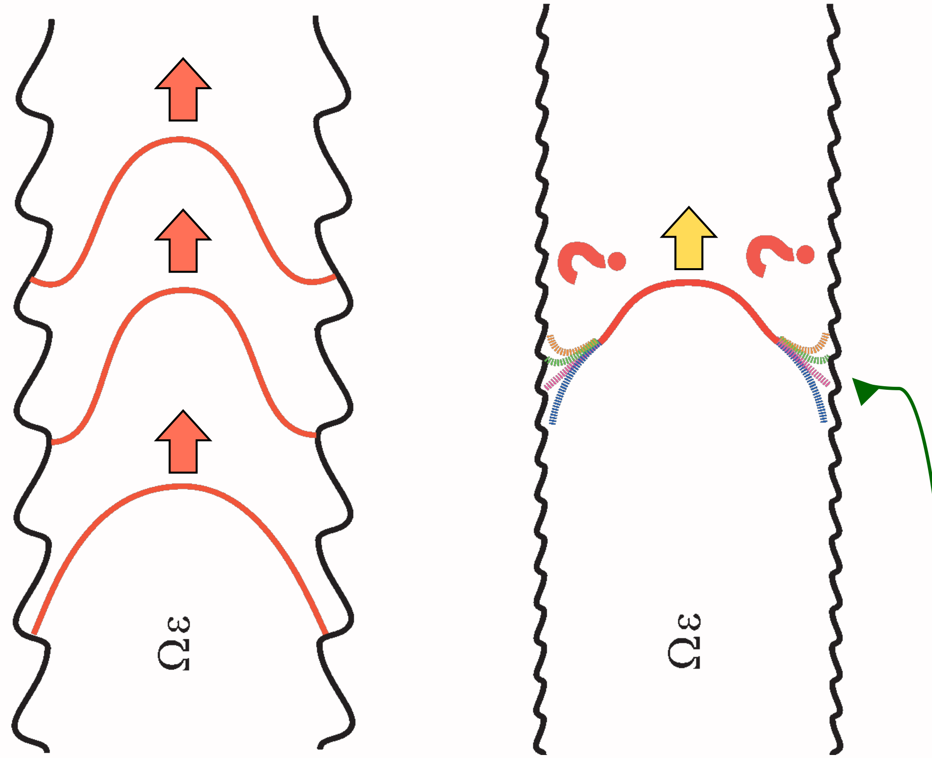
boundary shape

Homogenization limit

$$\gamma^{\varepsilon}(t) \to \varphi(x) + c_0 t \quad (\varepsilon \to 0)$$

$c_0$: limit speed

$\varphi$: limit profile



$\varepsilon$ large $\qquad$ $\varepsilon$ small

What determines the limit speed ?

What is the convergence rate ?

The limit contact angle plays the key role.

<u>Difficulty</u> The two ends of the curve flips back and forth rapidly, in a highly nonlinear manner.

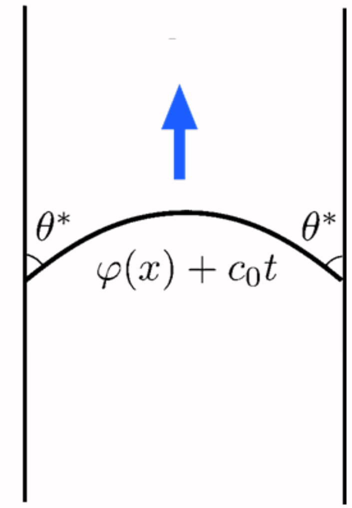
# 2. Basic concepts

- Definition of traveling waves
- Recurrence and ergodicity
- Average speed

# What is a traveling wave?

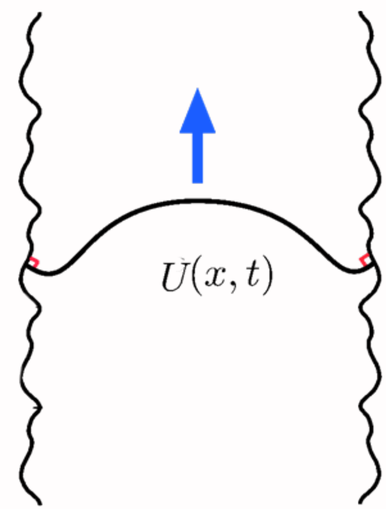
homogeneous environment

- constant speed
- constant profile




heterogeneous environment

- speed fluctuates
- profile fluctuates




More precise definition of TW is needed.

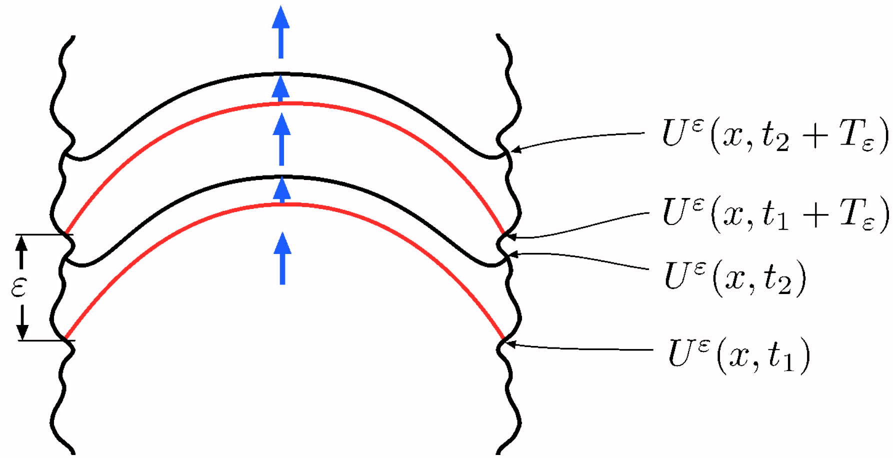
# What is a traveling wave?

The periodic case: $g_{\pm}(y+L) \equiv g_{\pm}(y)$, i.e. $g_{\pm}^{\varepsilon}(y+\varepsilon L) \equiv g_{\pm}^{\varepsilon}(y)$

Definition (periodic TW):

$$U^{\varepsilon}(x,t+T_{\varepsilon}) = U_{\varepsilon}(x,t) + \varepsilon L$$

$c_{\varepsilon} := \varepsilon L \,/\, T_{\varepsilon}$ average speed




The speed and profile may fluctuate, but periodically in time.

The above definition does not work in the non-periodic case!

To deal with non-periodic cases, it is useful to introduce the notion of the hull of a function.

# Hull of a function

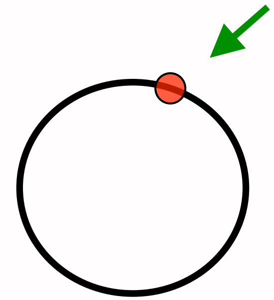
$g : \mathbb{R} \to \mathbb{R}$ bounded continuous function on ***R***

$$\mathcal{H}_g := \overline{\{\, \sigma_s g \mid s \in \mathbb{R} \,\}} \qquad \sigma_s : \ g(x) \mapsto g(x+s)$$

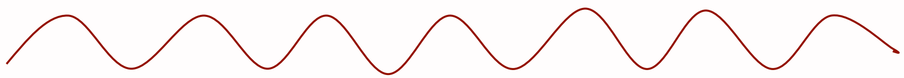
Closure of the set of all translations of $g(x)$ in the local uniform topology $L^{\infty}_{loc}$

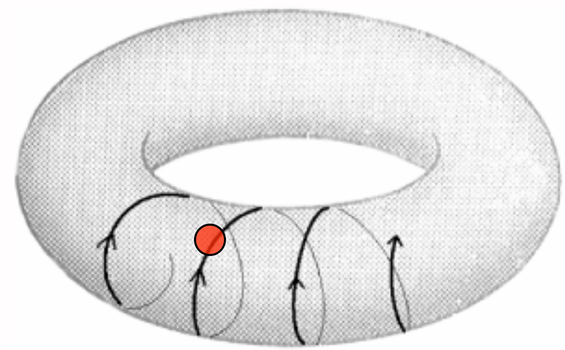
# Various classes of heterogeneity

$\sigma_s\, g, \quad s \in \mathbb{R}$

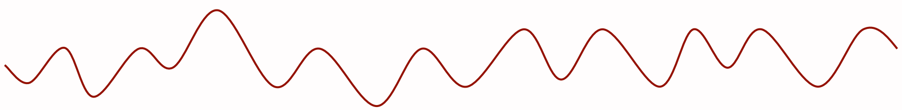


**periodic** $\quad \mathcal{H}_g := S^1$ (circle)





**quasi-periodic** $\quad \mathcal{H}_g := T^m$ (torus)



**almost periodic** $\quad \mathcal{H}_g :$ compact in $L^\infty(\mathbb{R})$

**strictly ergodic** = uniquely ergodic and recurrent

**recurrent** $\quad \mathcal{H}_g :$ compact in $L^\infty_{loc}(\mathbb{R})$

Moreover, every orbit is dense.

**random:** $g = g(x, \omega), \quad \omega \in \Omega$

London 2013

# Ergodicity

$g$ uniquely ergodic $\overset{def}{\Longleftrightarrow}$ $^{\exists}$uniqe shift-invariant measure on $\mathcal{H}_g$

A bounded uniformly continuous function $g : \mathbb{R} \to \mathbb{R}$ is uniquely ergodic if and only if, for any continuous map $F : \mathcal{H}_g \to \mathbb{R}$, the following limit exists uniformly in $a \in \mathbb{R}$:

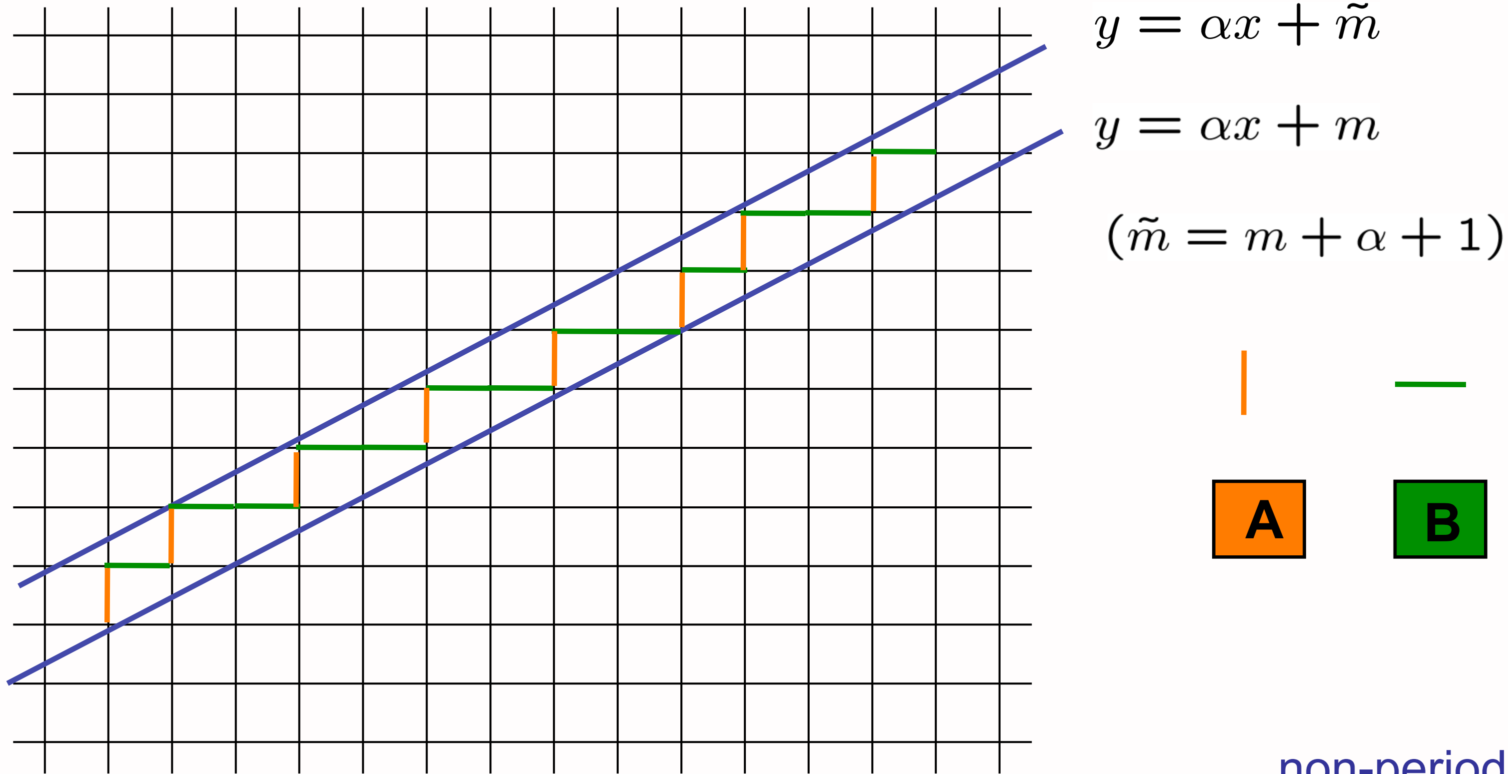
$$\lim_{L\to\infty} \frac{1}{L} \int_a^{a+L} F(\sigma_s g)ds. \quad = \int_{\mathcal{H}_g} F\, d\mu$$

## Remark

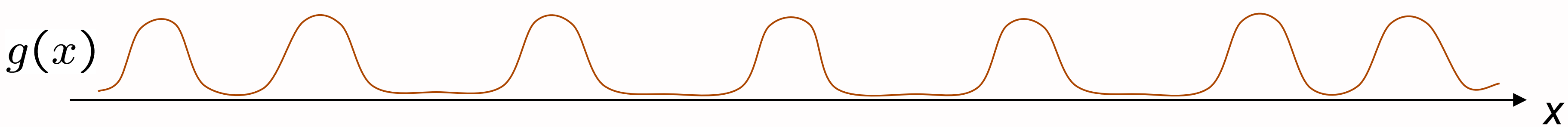
Ergodicity is preserved under continuous deformation of a function.

# Example of ergodic functions

$\alpha$ : irrational



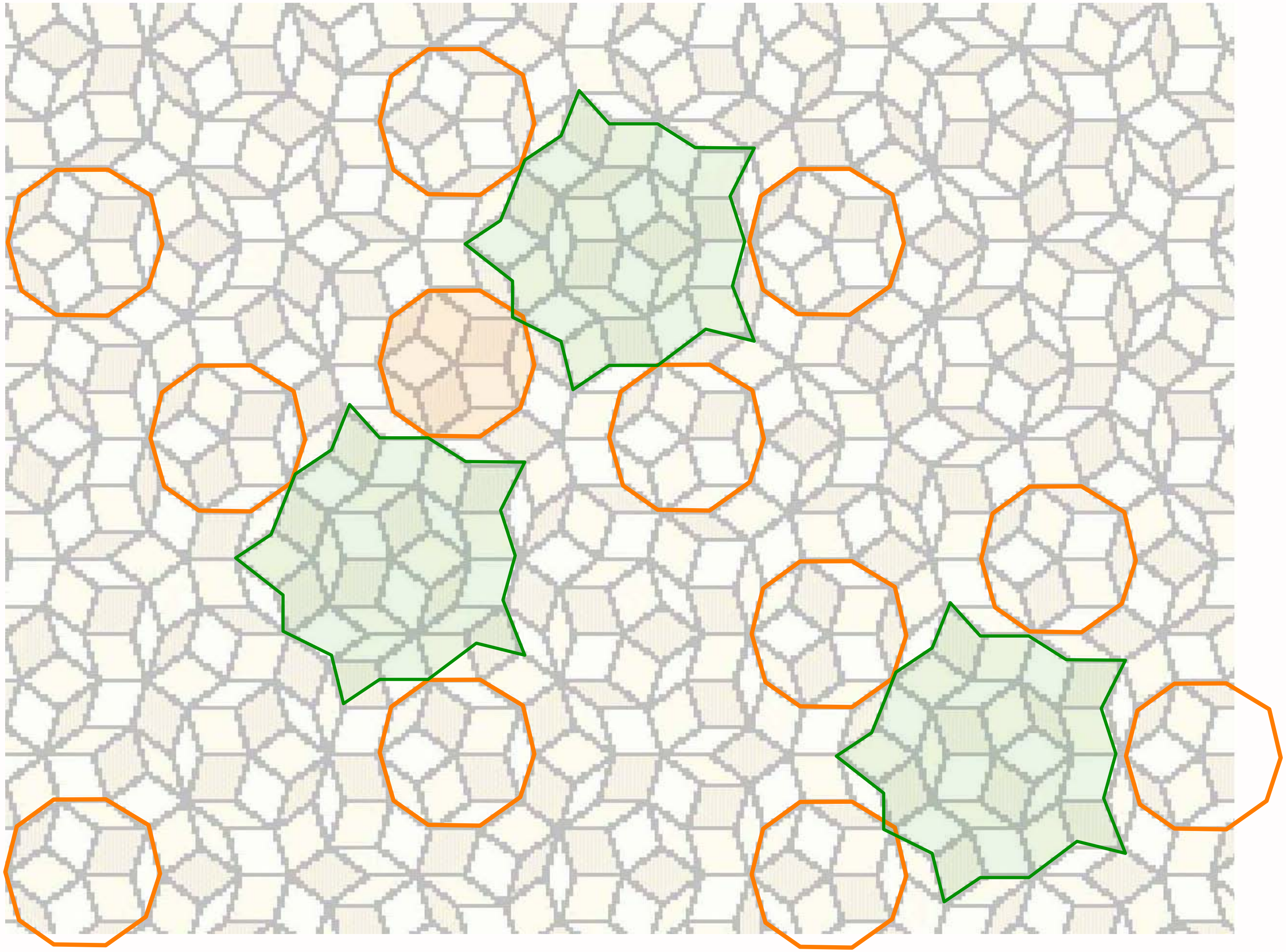

non-periodic

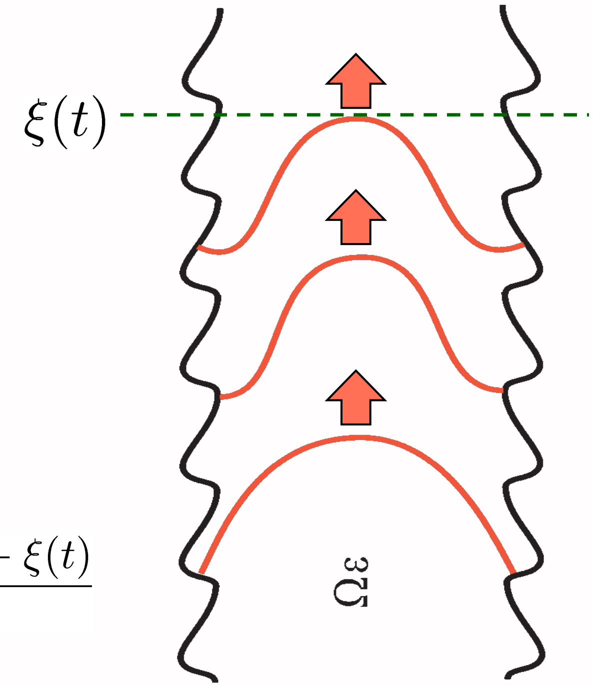

uniquely ergodic + recurrent = strictly ergodic

# Penrose tiling (2D ergodic)

Any finite pattern is distributed uniformly.



# Propagation speed




$\xi(t)$ : front position at time $t$

**Average speed** $$c := \lim_{T\to\infty} \frac{\xi(t+T)-\xi(t)}{T}$$

More precisely,

$$c_- := \lim_{T\to\infty} \inf_t \frac{\xi(t+T)-\xi(t)}{T}$$
lower average

$$c_+ := \lim_{T\to\infty} \sup_t \frac{\xi(t+T)-\xi(t)}{T}$$
upper average

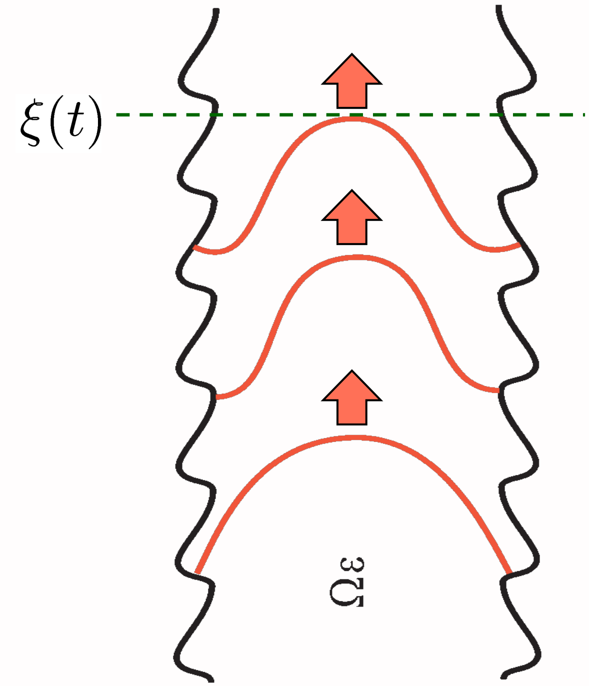
We say that the <u>average speed exists</u> if $c_- = c_+$

**Proposition**. The average speed exists if $g$ is uniquely ergodic and $c_- > 0$.

**Law of motion** $$\dot{\xi}(t) = p(\xi(t))$$

The function $p$ is determined uniquely by $g$. Hence ergodicity of $g$ implies that of $p$.

# Classification of front behaviors




## Pinning (propagation failure)

$$\lim_{t\to-\infty} \xi(t) > -\infty \quad \text{or} \quad \lim_{t\to+\infty} \xi(t) < +\infty$$

## Propagation

$$\lim_{t\to\pm\infty} \xi(t) = \pm\infty$$

regular propagation $\quad c_- > 0$

virtual pinning $\quad c_- = 0$

<u>Note</u>

1. Pinning occurs if and only if there exists a stationary solution.
2. Virtual pinning never occurs if $g$ is periodic.

# 3. Main results

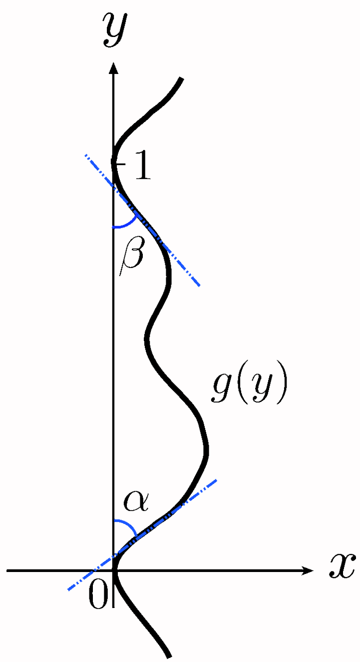
Joint work with

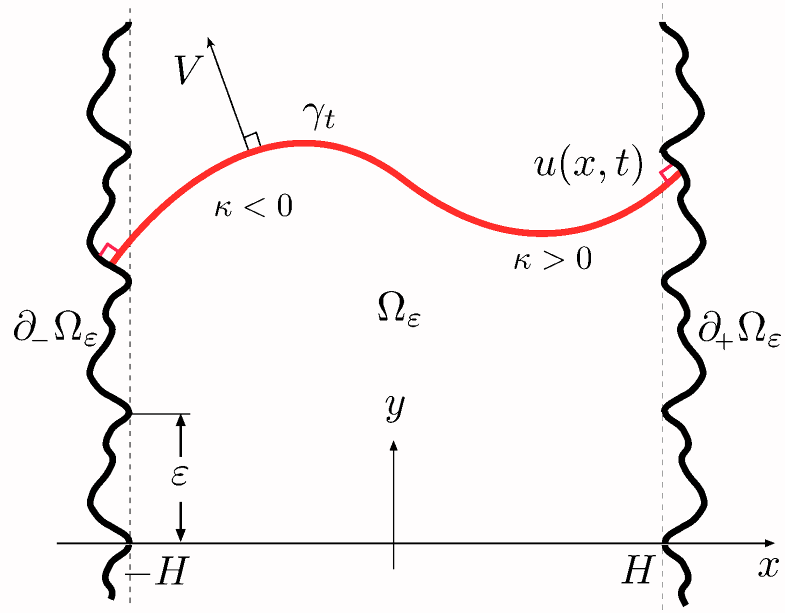
Bendong Lou and Ken-Ichi Nakamura

# Notation

$$\tan \alpha_\pm = \sup_y g'_\pm(y)$$ maximal opening angles

$$\tan \beta_\pm = -\inf_y g'_\pm(y)$$ maximal closing angles




Standing assumption: $\alpha_\pm, \beta_\pm \in (0, \pi/4)$ $\Rightarrow$ Global existence of classical solutions.

# Existence of TW

$\alpha_\pm, \beta_\pm \in (0, \pi/4)$

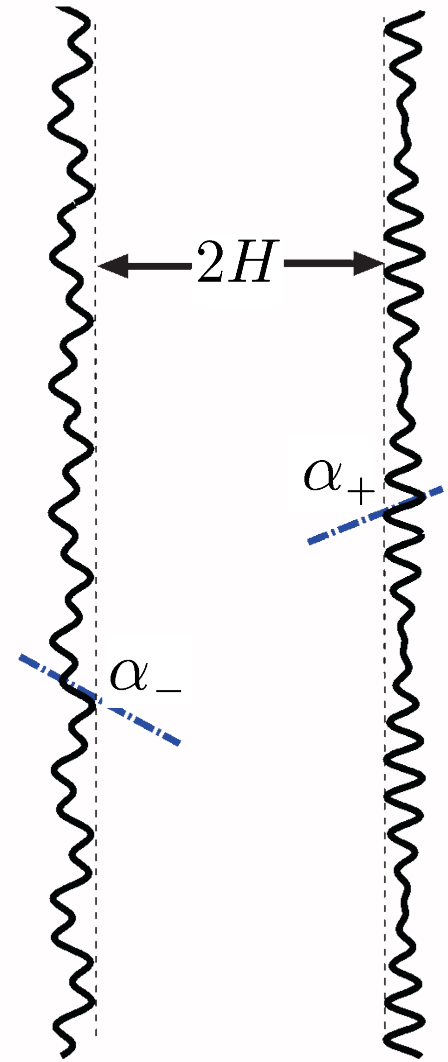
**Theorem 1 (existence and stability).**

If $2AH \geq \sin\alpha_- + \sin\alpha_+$ , then a recurrent TW exists for all small $\varepsilon > 0$ . This TW is unique up to time shift and is asymptotically stable.

**Theorem 2 (non-existence).**

If $2AH < \sin\alpha_- + \sin\alpha_+$ , then no TW exists. Moreover, any time-global solution $\gamma_t$ converges to a stationary solution as $t \to \infty$ .

**Proposition.** The average speed exists if $g$ is uniquely ergodic and $c_- > 0$.



# Virtual pinning $(\varepsilon = 1)$

Propagation $\lim_{t\to\pm\infty} \xi(t) = \pm\infty$ 
- regular propagation $c_- > 0$
- virtual pinning $c_- = 0$

**Theorem 3 (Virtual pinning).**

Virtual pinning occurs if and only if

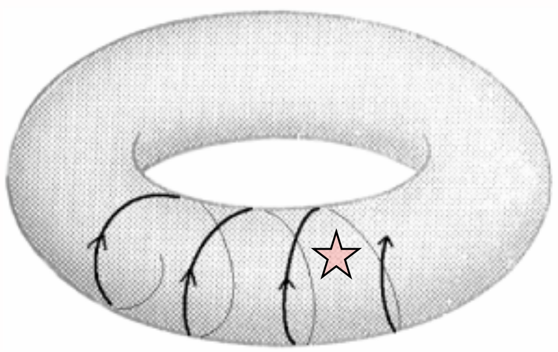
(1) there exists no stationary solution in $\Omega = \Omega_g$

(2) there exists a stationary solution in $\Omega_h$ for some $h \in \mathcal{H}_g$

$$\Omega = \{-H - g(y) < x < H + g(y),\ y \in \mathbf{R}\}$$

$$\Omega_h = \{-H - h(y) < x < H + h(y),\ y \in \mathbf{R}\}$$

$$\sigma_{a_j}\Omega \to \Omega_h \ (j \to \infty)$$



# Homogenization limit

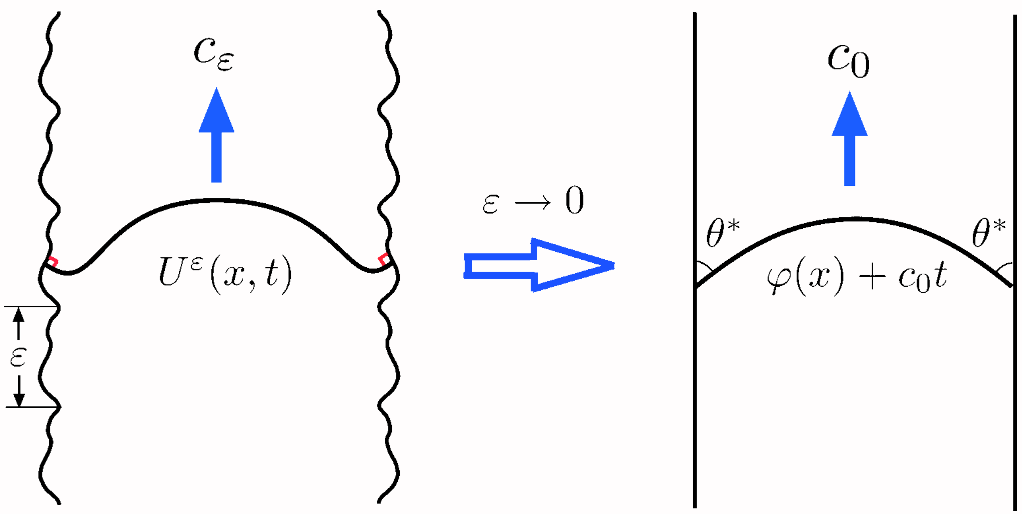
Here $\alpha_- = \alpha_+$ for simplicity.

Theorem 4 (homogenization).

Assume $AH > \sin\alpha$ and let $U^\varepsilon(x,t)$ be the recurrent TW that is normalized to satisfy $U^\varepsilon(0,0)=0$ . Then

(i) $U^\varepsilon(x,t)$ converges to a function of the form $\varphi(x)+c_0 t$ as $\varepsilon \to 0$ whose contact angle is $\theta^* = \pi/2 - \alpha$ .

(ii) The limit speed $c_0$ is determined by $H = \int_0^\alpha \frac{\cos\eta}{A - c_0\cos\eta} d\eta.$

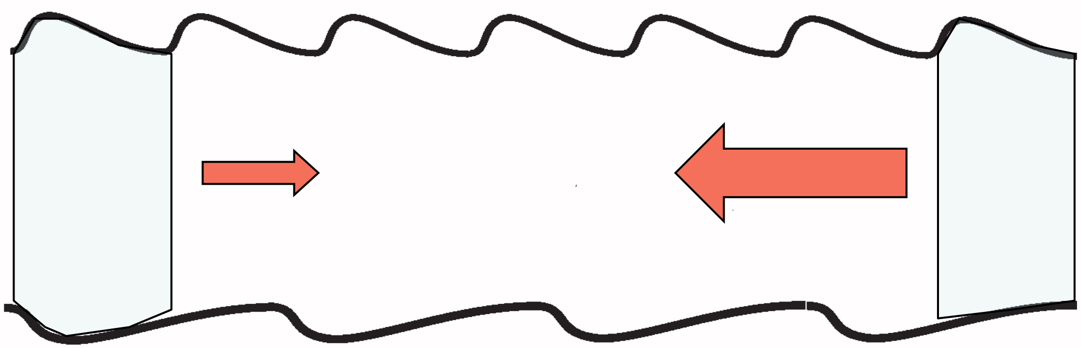


$$H = \int_0^{\alpha} \frac{\cos\eta}{A - c_0 \cos\eta} d\eta.$$

Corollary. The limit speed $c_0$ satisfies

$$\frac{\partial c_0}{\partial \alpha} < 0, \quad \frac{\partial c_0}{\partial A} > 0, \quad \frac{\partial c_0}{\partial H} > 0.$$

The larger the opening angle $\alpha$, the slower the speed $c_0$ .



# Convergence rate

$f(z_1, z_2, \ldots, z_m)$ : 1-periodic in each $z_i$
with non-degenerate maximum

Theorem 5 (convergence rate for quasi-periodic $g$).

Assume $g_+ \equiv g_- = f(\omega_1 y, \omega_2 y, \ldots, \omega_m y)$ and let $c_\varepsilon$ denote the average speed of $U^\varepsilon(x,t)$. Then

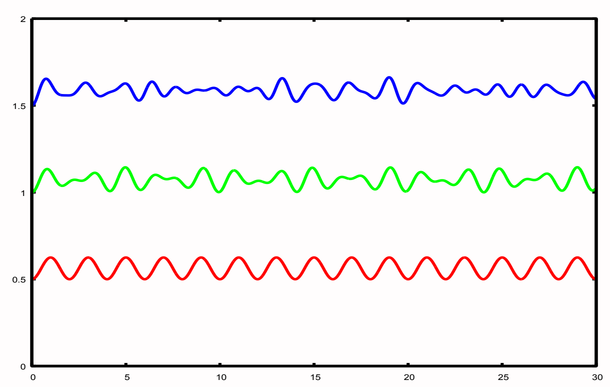
$$c_0 + P\,\varepsilon^{\frac{2}{m+3}} \leq c_\varepsilon \leq c_0 + Q\,\varepsilon^{\frac{2}{m+3}}$$

for some constants *P, Q* > 0. If, in particular, $g$ is periodic, then

$$c_0 + P\sqrt{\varepsilon} \leq c_\varepsilon \leq c_0 + Q\sqrt{\varepsilon}$$

## Examples of QP

$$C_1 \sin^2 \omega_1 y + C_2 \sin^2 \omega_2 y$$

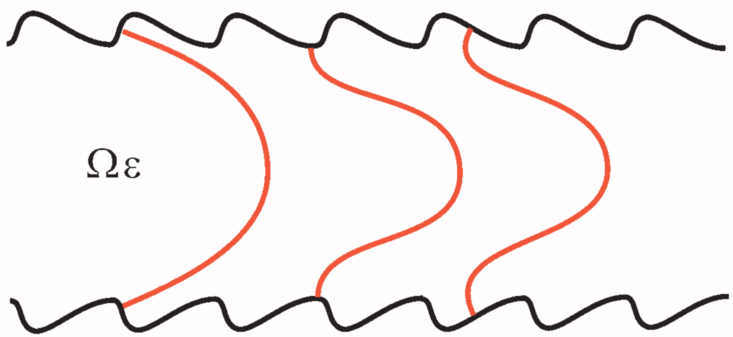


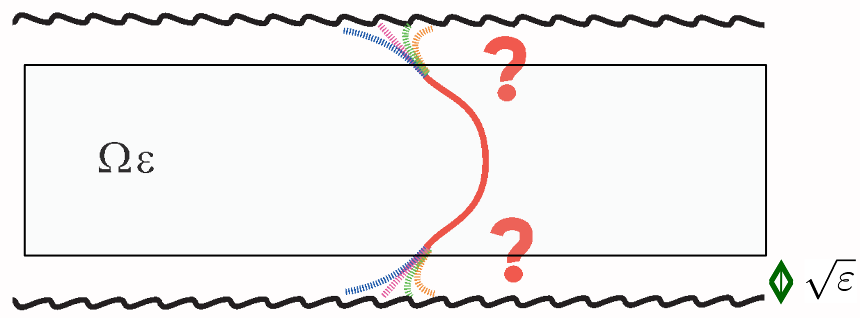

m = 4
m = 2
m = 1

# 4. Outline of the proof

(for homogenization)

$$g_i^{\varepsilon}(y) = \varepsilon g_i(y/\varepsilon) \to 0$$



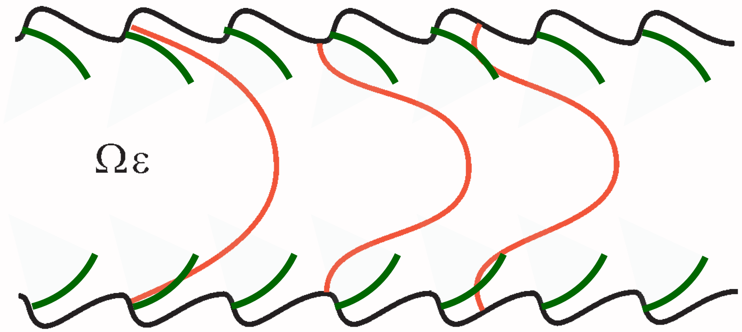

**Difficulty:** The two ends of the curve flips back and forth very rapidly, in a highly nonlinear manner. This makes it difficult to estimate the average speed.

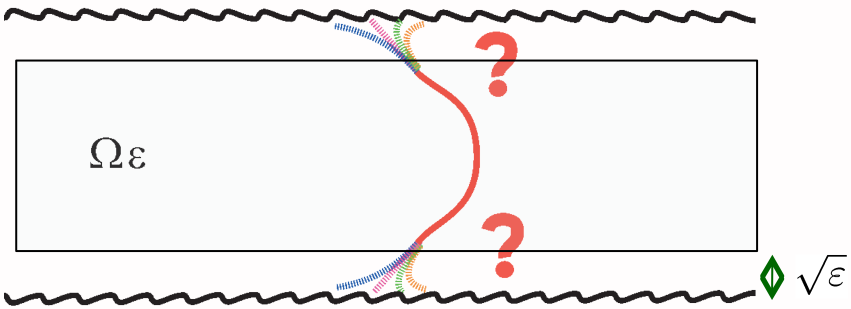
## Strategy

1. Estimate the gradient slightly away from the boundary. $O(\sqrt{\varepsilon})$

The derivatives stabilize in this zone as $\varepsilon$ tends to 0.

$$g_i^\varepsilon(y) = \varepsilon g_i(y/\varepsilon) \to 0$$



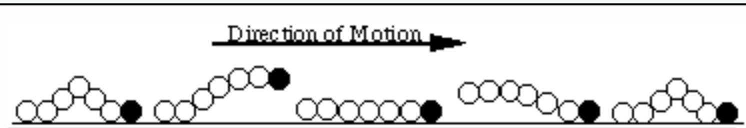




## Strategy

1. Estimate the gradient slightly away from the boundary. $O(\sqrt{\varepsilon})$

This can be done by placing circular arcs of curvature A at points where the opening angle is close to its supremum.

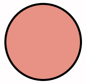
2. Construct a sub-solution in this zone whose motion mimicks that of an inchworm.



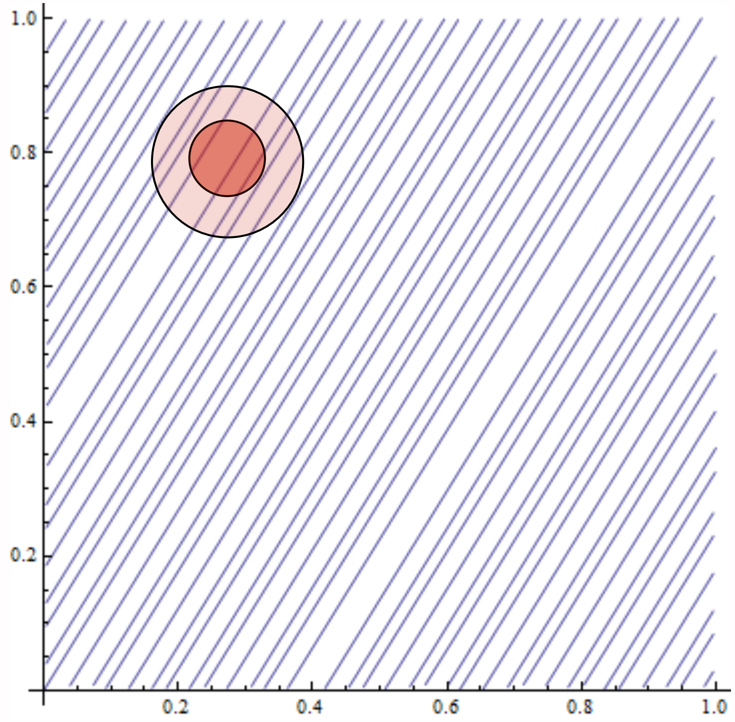

London 2013

Orbit of $(\omega_1 y, \omega_2 y)$ in $\mathbf{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$

$$g'(y) = f(\omega_1 y, \omega_2 y)$$

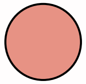


The parameter region where the opening angle is large (which slows down the speed).

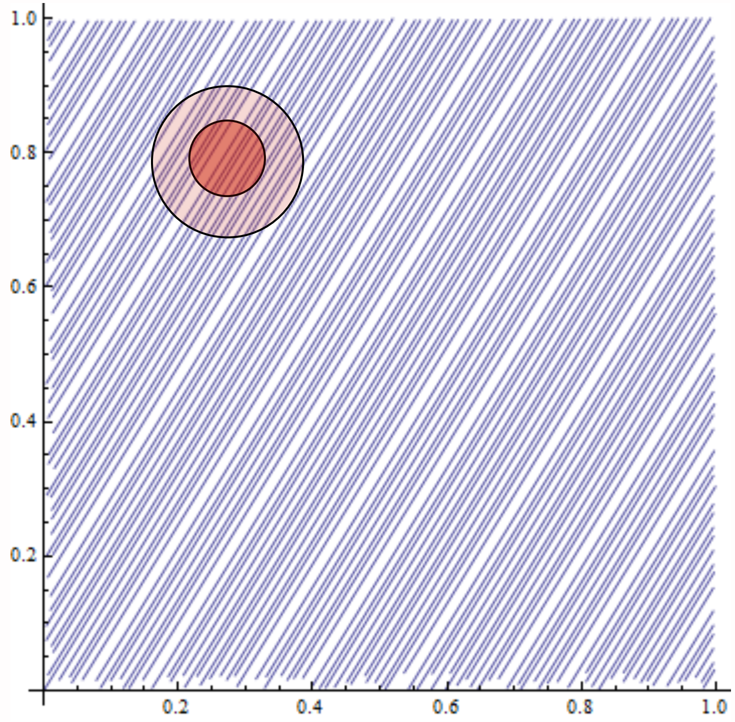


Orbit of $(\omega_1 y, \omega_2 y)$ in $\mathbf{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$

$$g'(y) = f(\omega_1 y, \omega_2 y)$$



The parameter region where the opening angle is large (which slows down the speed).



# 5. The random case

Joint work with James Nolen

# Assumptions

☆ $g = g(y,\omega) : \mathbf{R} \times \Omega \to \mathbf{R}$ random stationary ergodic

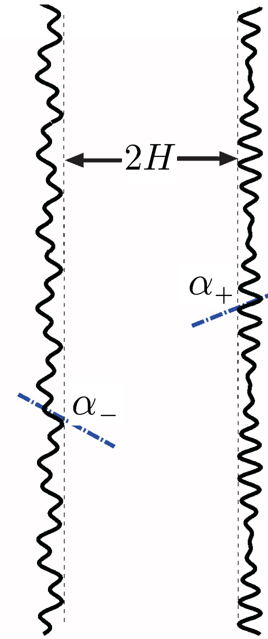
$$g(y+s,\omega) = g(y,\tau_s\omega) \ (s \in \mathbf{R})$$

$\tau_s : \Omega \to \Omega, \ \tau_s \circ \tau_{s'} = \tau_{s+s'}$ measure preserving and ergodic

☆ $\tan\alpha_\pm = \sup_y g'_\pm(y,\omega), \quad -\tan\beta_\pm = \inf_y g'_\pm(y,\omega)$ almost surely

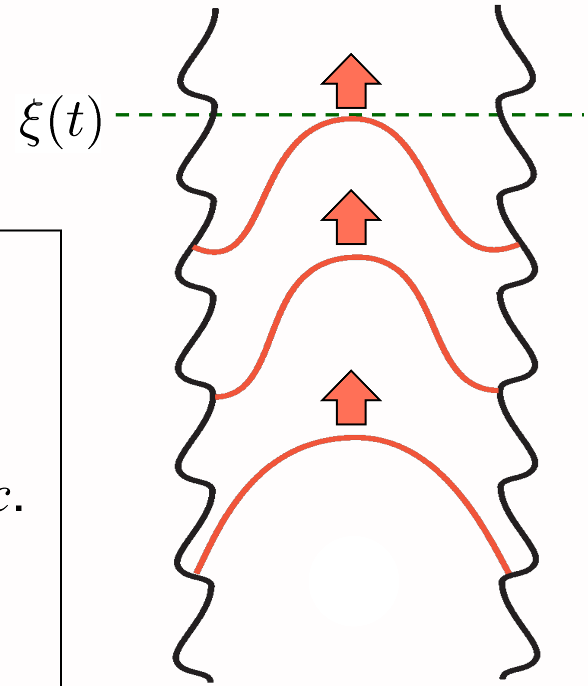
where the constants $\alpha_\pm, \ \beta_\pm$ satisfy

$$\alpha_\pm, \beta_\pm \in (0,\pi/4), \quad \sin\alpha_- + \sin\alpha_+ < 2AH.$$



## Notation

$$\xi(t,\omega) = \max_x U(x,t,\omega)$$




### Theorem 5 (Existence of average speed).

The following limit exists almost surely (i.e. with probability one) for some deterministic constant $c$.

$$c := \lim_{T\to\infty} \frac{\xi(t,\omega)}{t}$$

### Theorem 6 (Central limit theorem).

If *g* has a certain mixing property, then there is $\sigma \geq 0$ such that

$$\frac{\xi(t,\omega) - ct}{\sqrt{t}} \to N(0,\sigma^2)$$ (normal distribution)

# Concluding remarks:

general

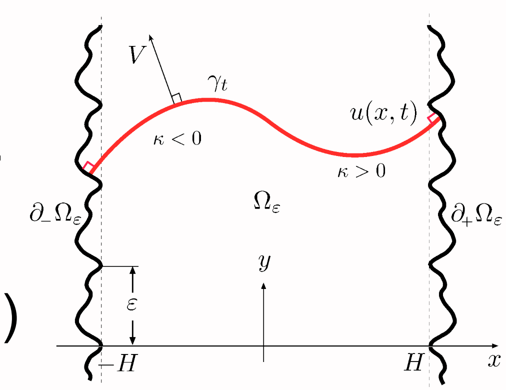
- TW is unique and stable if it exists.
- TW has a well-defined average speed in ergodic enviromnents.
- In non-periodic environments, “virtual pinning” can occur.

homogenization

- The limit speed of the homogenized TW is determined only by the maximal opening angle.
- The wider the maximal opening angle, the slower the limit speed.
- If $g$ is quasi-periodic, the rate of convergence of the speed is slower than in the periodic case.

# Open problems:

- What if we allow the propagating curve to be non-graphical? (Viscosity solution framework needed.)
- The case of random undulation? (Partially solved.)



# Thank you

&

# Happy New Year!