Finite-time blowup for the Zakharov system on 2D torus

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Zakharov system

$$
\text{(Z)} \qquad \begin{cases} i\partial_t u + \Delta u = nu, & u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}, \\ \partial_t^2 n - \Delta n = \Delta(|u|^2), & n : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}, \\ (u, n, \partial_t n)\big|_{t=0} = (u_0, n_0, n_1). \end{cases}
$$

$$
i = \sqrt{-1}, \ \Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \ \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2.
$$

- Conserved quantities :

$$
\begin{aligned} M(u)(t) &:= \big\|u(t)\big\|_{L^2}^2 \\ &= M(u_0), \\ H(u,n)(t) &:= \big\|\nabla u(t)\big\|_{L^2}^2 + \frac{1}{2}\Big(\big\|n(t)\big\|_{L^2}^2 + \big\||\nabla|^{-1}\partial_t n(t)\big\|_{L^2}^2\Big) + \int n(t)|u(t)|^2\,dx \\ &= H(u_0, n_0). \end{aligned}
$$

- Energy space : $H^1 \times L^2 \times |\nabla| L^2 \quad (\subset H^1 \times L^2 \times H^{-1})$

▶ **Physical background :**

Model of *Langmuir turbulence* in plasma fluid.

u ... Envelope of rapidly oscillating electric field

n ... Deviation of ion density from its mean value

- **Langmuir turbulence :**

1. **'Bubble' of low ion density region (*caviton*) is generated.**
2. **Caviton becomes smaller and smaller, with amplitude of electric field and deviation of ion density larger and larger. (caviton collapse)**
3. **When the collapse proceeds to some extent, energy dissipation begins, and eventually caviton disappears.**
4. **Such a process is repeated.**

▶ **Mathematical interest :** **Dispersion vs Nonlinear effect**

- **Linear part has *dispersive effects* ;**

  Nonlinearity has attractive (focusing) sign and causes *localisation* of waves.

  $\rightsquigarrow$ **In the Cauchy problem, one may expect :**

— ***global well-posedness*** **(and scattering to linear sol.) for small data,**

— ***finite-time blowup*** **for large data.**

- **Aim : To predict long-time behaviour for given data.**

  (Classification of data space)

Aim in this talk

To determine the *threshold* for global solvability.

Nonlinear Schrödinger equation

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta u = -|u|^2 u, \\ u\big|_{t=0} = u_0. \end{cases}$$

- **Relation to Zakharov system :**

— **Subsonic limit of** (Z).

— $u = e^{i\theta(t)}\phi(x)$ **: sol. to** (NLS)

$\Rightarrow (u, n) = (e^{i\theta(t)}\phi(x), -|\phi(x)|^2)$ **: sol. to** (Z).

- **Conserved quantities :**

$$M(u)(t) = \|u(t)\|_{L^2}^2 = M(u_0),$$
$$E(u)(t) := \|\nabla u(t)\|_{L^2}^2 - \frac{1}{2}\|u(t)\|_{L^4}^4 = E(u_0).$$

- **Energy space :** H^1.

Local well-posedness in the energy space

- (NLS)
  - **On $\mathbb{R}^2$ ... Kato (1987)**
  - **On $\mathbb{T}^2$ ... Bourgain (1993)**
- (Z)
  - **On $\mathbb{R}^2$ ... Bourgain-Colliander (1996)**
  - **On $\mathbb{T}^2$ ... K. (to appear)**

Remark

In 1D, Hamiltonian and the L^2 norm control the energy norm.

$\rightsquigarrow$ Global well-posedness in energy space follows from local results.

In higher dim., Hamiltonian does not necessarily control the energy norm, so finite-time blowup may occur.

(In 2D or 3D, Hamiltonian control is valid for small data.)

Global well-posedness in the energy space

- **Sharp Gagliardo-Nirenberg inequality**

  (Weinstein 1982 for $\mathbb{R}^2$ / Ceccon-Montenegro 2008 for $\mathbb{T}^2$)

$$\|u\|_{L^4}^4 \leq \frac{2}{\|Q\|_{L^2(\mathbb{R}^2)}^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 \,\Big(+ C\|u\|_{L^2}^4\Big),$$

  where $Q : \mathbb{R}^2 \to \mathbb{R}_+$ is pos. rad. solution of $-Q + \Delta Q + |Q|^2 Q = 0$ on $\mathbb{R}^2$.

  - (NLS) on $\mathbb{R}^2$ has a *standing wave solution* $u = e^{it}Q(x)$.

    (Z) on $\mathbb{R}^2$ has a similar solution $(u, n) = (e^{it}Q(x), -|Q(x)|^2)$.

- **Corollary : Hamiltonian control for solutions s.t. $M(u) < M(Q)$.**

  - (NLS) ... $E(u)(t) + CM(u)(t) \sim \|u(t)\|_{H^1}^2$
  - (Z) ... $H(u,n)(t) + CM(u)(t) \sim \|(u, n, \partial_t n)(t)\|_{H^1 \times L^2 \times |\nabla| L^2}^2$

$\Rightarrow$ Global well-posedness in energy space for data $M(u_0) < M(Q)$.

Results for $M(u_0) \geq M(Q)$

- **For** (NLS)
  - **On $\mathbb{R}^2$ ... Explicit blow-up solution with $M(u_0) = M(Q)$**

    $$u_\lambda(t,x) = \frac{1}{\lambda t} e^{-i\left(\frac{1}{\lambda^2 t} - \frac{|x|^2}{4t}\right)} Q\left(\frac{x}{\lambda t}\right),\ \lambda > 0,\ (t,x) \in (0,\infty) \times \mathbb{R}^2$$

    This solution is obtained from the standing wave sol $u = e^{it}Q(x)$ via pseudo-conformal (and scaling) transform.

  - **On $\mathbb{T}^2$ ... $\exists$ blow-up solution with $M(u_0) = M(Q)$**

    (Burq-Gérard-Tzvetkov 2003)

    Idea : $u = \psi(x)u_\lambda + r$, solve eq. for r via energy method.

    (cf. Ogawa-Tsutsumi 1990 for 1D quintic NLS)

▶ **For** (Z)

- $(u_0, n_0, n_1) \in H^1 \times L^2 \times H^{-1}$, $M(u_0) = M(Q) \Rightarrow$ **global.**

  (Glangetas-Merle (1994) for $\mathbb{R}^2$**, K.-Maeda (to appear) for** $\mathbb{T}^2$**)**

- **On** $\mathbb{R}^2$, $\forall \epsilon > 0$, $\exists$ **blow-up sol** (u, n) **in energy sp. with**

$$M(Q) < M(u)(t) < M(Q) + \epsilon.$$

  (Glangetas-Merle 1994)

▶ **Summary :**

<table>
<tr><th></th><th>(NLS) on $\mathbb{R}^2$</th><th>(NLS) on $\mathbb{T}^2$</th><th>(Z) on $\mathbb{R}^2$</th><th>(Z) on $\mathbb{T}^2$</th></tr>
<tr><td>$M(u_0) < M(Q)$</td><td colspan="2">global</td><td colspan="2" rowspan="2">global</td></tr>
<tr><td>$M(u_0) = M(Q)$</td><td colspan="2" rowspan="2">$\exists$ blowup</td></tr>
<tr><td>$M(u_0) > M(Q)$</td><td>$\exists$ blowup</td><td>??</td></tr>
</table>

Main result

Theorem (K.-Maeda, to appear)

On $\mathbb{T}^2$, $\forall \epsilon > 0$, $\exists$ blow-up sol (u, n) to (Z) in energy sp. with

$$M(Q) < M(u)(t) < M(Q) + \epsilon.$$

Remarks

- Solution is defined on $(0, T] \times \mathbb{T}^2$ for some $T = T(\epsilon) > 0$ and blows up at $x = 0$ as $t \to 0$. Moreover,

$$\|\nabla u(t)\|_{L^2} \sim \|n(t)\|_{L^2} \sim \||\nabla|^{-1}\partial_t n(t)\|_{L^2} \sim t^{-1}.$$

- For given p points on $\mathbb{T}^2$, we can construct a sol of (Z) which blows up at these points simultaneously and

$$pM(Q) < M(u)(t) < pM(Q) + \epsilon.$$

- We can also construct a blow-up solution on bounded domain $(u|_{\partial\Omega} = 0)$.

Outline of proof

▶ **Blow-up sol to (NLS) on $\mathbb{T}^2$ (Burq-Gérard-Tzvetkov 2003)**

- **Recall that** $\widetilde{U}(t,x) := \frac{1}{t}e^{-i(\frac{1}{t}-\frac{|x|^2}{4t})}Q(\frac{x}{t})$ **is a blow-up sol on $\mathbb{R}^2$.**

⇝ **Set** $u = \psi(x)\widetilde{U} + v$, **where ψ is a smooth cutoff such that**

$$\psi \equiv 1 \text{ in a nbd. of } 0, \qquad \text{supp } \psi \subset (-\pi,\pi)^2.$$

u is a blow-up solution on $\mathbb{T}^2$ if v solves

$$\begin{cases} i\partial_t v + \Delta v = & -|v|^2 v - (2U|v|^2 + \overline{U}v^2) \\ & -(U^2\overline{v} + 2|U|^2 v) + F \quad \text{in } (0,T] \times \mathbb{T}^2, \\ v(0,x) = & 0, \end{cases}$$

$$U := \psi(x)\widetilde{U},$$

$$F := (1-\psi^2)\psi|\widetilde{U}|^2\widetilde{U} - 2\nabla\psi \cdot \nabla\widetilde{U} - (\Delta\psi)\widetilde{U}.$$

— ***Singular coeff.*** **in RHS :** $\|U(t)^2\|_{L^\infty(\mathbb{T}^2)} = O(t^{-2})$ $(t \to 0)$.

$$\begin{cases} i\partial_t v + \Delta v = & -|v|^2 v - (2U|v|^2 + \overline{U} v^2) \\ & -(U^2 \overline{v} + 2|U|^2 v) + F \quad \text{in } (0,T] \times \mathbb{T}^2, \\ v(0,x) = & 0, \end{cases}$$

$$U := \psi(x)\widetilde{U}, \qquad \widetilde{U} = \frac{1}{t} e^{-i(\frac{1}{t} - \frac{|x|^2}{4t})} Q(\tfrac{x}{t}),$$

$$F := (1 - \psi^2)\psi|\widetilde{U}|^2\widetilde{U} - 2\nabla\psi \cdot \nabla\widetilde{U} - (\Delta\psi)\widetilde{U}.$$

- $\widetilde{U}(t,0) = O(t^{-1})$, but $\widetilde{U}(t,x) = O(e^{-\frac{c}{t}})$ outside nbd of $x = 0$.

  (Note that $Q(x)$ decays *exponentially* as $|x| \to \infty$.)

  Since $F \equiv 0$ around $x = 0$, we have $F(t) = O(e^{-\frac{c}{t}})$ $(t \to 0)$.

  Then, one can also expect that $v(t) = O(e^{-\frac{c}{t}})$.

  $\rightsquigarrow$ Solve the equation by the iteration with the norm, e.g.,

$$\|v\|_X = \sup_{0<t<T} e^{\frac{c}{t}} \|v(t)\|_{H^3(\mathbb{T}^2)}.$$

- *Exponential decay is essential* to control t^{-2} singularity!

- **Blow-up sol to (Z) on $\mathbb{R}^2$ (Glangetas-Merle 1996)**

$\exists$ Family of rad. functions $(P_\lambda, N_\lambda) : \mathbb{R}^2 \to \mathbb{R}^2$, $0 < \lambda \ll 1$ s.t.

(1) $\widetilde{U}_\lambda(t,x) := \frac{1}{\lambda t} e^{-i(\frac{1}{\lambda^2 t} - \frac{|x|^2}{4t})} P_\lambda(\frac{x}{\lambda t})$, $\widetilde{W}_\lambda(t,x) := (\frac{1}{\lambda t})^2 N_\lambda(\frac{x}{\lambda t})$

$\Rightarrow$ $(u,n) = (\widetilde{U}_\lambda, \widetilde{W}_\lambda)$: blow-up sol to (Z) on $(0,\infty) \times \mathbb{R}^2$.

(2) $\forall k \geq 0$, $\exists C, \mu > 0$ s.t. $\forall x \in \mathbb{R}^2$,

$$|\nabla^k P_\lambda(x)| \leq C e^{-\mu|x|}, \qquad |\nabla^k N_\lambda(x)| \leq \frac{C}{1 + |x|^{3+k}}.$$

(3) $(P_\lambda, N_\lambda) \to (Q, -Q^2)$ in $H^1 \times L^2$ $\quad (\lambda \to 0)$

- P_λ decays *exponentially* as Q, but N_λ decays *polynomially*.
- Choose λ small so that $M(Q) < M(P_\lambda) < M(Q) + \epsilon$.
- Set $(u,n) = (U + v, W + w)$, $(U, W) := (\psi(x)\widetilde{U}, \psi(x)\widetilde{W})$,

  and solve the system for (v, w).

▶ System for the perturbation (v, w) :

$$\begin{cases} (i\partial_t + \Delta)v = vw + Wv + Uw + O(e^{-\frac{c}{t}}), \\ (\partial_t^2 - \Delta)w = \Delta(|v|^2 + \overline{U}v + U\overline{v}) + O(t^3), \\ \ (v, w)\big|_{t=0} = (0, 0), \qquad (t, x) \in (0, T] \times \mathbb{T}^2. \end{cases}$$

- $Wv \sim t^{-2}v$ has a strong singularity as $t \to 0$.

  To treat this term as a perturbation, $v(t) = O(e^{-\frac{c}{t}})$ is required.

Difficulty 1 One cannot expect exp. decay for (v, w) !

▶ Decomposition of the wave equation :

$$w = Z + z, \quad \begin{cases} (\partial_t^2 - \Delta)Z = O(t^3), \\ (\partial_t^2 - \Delta)z = \Delta(|v|^2 + \bar{U}v + U\bar{v}). \end{cases}$$

- $Z(t)$ decays only polynomially, but $Z(t, x) \equiv 0$ around $x = 0$ for some time due to *finite speed of propagation*.

- **System for (v, z) :**

$$\begin{cases} (i\partial_t + \Delta)v = vz + (W + Z)v + Uz + UZ + O(e^{-\frac{c}{t}}), \\ (\partial_t^2 - \Delta)z = \Delta(|v|^2 + \bar{U}v + U\bar{v}), \\ (v, w)\big|_{t=0} = (0, 0), \qquad (t, x) \in (0, T] \times \mathbb{T}^2. \end{cases}$$

One can expect the exponential decay for (v, z) !

- **Reduction to 1st order system :**

Set $r = z + i|\nabla|^{-1}\partial_t z$ and solve the system for (v, r) :

$$\begin{cases} (i\partial_t + \Delta)v = vr_R + (W + Z)v + Ur_R + O(e^{-\frac{c}{t}}), \\ (i\partial_t - |\nabla|)r = |\nabla|(|v|^2 + \bar{U}v + U\bar{v}), \\ (v(t), r(t)) = O(e^{-\frac{c}{t}}) \text{ in } H^1 \times L^2 \quad (t \to 0). \end{cases}$$

where $r_R := \mathrm{Re}\ r$.

Difficulty 2 **Loss of one derivative in the wave equation !**

- Parabolic regularisation

For $\epsilon > 0$, consider sol (v^ϵ, r^ϵ) of the approximate system :

$$\begin{cases} (i\partial_t + \Delta + i\epsilon\Delta^2)v^\epsilon = v^\epsilon r_R^\epsilon + (W+Z)v^\epsilon + U r_R^\epsilon + O(e^{-\frac{c}{t}}), \\ (i\partial_t - |\nabla| + i\epsilon\Delta^2)r^\epsilon = |\nabla|(|v^\epsilon|^2 + \bar{U}v^\epsilon + U\bar{v}^\epsilon), \\ (v^\epsilon(t), r^\epsilon(t)) = O(e^{-\frac{c}{t}}) \text{ in } H^1 \times L^2 \quad (t \to 0). \end{cases}$$

- Easily solved by iteration with the norm, e.g.,

$$\big\|(v^\epsilon, r^\epsilon)\big\|_{X_{T_\epsilon}} := \sup_{0<t<T_\epsilon} \mathcal{H}[v^\epsilon, r^\epsilon](t),$$

$$\mathcal{H}[v^\epsilon, r^\epsilon](t)^2 = e^{\frac{2c}{t}}\Big(\big\|v^\epsilon(t)\big\|_{H^3}^2 + \big\|r^\epsilon(t)\big\|_{H^2}^2\Big)$$

- $T_\epsilon = O(\epsilon^{3/2})$ depends on ϵ

$\rightsquigarrow$ One needs an *a priori estimate* on $\mathcal{H}[v^\epsilon, r^\epsilon](t)$ uniform in $\epsilon > 0$.

▶ **Energy estimate with modified energy (Kwon 2008, Segata 2012)**

- **For (v^ϵ, r^ϵ) : solution of the approximate system, one has**

$$\frac{d}{dt}\mathcal{H}^2[v^\epsilon, r^\epsilon] \leq C(1+\mathcal{H}^3[v^\epsilon, r^\epsilon]) + 2e^{\frac{2c}{t}}\mathrm{Im}\int (v^\epsilon + U)\nabla^3 r_R^\epsilon \cdot \nabla^3 \bar{v}^\epsilon$$

On the other hand,

$$\frac{d}{dt}\Big[2e^{\frac{2c}{t}}\mathrm{Re}\int (v^\epsilon + U)\nabla^2 r_R^\epsilon \cdot \nabla^2 \bar{v}^\epsilon\Big] \leq -2e^{\frac{2c}{t}}\mathrm{Im}\int (v^\epsilon + U)\nabla^3 r_R^\epsilon \cdot \nabla^3 \bar{v}^\epsilon \quad + \ (\text{l.o.t.})$$

- **Define *modified energy* $\mathcal{E}$ as**

$$\mathcal{E}[v,r] := \mathcal{H}^2[v,r] + 2e^{\frac{2c}{t}}\mathrm{Re}\int (v+U)\nabla^2 r_R \cdot \nabla^2 \bar{v} \ + e^{\frac{2c}{t}}\|u(t)\|_{H^1}^{10}$$

(1) $\mathcal{H}^2(t) \lesssim \mathcal{E}(t) \lesssim \mathcal{H}^2(t) + \mathcal{H}^{10}(t)$ for $0 < t \ll 1$, uniformly in ϵ.

(2) $\frac{d}{dt}\mathcal{E}(t) \leq C(1+\mathcal{E}^3(t))$ for $0 < t \ll 1$, uniformly in ϵ.

$\Rightarrow$ A priori estimate follows. □

That's all.

Thank you for your attention!