Painlevé Equations — Nonlinear Special Functions

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Outline

1. Introduction

2. Review of some properties of the **fourth Painlevé equation**

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - A)w + \frac{B}{w} \qquad \mathrm{P_{IV}}$$

and the **fourth Painlevé σ-equation**

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

- Hamiltonian formulation
- Bäcklund and Schlesinger transformations
- Classical solutions

3. Application of $\mathrm{P_{IV}}$ to orthogonal polynomials

4. Application of $\mathrm{P_{IV}}$ to vortex dynamics

5. Symmetric form of $\mathrm{P_{IV}}$

6. Numerics and Asymptotics for $\mathrm{P_{IV}}$

7. Discussion

Classical Special Functions

- **Airy, Bessel, Whittaker, Kummer, hypergeometric functions**
- Special solutions in terms of rational and elementary functions (for certain values of the parameters)
- Solutions satisfy **linear** ordinary differential equations and **linear** difference equations
- Solutions related by **linear** recurrence relations

Painlevé Transcendents — Nonlinear Special Functions

- Special solutions such as rational solutions, algebraic solutions and special function solutions (for certain values of the parameters)
- Solutions satisfy **nonlinear** ordinary differential equations and **nonlinear** difference equations
- Solutions related by **nonlinear** recurrence relations

Definition

An ODE has the **Painlevé property** if its solutions have **no movable branch points**.

- **Single-valued**

$$w(z) = \frac{1}{z - z_0} \qquad \textbf{pole}$$

$$w(z) = \exp\left(\frac{1}{z - z_0}\right) \qquad \textbf{essential singularity}$$

- **Multi-valued**

$$w(z) = \sqrt{z - z_0} \qquad \textbf{algebraic branch point}$$

$$w(z) = \ln(z - z_0) \qquad \textbf{logarithmic branch point}$$

$$w(z) = \tan[\ln(z - z_0)] \qquad \textbf{essential singularity}$$

Reference

- **Cosgrove**, “Painlevé classification problems featuring essential singularities", *Stud. Appl. Math.*, **98** (1997) 355–433. [See also **Cosgrove**, *Stud. Appl. Math.*, **104** (2000) 1–65; **104** (2000) 171–228; **116** (2006) 321–413.]

Second Order Equations

Painlevé, Gambier, R Fuchs *et al.* [1893–1906] studied

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = F\left(\frac{\mathrm{d}w}{\mathrm{d}z}, w, z\right) \tag{1}$$

where F is rational in $\dfrac{\mathrm{d}w}{\mathrm{d}z}$ and w, and analytic in z.

- Fifty canonical types whose solutions have no movable critical points.
- Forty-four of these are integrable in terms of previously known functions, such as elliptic functions and linear equations, or were reducible to one of six new nonlinear ordinary differential equations, namely the **Painlevé equations**.
- The fifty canonical types are generalizable by the **Möbius transformation**

$$W(\zeta) = \frac{a(z)w + b(z)}{c(z)w + d(z)}, \qquad \zeta = \phi(z)$$

- The most interesting of the fifty canonical equations are those which require the introduction of new transcendental functions for their solution. These are the six Painlevé equations.

Painlevé Equations

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 6w^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \alpha w^2 + \beta z + \gamma w^3 + \frac{\delta}{w} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \qquad \mathrm{P_V}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

where α, β, γ and δ are arbitrary constants.

Higher Degree and Higher Order Equations

$$\frac{\mathrm{d}w_j}{\mathrm{d}z} = F_j(w_1, w_2, \ldots, w_n, z), \qquad j = 1, 2, \ldots, n$$
$$\frac{\mathrm{d}^n w}{\mathrm{d}z^n} = F\left(\frac{\mathrm{d}^{n-1}w}{\mathrm{d}z^{n-1}}, \ldots, \frac{\mathrm{d}w}{\mathrm{d}z}, w, z\right)$$

- At present there are no comprehensive results for third and higher order equations.
- Partial classifications for the third order equation (**Chazy [1911], Garnier [1907, 1912], Bureau [1964, 1972], Lukashevich [1982], Martynov [1982], Cosgrove [1997, 2000, 2001]**)

$$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} = F\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}, \frac{\mathrm{d}w}{\mathrm{d}z}, w, z\right)$$

  To date, **no** new transcendental third-order equations have been discovered.
- No comprehensive results either for the second order, second degree equation

$$\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 = F\left(\frac{\mathrm{d}w}{\mathrm{d}z}, w, z\right)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + G\left(\frac{\mathrm{d}w}{\mathrm{d}z}, w, z\right)$$

  Cosgrove and Scoufis [1993] have done the special case when $F \equiv 0$.

Painlevé σ-Equations

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma = 0 \qquad \mathrm{S_I}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + \left[4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - 1\right]\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) + \vartheta_0\vartheta_\infty\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \tfrac{1}{4}\left(\vartheta_0^2 + \vartheta_\infty^2\right) \qquad \mathrm{S_{III}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left[2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \vartheta_j\right) = 0 \qquad \mathrm{S_V}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left[z(z-1)\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right]^2 + \left[\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left\{2\sigma - (2z-1)\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right\} + \vartheta_1\vartheta_2\vartheta_3\vartheta_4\right]^2 = \prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \vartheta_j^2\right) \qquad \mathrm{S_{VI}}$$

where α, ϑ_0, ϑ_1, ϑ_2, ϑ_3, ϑ_4 and ϑ_∞ are arbitrary constants.

History of the Painlevé Equations

- Derived by **Painlevé, Gambier** and colleagues in the late 19th/early 20th centuries.
- Studied in Minsk, Belarus by **Erugin, Lukashevich, Gromak** *et al.* since 1950's; much of their work is published in the journal *Diff. Eqns.*, translation of *Diff. Urav.*.
- **Barouch, McCoy, Tracy & Wu [1973, 1976]** showed that the correlation function of the two-dimensional Ising model is expressible in terms of solutions of P_{III}.
- **Ablowitz & Segur [1977]** demonstrated a close connection between completely integrable PDEs solvable by inverse scattering, the so-called **soliton equations**, such as the **Korteweg-de Vries equation** and the **nonlinear Schrödinger equation**, and the Painlevé equations.
- **Flaschka & Newell [1980]** introduced the **isomonodromy deformation method** (inverse scattering for ODEs), which expresses the Painlevé equation as the compatibility condition of two **linear** systems of equations and are studied using **Riemann-Hilbert** methods. Subsequent developments by **Deift, Fokas, Its, Zhou, . . .**
- Algebraic and geometric studies of the Painlevé equations by **Okamoto** in 1980's. Subsequent developments by **Noumi, Umemura, Yamada, . . .**
- The Painlevé equations are a chapter in the "**Digital Library of Mathematical Functions**", which is a rewrite/update of **Abramowitz & Stegun's** "**Handbook of Mathematical Functions**" — see `http://dlmf.nist.gov`.

Some Properties of the Painlevé Equations

- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **Bäcklund transformations** which relate solutions of a given Painlevé equation to solutions of the same Painlevé equation, though with different values of the parameters with associated **Affine Weyl groups** that act on the parameter space.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **rational**, **algebraic** and **special function solutions** expressed in terms of the classical special functions [$\mathrm{P_{II}}$: **Airy** $\mathrm{Ai}(z)$**,** $\mathrm{Bi}(z)$; $\mathrm{P_{III}}$: **Bessel** $J_\nu(z)$**,** $Y_\nu(z)$**,** $J_\nu(z)$**,** $K_\nu(z)$; $\mathrm{P_{IV}}$: **parabolic cylinder** $D_\nu(z)$; $\mathrm{P_{V}}$: **confluent hypergeometric** ${}_1F_1(a;c;z)$ [equivalently **Kummer** $M(a,b,z)$**,** $U(a,b,z)$ or **Whittaker** $M_{\kappa,\mu}(z)$**,** $W_{\kappa,\mu}(z)$]; $\mathrm{P_{VI}}$: **hypergeometric** ${}_2F_1(a,b;c;z)$], for certain values of the parameters.
- These rational, algebraic and special function solutions of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$, called **classical solutions**, can usually be written in **determinantal form**, frequently as **wronskians**. Often these can be written as **Hankel determinants** or **Toeplitz determinants**.
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ can be written as a (non-autonomous) **Hamiltonian system** and the Hamiltonians satisfy a second-order, second-degree differential equations ($\mathrm{S_{I}}$–$\mathrm{S_{VI}}$).
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ possess **Lax pairs** (**isomonodromy problems**).
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ and $\mathrm{S_{I}}$–$\mathrm{S_{VI}}$ form a **coalescence cascade**

$$\begin{array}{ccccccc} \mathrm{P_{VI}} & \longrightarrow & \mathrm{P_{V}} & \longrightarrow & \mathrm{P_{IV}} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{P_{III}} & \longrightarrow & \mathrm{P_{II}} & \longrightarrow & \mathrm{P_{I}} \end{array} \qquad \begin{array}{ccccccc} \mathrm{S_{VI}} & \longrightarrow & \mathrm{S_{V}} & \longrightarrow & \mathrm{S_{IV}} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{S_{III}} & \longrightarrow & \mathrm{S_{II}} & \longrightarrow & \mathrm{S_{I}} \end{array}$$

Properties of the fourth Painlevé equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}$$

- Hamiltonian formulation
- Bäcklund and Schlesinger transformations
- Classical solutions

Hamiltonian Representation of $\mathrm{P_{IV}}$

$\mathrm{P_{IV}}$ can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\mathrm{IV}}}{\partial p} = 4qp - q^2 - 2zq - 2\vartheta_0$$
$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\mathrm{IV}}}{\partial q} = -2p^2 + 2pq + 2zp - \vartheta_\infty$$

where $\mathcal{H}_{\mathrm{IV}}(q,p,z;\vartheta_0,\vartheta_\infty)$ is the Hamiltonian defined by

$$\mathcal{H}_{\mathrm{IV}}(q,p,z;\vartheta_0,\vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

Eliminating p then $w = q$ satisfies

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \tfrac{3}{2}q^3 + 4zq^2 + 2(z^2 + \vartheta_0 - 2\vartheta_\infty - 1)q - \frac{2\vartheta_0^2}{q}$$

which is $\mathrm{P_{IV}}$ with $\alpha = 1 - \vartheta_0 + 2\vartheta_\infty$ and $\beta = -2\vartheta_0^2$, whilst eliminating q then p satisfies

$$\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2p}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + 6p^3 - 8zp^2 + 2(z^2 - 2\vartheta_0 + \vartheta_\infty + 1)p - \frac{\vartheta_\infty^2}{2p}$$

and letting $p = -\tfrac{1}{2}w$ gives $\mathrm{P_{IV}}$ with $\alpha = 2\vartheta_0 - \vartheta_\infty - 1$ and $\beta = -2\vartheta_\infty^2$.

Theorem (Okamoto [1986])

The function

$$\sigma(z;\vartheta_0,\vartheta_\infty) = \mathcal{H}_{\rm IV} \equiv 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

where q *and* p *satisfy the Hamiltonian system*

$$\frac{\mathrm{d}q}{\mathrm{d}z} = 4qp - q^2 - 2zq - 2\vartheta_0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -2p^2 + 2pq + 2zp - \vartheta_\infty \qquad \mathrm{H}_{\rm IV}$$

satisfies the second-order, second-degree equation

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S}_{\rm IV}$$

Conversely, if $\sigma(z;\vartheta_0,\vartheta_\infty)$ *is a solution of* $\mathrm{S}_{\rm IV}$*, then*

$$q(z;\vartheta_0,\vartheta_\infty) = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\vartheta_\infty)}, \qquad p(z;\vartheta_0,\vartheta_\infty) = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\vartheta_0)}$$

are solutions of the Hamiltonian system $\mathrm{H}_{\rm IV}$.

Bäcklund Transformations

Definition

- A **Bäcklund transformation** maps solutions of a given Painlevé equation to solutions of the same Painlevé equation, though with different values of the parameters.

Bäcklund Transformations of $\mathrm{P_{IV}}$

Theorem

Let $w = w(z;\alpha,\beta)$ *and* $w_j^{\pm} = w(z;\alpha_j^{\pm},\beta_j^{\pm})$, $j = 1,2,3,4$ *be solutions of* $\mathrm{P_{IV}}$ *with*

$$
\begin{aligned}
\alpha_1^{\pm} &= \tfrac14\left(2 - 2\alpha \pm 3\sqrt{-2\beta}\right), & \beta_1^{\pm} &= -\tfrac12\left(1 + \alpha \pm \tfrac12\sqrt{-2\beta}\right)^2\\
\alpha_2^{\pm} &= -\tfrac14\left(2 + 2\alpha \pm 3\sqrt{-2\beta}\right), & \beta_2^{\pm} &= -\tfrac12\left(1 - \alpha \pm \tfrac12\sqrt{-2\beta}\right)^2\\
\alpha_3^{\pm} &= \tfrac32 - \tfrac12\alpha \mp \tfrac34\sqrt{-2\beta}, & \beta_3^{\pm} &= -\tfrac12\left(1 - \alpha \pm \tfrac12\sqrt{-2\beta}\right)^2\\
\alpha_4^{\pm} &= -\tfrac32 - \tfrac12\alpha \mp \tfrac34\sqrt{-2\beta}, & \beta_4^{\pm} &= -\tfrac12\left(-1 - \alpha \pm \tfrac12\sqrt{-2\beta}\right)^2
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{T}_1^{\pm}: \qquad & w_1^{\pm} = \frac{w' - w^2 - 2zw \mp \sqrt{-2\beta}}{2w}\\
\mathcal{T}_2^{\pm}: \qquad & w_2^{\pm} = -\frac{w' + w^2 + 2zw \mp \sqrt{-2\beta}}{2w}\\
\mathcal{T}_3^{\pm}: \qquad & w_3^{\pm} = w + \frac{2\left(1 - \alpha \mp \frac12\sqrt{-2\beta}\right)w}{w' \pm \sqrt{-2\beta} + 2zw + w^2}\\
\mathcal{T}_4^{\pm}: \qquad & w_4^{\pm} = w + \frac{2\left(1 + \alpha \pm \frac12\sqrt{-2\beta}\right)w}{w' \mp \sqrt{-2\beta} - 2zw - w^2}
\end{aligned}
$$

which are valid when the denominators are non-zero, and where the upper signs or the lower signs are taken throughout each transformation.

Schlesinger Transformations of $\mathrm{P_{IV}}$

(**Fokas, Mugan & Ablowitz [1988]**)

	α	β	κ_0	κ_∞
$\mathcal{R}_1$	$\alpha+1$	$-\frac{1}{2}\left(2-\sqrt{-2\beta}\right)^2$	κ_0-1	$\kappa_\infty+1$
$\mathcal{R}_2$	$\alpha-1$	$-\frac{1}{2}\left(2+\sqrt{-2\beta}\right)^2$	κ_0+1	$\kappa_\infty-1$
$\mathcal{R}_3$	$\alpha+1$	$-\frac{1}{2}\left(2+\sqrt{-2\beta}\right)^2$	κ_0+1	$\kappa_\infty+1$
$\mathcal{R}_4$	$\alpha-1$	$-\frac{1}{2}\left(2-\sqrt{-2\beta}\right)^2$	κ_0-1	$\kappa_\infty-1$

$$\mathcal{R}_1: \quad w_1 = \frac{\left(w'+\sqrt{-2\beta}\right)^2 + \left(4\alpha+4-2\sqrt{-2\beta}\right)w^2 - w^2(w+2z)^2}{2w\left(w^2+2zw-w'-\sqrt{-2\beta}\right)},$$

$$\mathcal{R}_2: \quad w_2 = \frac{\left(w'-\sqrt{-2\beta}\right)^2 + \left(4\alpha-4-2\sqrt{-2\beta}\right)w^2 - w^2(w+2z)^2}{2w\left(w^2+2zw+w'-\sqrt{-2\beta}\right)},$$

$$\mathcal{R}_3: \quad w_3 = \frac{\left(w'-\sqrt{-2\beta}\right)^2 - \left(4\alpha+4+2\sqrt{-2\beta}\right)w^2 - w^2(w+2z)^2}{2w\left(w^2+2zw-w'+\sqrt{-2\beta}\right)},$$

$$\mathcal{R}_4: \quad w_4 = \frac{\left(w'+\sqrt{-2\beta}\right)^2 + \left(4\alpha-4+2\sqrt{-2\beta}\right)w^2 - w^2(w+2z)^2}{2w\left(w^2+2zw+w'+\sqrt{-2\beta}\right)},$$

Fokas, Mugan & Ablowitz [1988] defined the composite transformations

	α	β	κ_0	κ_∞
$\mathcal{R}_5 \equiv \mathcal{R}_1\mathcal{R}_3$	$\alpha+2$	β	κ_0	$\kappa_\infty+2$
$\mathcal{R}_6^+ \equiv \mathcal{R}_2\mathcal{R}_3$	α	$-\frac{1}{2}\left(4+\sqrt{-2\beta}\right)^2$	κ_0+2	κ_∞
$\mathcal{R}_6^- \equiv \mathcal{R}_1\mathcal{R}_4$	α	$-\frac{1}{2}\left(4-\sqrt{-2\beta}\right)^2$	κ_0-2	κ_∞
$\mathcal{R}_7 \equiv \mathcal{R}_2\mathcal{R}_4$	$\alpha-2$	β	κ_0	$\kappa_\infty-2$

$$\mathcal{R}_5: \qquad w_5 = \frac{\left(w'-w^2-2zw\right)^2+2\beta}{2w\left\{w'-w^2-2zw+2(\alpha+1)\right\}},$$

$$\mathcal{R}_6^\pm: \qquad w_6 = w + \frac{\left(2\alpha-2\mp\sqrt{-2\beta}\right)wM^\pm(w,w',z;\alpha,\beta)}{w(4\pm2\sqrt{-2\beta})-M^\pm(w,w',z)\left(w'-2zw-w^2\mp\sqrt{-2\beta}\right)}$$
$$+\frac{\left(2+2\alpha\pm\sqrt{-2\beta}\right)w}{w'-2zw-w^2\mp\sqrt{-2\beta}}+\frac{2\pm\sqrt{-2\beta}}{M^\pm(w,w',z;\alpha,\beta)},$$

$$\mathcal{R}_7: \qquad w_7 = -\frac{\left(w'+w^2+2zw\right)^2+2\beta}{2w\left\{w'+w^2+2zw-2(\alpha-1)\right\}},$$

where

$$M^\pm(w,w',z;\alpha,\beta) = \tfrac{1}{2}w+z+\frac{\left(2+2\alpha\pm\sqrt{-2\beta}\right)w}{w'-2zw-w^2\mp\sqrt{-2\beta}}+\frac{w'\mp\sqrt{-2\beta}}{2w}.$$

Remark: **Murata [1985]** derived the transformations $\mathcal{R}_5$ and $\mathcal{R}_7$.

Classical Solutions of the Fourth Painlevé Equation and the Fourth Painlevé σ-Equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

Classical Solutions of $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

Theorem

- $\mathrm{P_{IV}}$ *has* ***rational solutions*** *if and only if*

  $$(\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\boldsymbol{m}, -\mathbf{2}(\mathbf{2}\boldsymbol{n}-\boldsymbol{m}+\mathbf{1})^{\mathbf{2}}\right) \quad \text{or} \quad (\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\boldsymbol{m}, -\mathbf{2}(\mathbf{2}\boldsymbol{n}-\boldsymbol{m}+\tfrac{\mathbf{1}}{\mathbf{3}})^{\mathbf{2}}\right)$$

  with $m, n \in \mathbb{Z}$*. Further these rational solutions are unique.*

- $\mathrm{P_{IV}}$ *has* ***special function solutions*** *in terms of* ***parabolic cylinder functions*** *through the Riccati equation*

  $$z\frac{\mathrm{d}w}{\mathrm{d}z} = \varepsilon(w^2+2zw) - 2(1+\varepsilon\alpha), \qquad \varepsilon = \pm 1$$

  if and only if

  $$\beta = -2(2n+1+\varepsilon\alpha)^2 \qquad \text{or} \qquad \beta = -2n^2$$

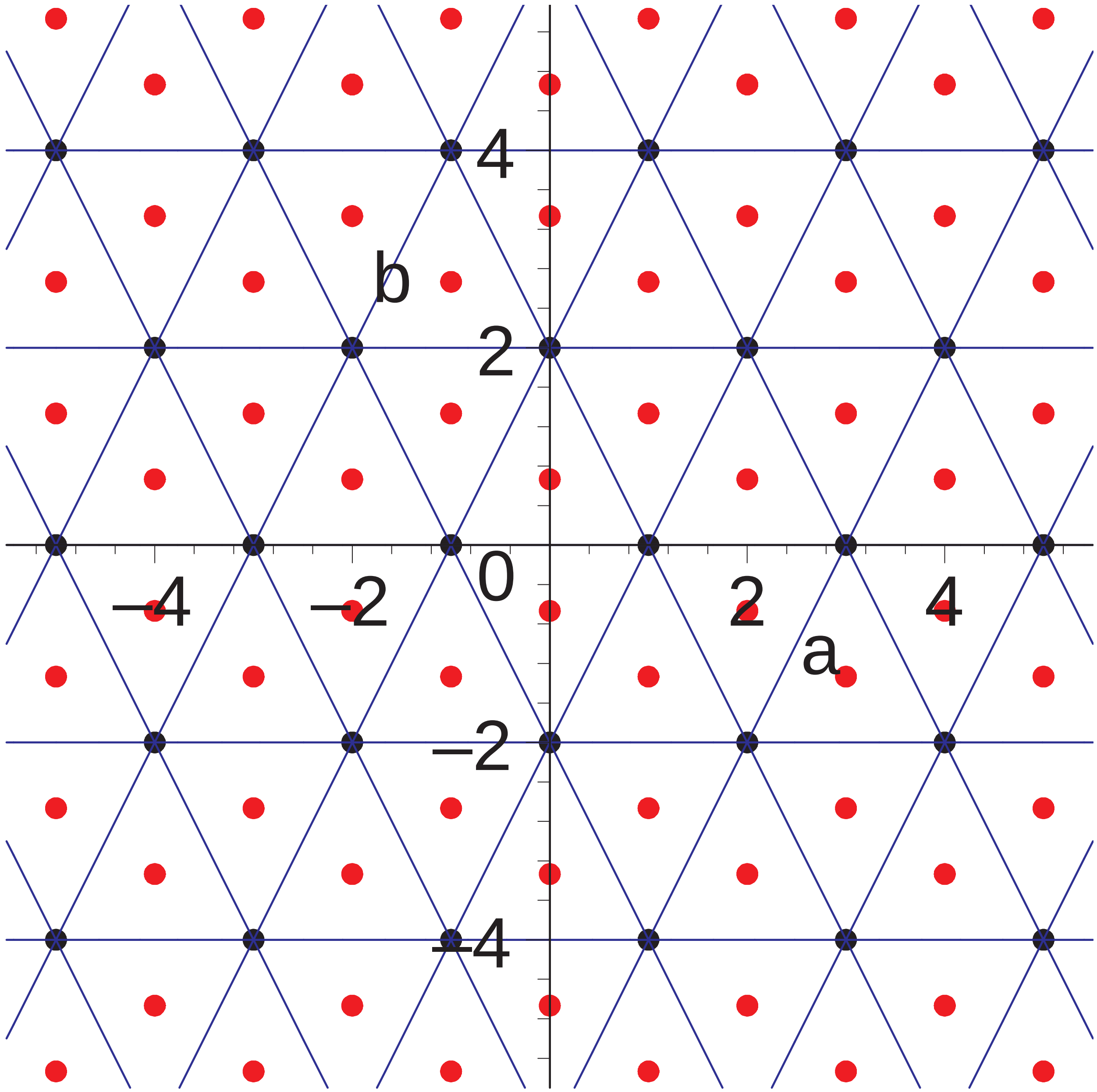
  with $n \in \mathbb{Z}$ *which has solution*

  $$w(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\varphi_\nu(z;\varepsilon)$$

  where $\varphi_\nu(z;\varepsilon)$ *satisfies the* ***Weber-Hermite equation***

  $$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0$$

Rational and Special Function Solutions of S_{IV}



$\mathrm{P_{IV}}$ — Generalized Hermite Polynomials

Theorem (**Kajiwara & Ohta [1998], Noumi & Yamada [1998]**)

Define the ***generalized Hermite polynomial*** $\boldsymbol{H_{m,n}(z)}$, *which has degree* mn, *by*

$$H_{m,n}(z) = a_{m,n}\,\mathcal{W}\left(H_m(z), H_{m+1}(z), \ldots, H_{m+n-1}(z)\right), \qquad m,n \geq 1$$

where $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ *is the Wronskian,* $H_n(z)$ *is the* n^{th} *Hermite polynomial and* $a_{m,n}$ *is a constant. Then*

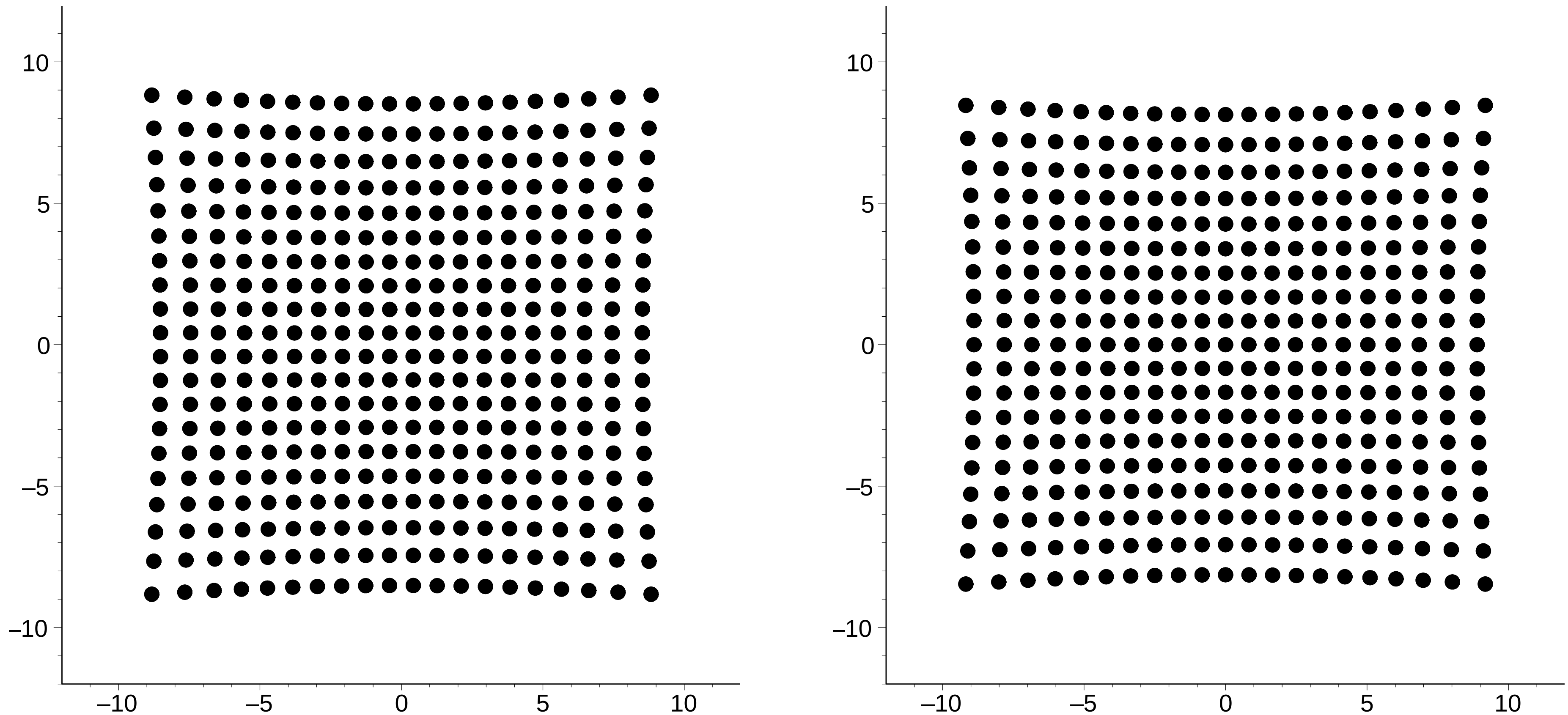
$$w^{(\mathrm{i})}_{m,n}(z) = w(z; \alpha^{(\mathrm{i})}_{m,n}, \beta^{(\mathrm{i})}_{m,n}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{m+1,n}(z)}{H_{m,n}(z)}$$
$$w^{(\mathrm{ii})}_{m,n}(z) = w(z; \alpha^{(\mathrm{ii})}_{m,n}, \beta^{(\mathrm{ii})}_{m,n}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{m,n}(z)}{H_{m,n+1}(z)}$$
$$w^{(\mathrm{iii})}_{m,n}(z) = w(z; \alpha^{(\mathrm{iii})}_{m,n}, \beta^{(\mathrm{iii})}_{m,n}) = -2z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{m,n+1}(z)}{H_{m+1,n}(z)}$$

are respectively solutions of $\mathrm{P_{IV}}$ *for*

$$(\alpha^{(\mathrm{i})}_{m,n}, \beta^{(\mathrm{i})}_{m,n}) = (2m+n+1, -2n^2)$$
$$(\alpha^{(\mathrm{ii})}_{m,n}, \beta^{(\mathrm{ii})}_{m,n}) = (-m-2n-1, -2m^2)$$
$$(\alpha^{(\mathrm{iii})}_{m,n}, \beta^{(\mathrm{iii})}_{m,n}) = (n-m, -2(m+n+1)^2)$$

Roots of the Generalized Hermite Polynomials $H_{m,n}(z)$

(PAC [2003])



$H_{20,20}(z)$ $\qquad$ $H_{21,19}(z)$

$m \times n$ "rectangles"

Properties of the Generalized Hermite Polynomials $H_{m,n}(z)$

- $H_{m,n}(z)$ can be expressed as the multiple integral

$$H_{m,n}(z) = \frac{\pi^{m/2}\prod_{k=1}^{m} k!}{2^{m(m+2n-1)/2}} \int_{-\infty}^{\infty} \overset{n}{\cdots} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \prod_{j=i+1}^{n} (x_i - x_j)^2 \prod_{k=1}^{n} (z - x_k)^m \\ \times \exp\left(-x_1^2 - x_2^2 - \ldots - x_n^2\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

  which arises in random matrix theory (**Brézin & Hikami [2000], Forrester & Witte [2001], Kanzieper [2002]**).

- $H_{m,n}(z)$ satisfies the fourth order bilinear equation

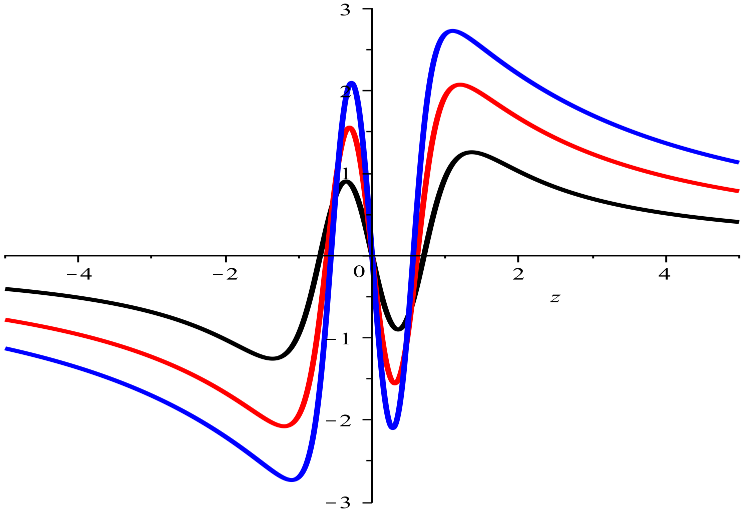
$$H_{m,n}H''''_{m,n} - 4H'_{m,n}H'''_{m,n} + 3\left(H''_{m,n}\right)^2 + 4zH_{m,n}H'_{m,n} - 8mnH^2_{m,n} \\ - 4(z^2 + 2n - 2m)\left\{H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2\right\} = 0$$

  and homogeneous difference equations (**PAC [2005]**).

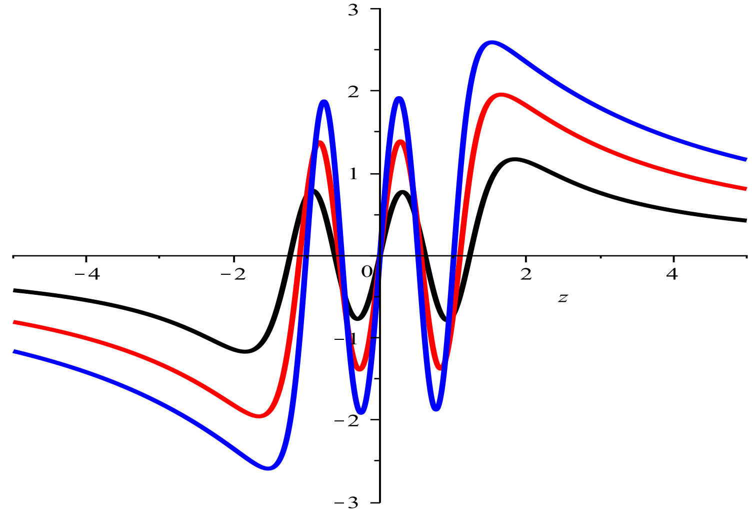
- $H_{m,n}(z)$ has a real zero unless n is a positive, even integer. Hence the only bounded rational solutions of $\mathrm{P_{IV}}$ are $w^{[1]}_{m,2n}(z)$, with $n \in \mathbb{Z}^+$, which have $2m+1$ real zeros and asymptotics, as $z \to \infty$

$$w^{[1]}_{m,2n}(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{m+1,2n}(z)}{H_{m,2n}(z)} \sim \frac{2n}{z} + \frac{(2m-2n+1)n}{z^3} + \mathcal{O}\left(\frac{1}{z^5}\right)$$

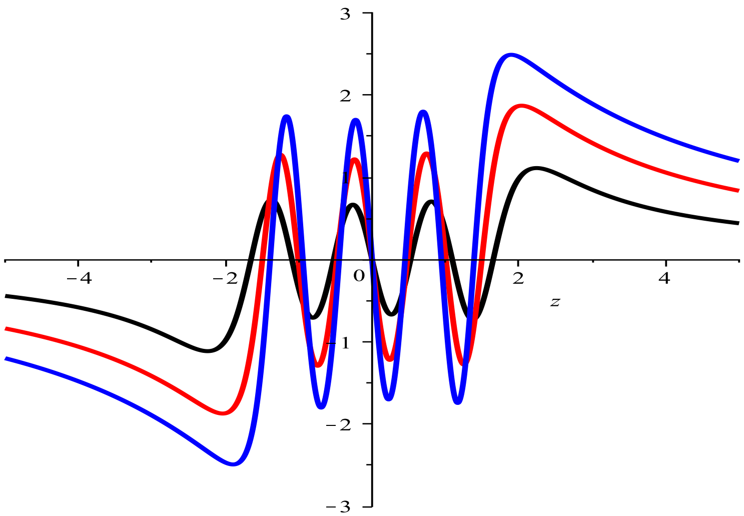
Plots of Bounded Rational Solutions of $\mathrm{P_{IV}}$



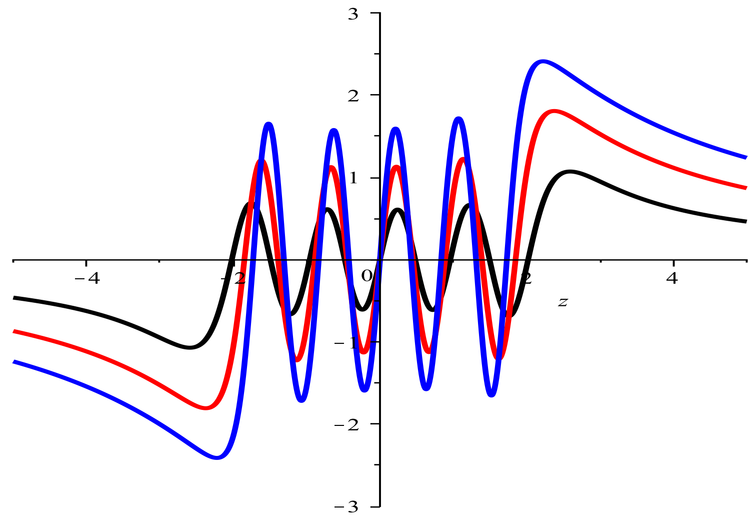
$$w_{1,2}^{[1]}(z), \quad w_{1,4}^{[1]}(z), \quad w_{1,6}^{[1]}(z)$$



$$w_{2,2}^{[1]}(z), \quad w_{2,4}^{[1]}(z), \quad w_{2,6}^{[1]}(z)$$



$$w_{3,2}^{[1]}(z), \quad w_{3,4}^{[1]}(z), \quad w_{3,6}^{[1]}(z)$$



$$w_{4,2}^{[1]}(z), \quad w_{4,4}^{[1]}(z), \quad w_{4,6}^{[1]}(z)$$

$\mathrm{P_{IV}}$ — Generalized Okamoto Polynomials

Theorem (**Kajiwara & Ohta [1998], Noumi & Yamada [1998], PAC [2006]**)

Let $\varphi_k(z) = 3^{k/2}\mathrm{e}^{-k\pi\mathrm{i}/2}H_k\left(\tfrac{1}{3}\sqrt{3}\,\mathrm{i}z\right)$*, with* $H_k(\zeta)$ *the* k^{th} *Hermite polynomial, then define the* ***generalized Okamoto polynomial*** $\boldsymbol{Q_{m,n}(z)}$ *by*

$$Q_{m,n}(z) = \mathcal{W}(\varphi_1, \varphi_4, \ldots, \varphi_{3m+3n-5}; \varphi_2, \varphi_5, \ldots, \varphi_{3n-4})$$

with $m, n \geq 1$*, where* $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ *is the Wronskian. Then*

$$\widetilde{w}^{(\mathrm{i})}_{m,n}(z) = w(z; \widetilde{\alpha}^{(\mathrm{i})}_{m,n}, \widetilde{\beta}^{(\mathrm{i})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m+1,n}(z)}{Q_{m,n}(z)}$$

$$\widetilde{w}^{(\mathrm{ii})}_{m,n}(z) = w(z; \widetilde{\alpha}^{(\mathrm{ii})}_{m,n}, \widetilde{\beta}^{(\mathrm{ii})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n}(z)}{Q_{m,n+1}(z)}$$

$$\widetilde{w}^{(\mathrm{iii})}_{m,n}(z) = w(z; \widetilde{\alpha}^{(\mathrm{iii})}_{m,n}, \widetilde{\beta}^{(\mathrm{iii})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n+1}(z)}{Q_{m+1,n}(z)}$$

are respectively solutions of $\mathrm{P_{IV}}$ *for*

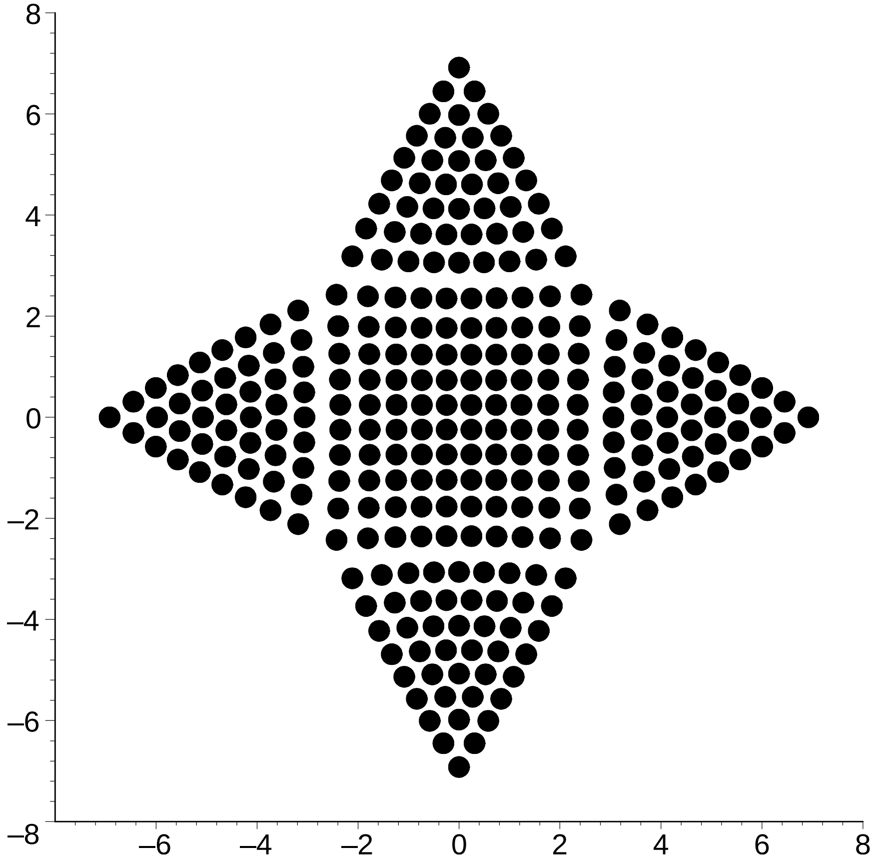
$$(\widetilde{\alpha}^{(\mathrm{i})}_{m,n}, \widetilde{\beta}^{(\mathrm{i})}_{m,n}) = (2m+n, -2(n-\tfrac{1}{3})^2)$$

$$(\widetilde{\alpha}^{(\mathrm{ii})}_{m,n}, \widetilde{\beta}^{(\mathrm{ii})}_{m,n}) = (-m-2n, -2(m-\tfrac{1}{3})^2)$$

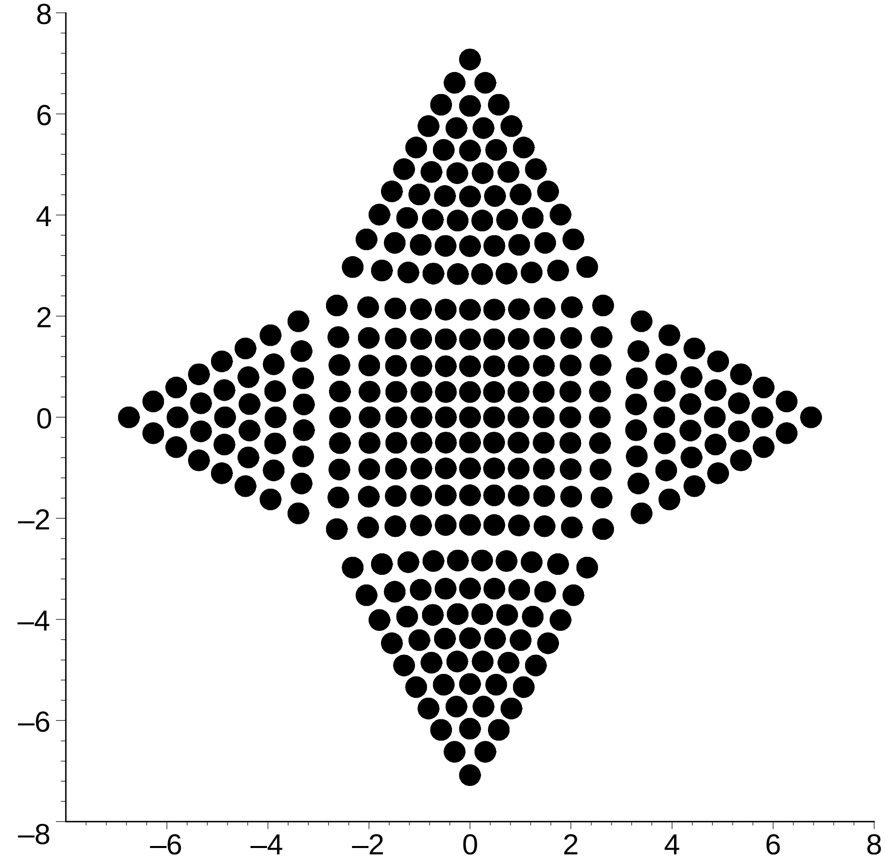
$$(\widetilde{\alpha}^{(\mathrm{iii})}_{m,n}, \widetilde{\beta}^{(\mathrm{iii})}_{m,n}) = (n-m, -2(m+n+\tfrac{1}{3})^2)$$

Roots of the Generalized Okamoto Polynomials $Q_{m,n}(z)$, $m,n > 0$

(PAC [2003])



$Q_{10,10}(z)$



$Q_{11,9}(z)$

$m \times n$ “rectangles" and “equilateral triangles" with sides $m-1$ and $n-1$

Rational and Rational-Oscillatory Solutions of the NLS Equation

Theorem (**PAC [2006]**)

The de-focusing NLS equation

$$\mathrm{i}u_t = u_{xx} - 2|u|^2u \tag{1}$$

has decaying rational solutions of the form

$$u_n(x,t) = \frac{n\mathrm{e}^{\pi\mathrm{i}/4}}{\sqrt{t}} \frac{H_{n+1,n-1}(z)}{H_{n,n}(z)}, \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{t}} \tag{2}$$

and non-decaying rational-oscillatory solutions of the forms

$$\widetilde{u}_n(x,t) = \frac{\mathrm{e}^{-\pi\mathrm{i}/4}}{3\sqrt{2t}} \frac{Q_{n+1,n-1}(z)}{Q_{n,n}(z)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{t}} \tag{3}$$

where $n \geq 1$.

- The rational solutions (2) generalize the results of **Hirota & Nakamura [1985]** (see also **Boiti & Pempinelli [1981]; Hone [1996]**).
- The rational-oscillatory solutions (3) are new solutions of the NLS equation (1).
- There are other rational-oscillatory solutions of the NLS equation (1), e.g. the **Ma-Peregrine solution**

$$u(x,t) = \left\{1 - \frac{4(1+4\mathrm{i}t)}{1-4x^2+16t^2}\right\}\mathrm{e}^{2\mathrm{i}t}$$

Parabolic Cylinder Function Solutions of $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}$$

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\psi_\nu(z;\varepsilon), \psi'_\nu(z;\varepsilon), \ldots, \psi_\nu^{(n-1)}(z;\varepsilon)\right), \qquad n \geq 1$$

where $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\psi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\psi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\psi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\psi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then ***parabolic cylinder function*** *solutions of* $\mathrm{P_{IV}}$ *are given by*

$$w^{[1]}_{\nu,n}(z) = -2z + \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \quad \left(\alpha^{[1]}_{\nu,n}, \beta^{[1]}_{\nu,n}\right) = \left(\varepsilon(2n-\nu), -2(\nu+1)^2\right)$$

$$w^{[2]}_{\nu,n}(z) = \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu+1,n}(z;\varepsilon)}, \quad \left(\alpha^{[2]}_{\nu,n}, \beta^{[2]}_{\nu,n}\right) = \left(-\varepsilon(n+\nu), -2(\nu-n+1)^2\right)$$

$$w^{[3]}_{\nu,n}(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu+1,n}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \quad \left(\alpha^{[3]}_{\nu,n}, \beta^{[3]}_{\nu,n}\right) = \left(\varepsilon(2\nu-n+1), -2n^2\right)$$

$$\frac{\mathrm{d}^2\psi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\psi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\psi_\nu = 0, \qquad \varepsilon^2 = 1 \tag{$*$}$$

- If $\nu \notin \mathbb{N}$ then $(*)$ has the solutions

$$\psi_\nu(z;\varepsilon) = \begin{cases} \left\{C_1 D_\nu(\sqrt{2}\,z) + C_2 D_\nu(-\sqrt{2}\,z)\right\}\exp\left(\tfrac{1}{2}z^2\right), & \text{if} \quad \varepsilon = 1 \\ \left\{C_1 D_{-\nu-1}(\sqrt{2}\,z) + C_2 D_{-\nu-1}(-\sqrt{2}\,z)\right\}\exp\left(-\tfrac{1}{2}z^2\right), & \text{if} \quad \varepsilon = -1 \end{cases}$$

with C_1 and C_2 arbitrary constants and where $D_\nu(\zeta)$ is the **parabolic cylinder function** which satisfies

$$\frac{\mathrm{d}^2 D_\nu}{\mathrm{d}\zeta^2} = (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})D_\nu$$

with boundary condition

$$D_\nu(\zeta) \sim \zeta^\nu \exp\left(-\tfrac{1}{4}\zeta^2\right), \qquad \text{as} \quad \zeta \to +\infty$$

- If $\nu = 0$ then $(*)$ has the solutions

$$\psi_0(z;\varepsilon) = \begin{cases} C_1 + C_2\,\mathrm{erfi}(z), & \text{if} \quad \varepsilon = 1 \\ C_1 + C_2\,\mathrm{erfc}(z), & \text{if} \quad \varepsilon = -1 \end{cases}$$

with C_1 and C_2 arbitrary constants, where $\mathrm{erfc}(z)$ is the **complementary error function** and $\mathrm{erfi}(z)$ the **imaginary error function**, which are defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-t^2)\,\mathrm{d}t, \qquad \mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(t^2)\,\mathrm{d}t$$

$$\frac{\mathrm{d}^2\psi_\nu}{\mathrm{d}z^2} - 2\varepsilon z \frac{\mathrm{d}\psi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\psi_\nu = 0, \qquad \varepsilon^2 = 1 \tag{$*$}$$

- If $\nu = n$, for $n \geq 1$, then $(*)$ has the solutions

$$\psi_n(z;\varepsilon) = \begin{cases} C_1 H_n(z) + C_2 \exp(z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \text{if} \quad \varepsilon = 1 \\ C_1(-\mathrm{i})^n H_n(\mathrm{i}z) + C_2 \exp(-z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \text{if} \quad \varepsilon = -1 \end{cases}$$

with C_1 and C_2 arbitrary constants, where $H_n(z)$ is the **Hermite polynomial**, $\operatorname{erfc}(z)$ the **complementary error function** and $\operatorname{erfi}(z)$ the **imaginary error function**.

- If $\nu = -n$, for $n \geq 1$, then $(*)$ has the solutions

$$\psi_{-n}(z;\varepsilon) = \begin{cases} C_1(-\mathrm{i})^{n-1} H_{n-1}(\mathrm{i}z)\exp(z^2) + C_2 \dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \text{if} \quad \varepsilon = 1 \\ C_1 H_{n-1}(z)\exp(-z^2) + C_2 \dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \text{if} \quad \varepsilon = -1 \end{cases}$$

with C_1 and C_2 arbitrary constants, where $H_n(z)$ is the **Hermite polynomial**, $\operatorname{erfc}(z)$ the **complementary error function** and $\operatorname{erfi}(z)$ the **imaginary error function**.

Special Cases

$$D_0\left(\sqrt{2}\,z\right) = \exp(-\tfrac{1}{2}z^2)$$

$$D_n\left(\sqrt{2}\,z\right) = (\tfrac{1}{2})^{n/2}\exp(-\tfrac{1}{2}z^2)\,H_n(z), \qquad n = 1, 2, \ldots$$

$$D_{-1}\left(\sqrt{2}\,z\right) = \tfrac{1}{2}\sqrt{2\pi}\,\exp(\tfrac{1}{2}z^2)\,\mathrm{erfc}(z)$$

$$D_{-n-1}\left(\sqrt{2}\,z\right) = \frac{(-1)^n\sqrt{\pi}}{n!\,2^{(n+1)/2}}\exp(-\tfrac{1}{2}z^2)\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\exp(z^2)\,\mathrm{erfc}(z)\right\}, \qquad n = 1, 2, \ldots$$

Integral Representation

$$D_\nu(z) = \frac{\exp\left(-\frac{1}{4}z^2\right)}{\Gamma(-\nu)}\int_0^\infty t^{-\nu-1}\exp\left(-\tfrac{1}{2}t^2 - zt\right)\mathrm{d}t, \qquad \nu < 0$$

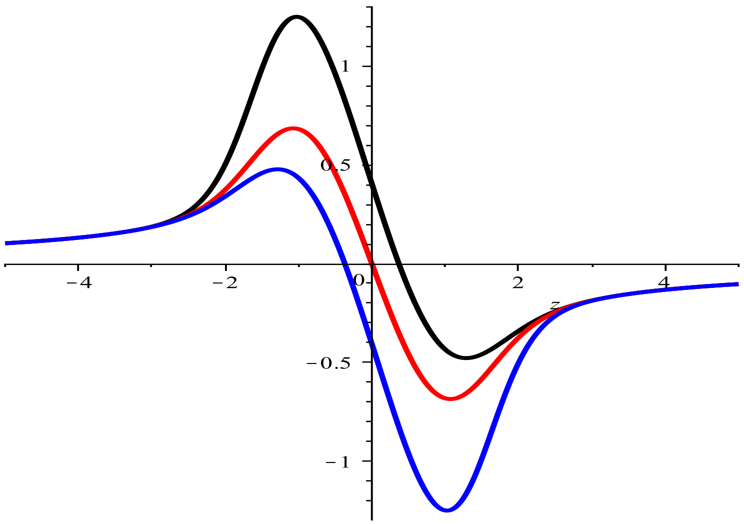
Property

- The **parabolic cylinder function** $D_\nu(z)$ has no real zeros if $\nu < 0$, so

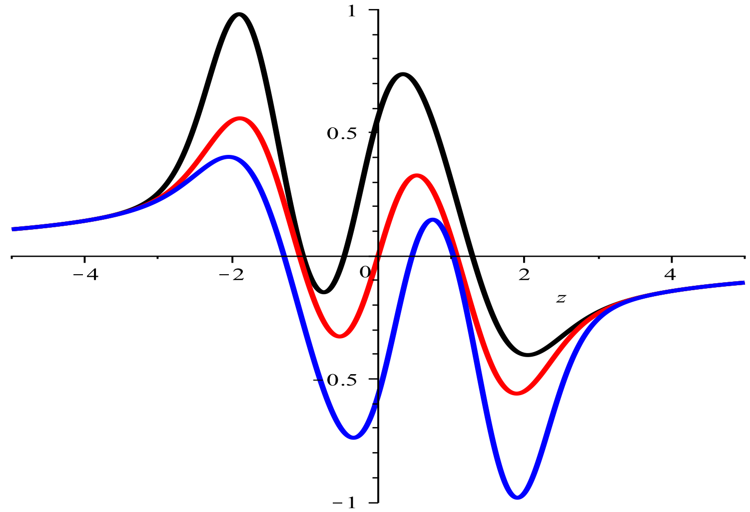
$$\psi_\nu(z) = \left\{C_1 D_{-\nu}(\sqrt{2}\,z) + C_2 D_{-\nu}(-\sqrt{2}\,z)\right\}\exp\left(\tfrac{1}{2}z^2\right)$$

has no real zeros if $\nu > 0$ and $C_1 C_2 > 0$.

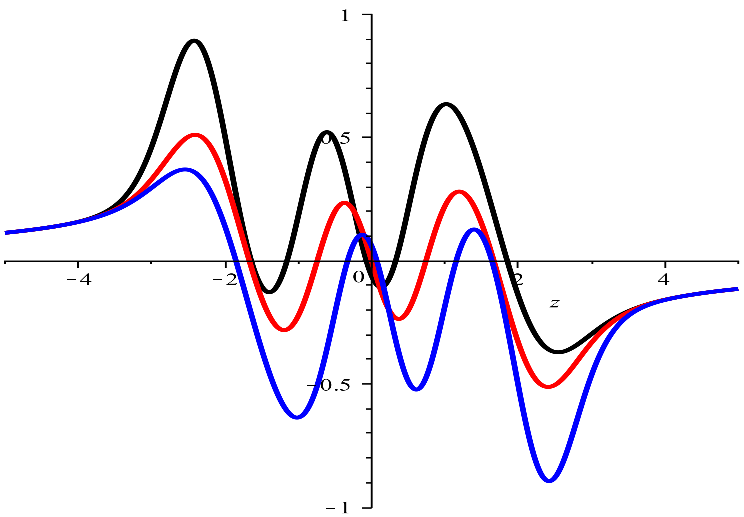
Plots of Parabolic Cylinder Function Solutions of $\mathrm{P_{IV}}$



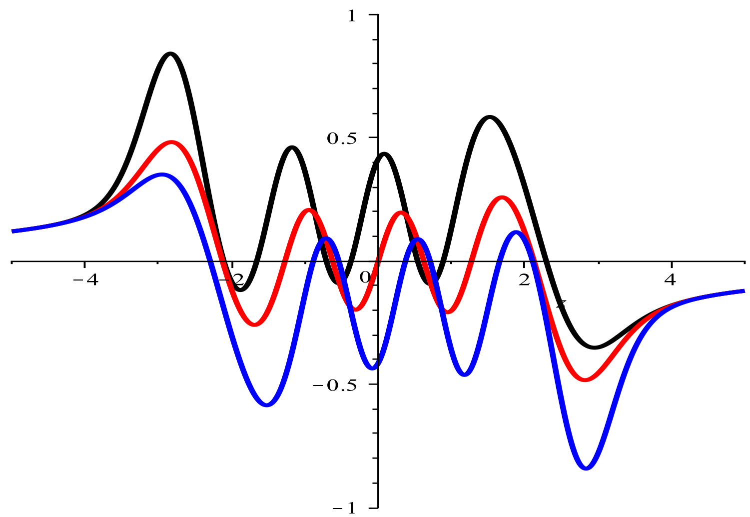
$w(z; \frac{1}{2}, -\frac{1}{2})$



$w(z; \frac{5}{2}, -\frac{1}{2})$

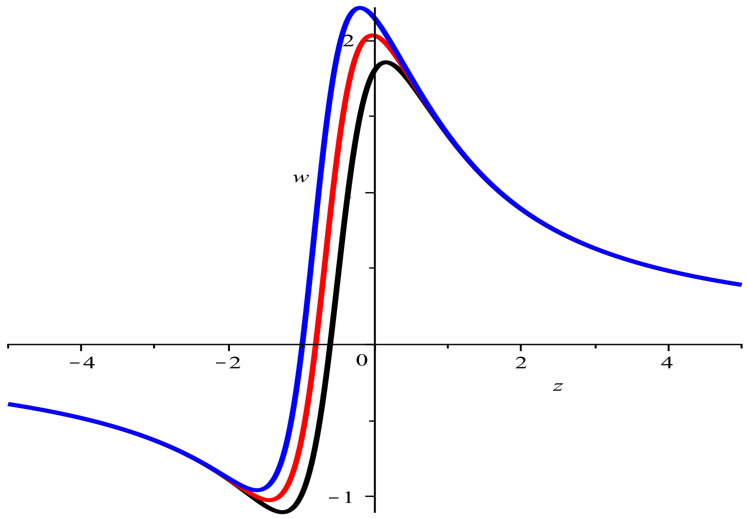


$w(z; \frac{9}{2}, -\frac{1}{2})$

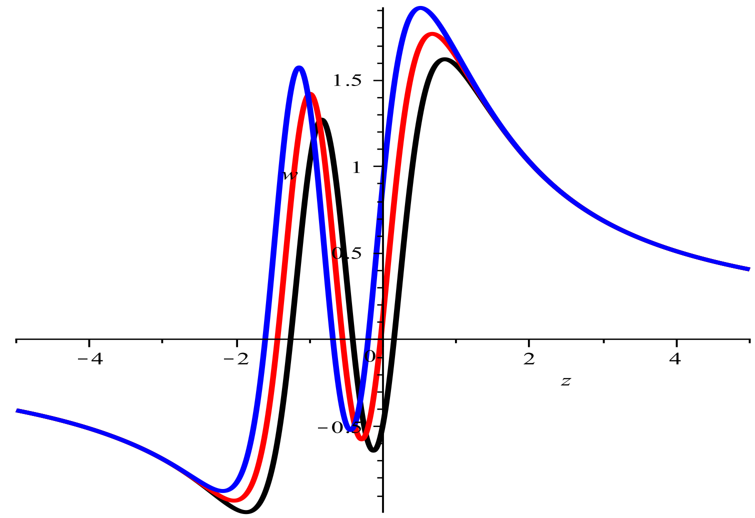


$w(z; \frac{13}{2}, -\frac{1}{2})$

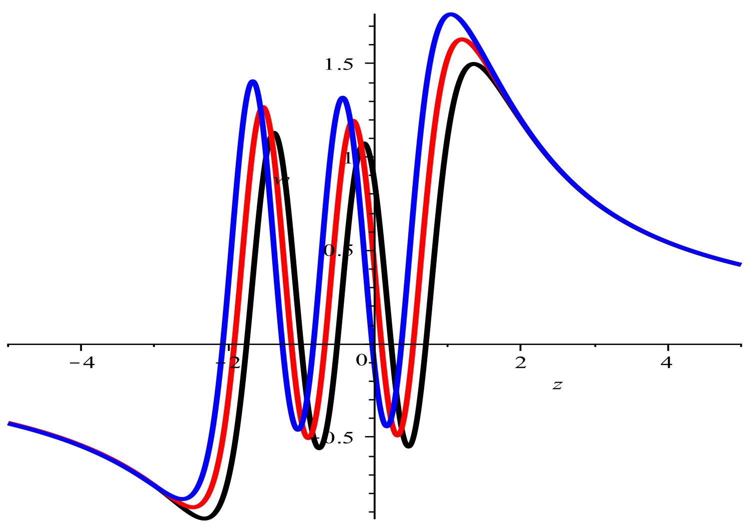
Plots of Error Function Solutions of $\mathrm{P_{IV}}$



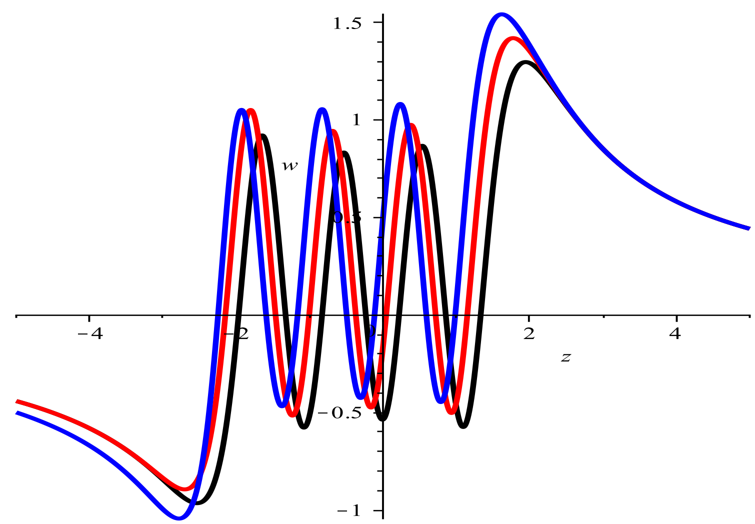
$w(z;3,-8)$



$w(z;5,-8)$

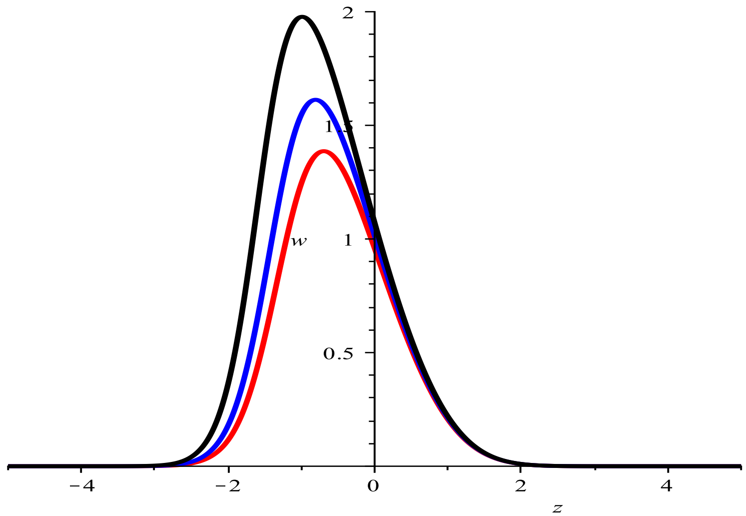


$w(z;7,-8)$

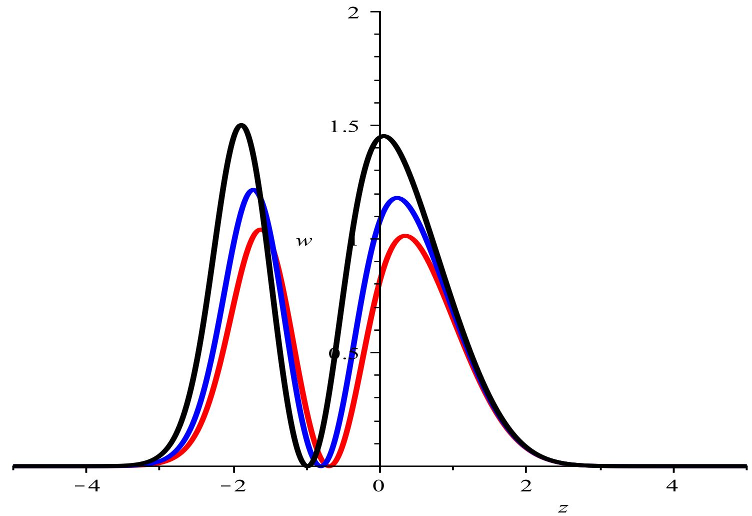


$w(z;9,-8)$

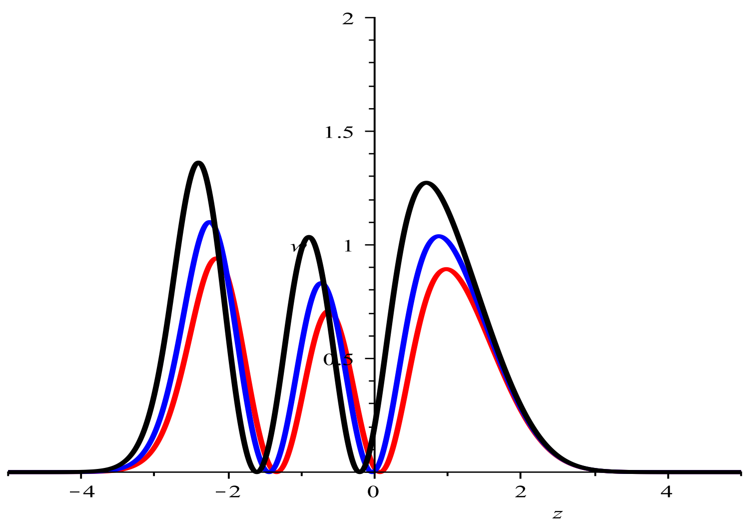
Plots of Bound State Solutions of P_{IV}



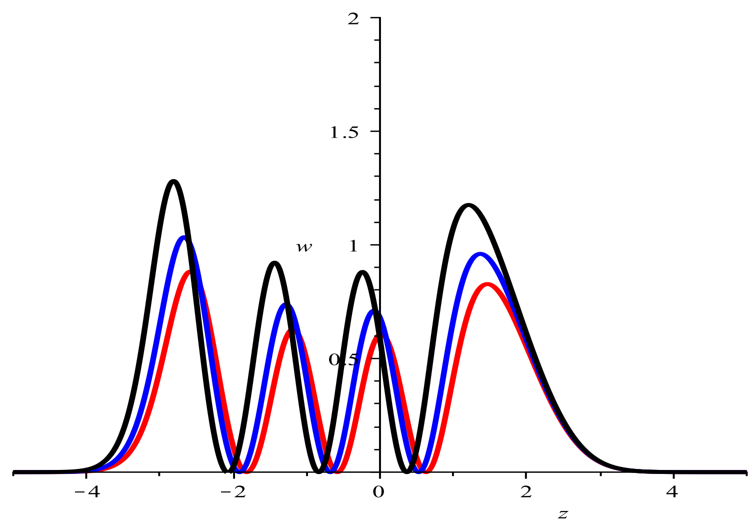
$w(z; 1, 0)$



$w(z; 3, 0)$



$w(z; 5, 0)$



$w(z; 7, 0)$

Classical Solutions of S_{IV}

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0$$

Theorem

- *Rational solutions of* S_{IV} *are given by*

$$\sigma_{m,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln H_{m,n}(z), \qquad (\vartheta_0, \vartheta_\infty) = (m, -n)$$

$$\widetilde{\sigma}_{m,n}(z) = \tfrac{4}{27}z^3 - \tfrac{2}{3}(m-n)z + \frac{\mathrm{d}}{\mathrm{d}z}\ln Q_{m,n}(z), \quad (\vartheta_0, \vartheta_\infty) = \left(m - \tfrac{1}{3}, -n + \tfrac{1}{3}\right)$$

where $H_{m,n}(z)$ *is the* ***generalized Hermite polynomial*** *and* $Q_{m,n}(z)$ *the* ***generalized Okamoto polynomial****.*

- *Suppose* $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\psi_\nu(z;\varepsilon), \psi_\nu'(z;\varepsilon), \ldots, \psi_\nu^{(n-1)}(z;\varepsilon)\right), \qquad n \geq 1$$

where $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\psi_\nu(z;\varepsilon)$ *satisfies*

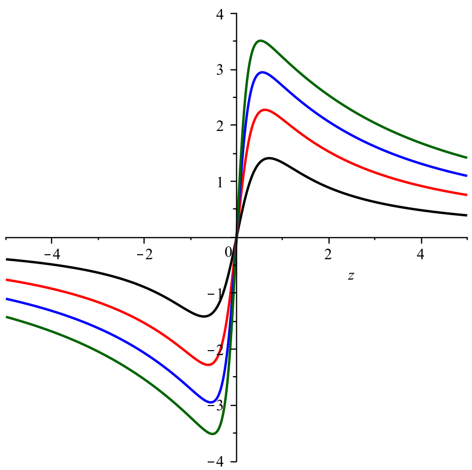
$$\frac{\mathrm{d}^2\psi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\psi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\psi_\nu = 0, \qquad \varepsilon^2 = 1$$

then ***parabolic cylinder function*** *solutions of* S_{IV} *are given by*

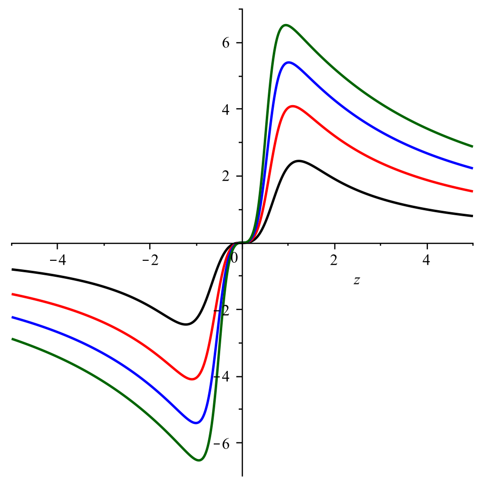
$$\sigma_{\nu,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_{\nu,n}(z;\varepsilon), \qquad (\vartheta_0, \vartheta_\infty) = \left(\varepsilon(\nu - n + 1), -\varepsilon n\right)$$

Plots of Bounded Rational Solutions of $\mathrm{S_{IV}}$

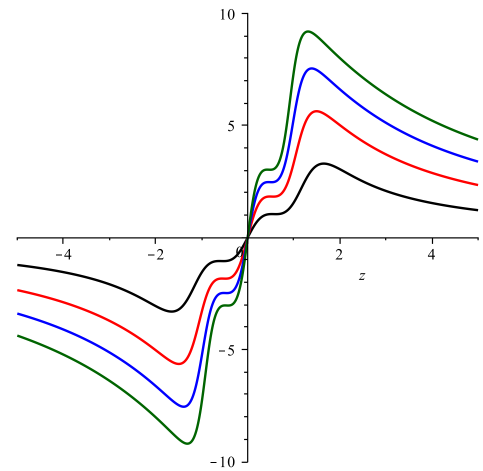
$$\sigma_{m,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln H_{m,n}(z), \qquad H_{m,n}(z) = \mathcal{W}\big(H_m, H_{m+1}, \ldots, H_{m+n-1}\big)$$



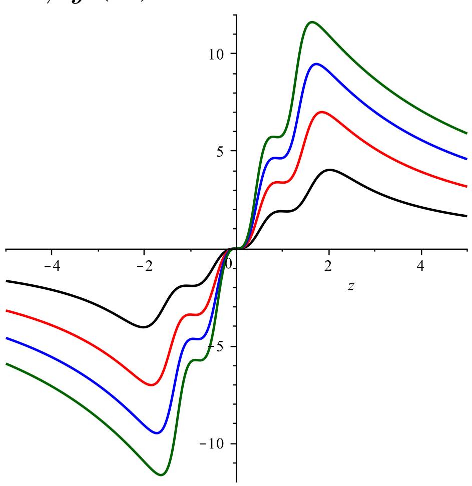
$\sigma_{1,2j}(z), \quad j = 1, 2, 3, 4$



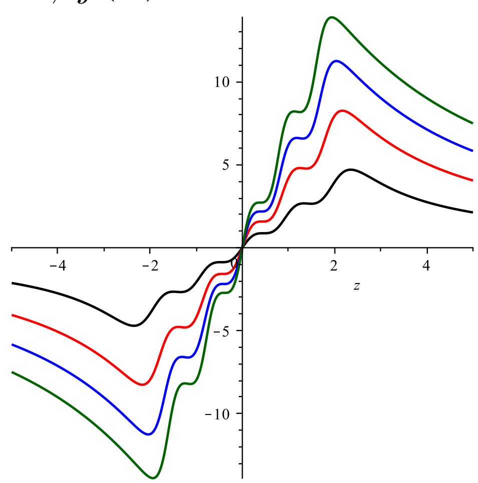
$\sigma_{2,2j}(z), \quad j = 1, 2, 3, 4$



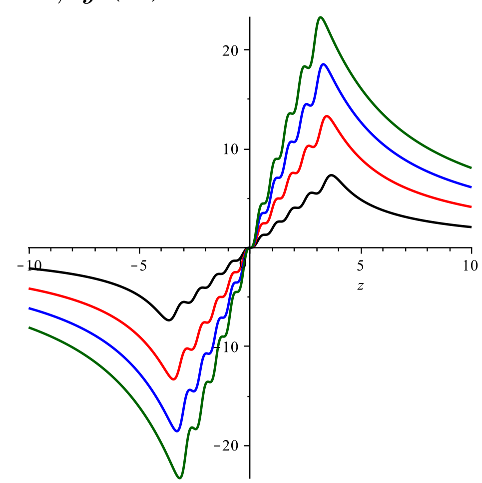
$\sigma_{3,2j}(z), \quad j = 1, 2, 3, 4$



$\sigma_{4,2j}(z), \quad j = 1, 2, 3, 4$



$\sigma_{5,2j}(z), \quad j = 1, 2, 3, 4$

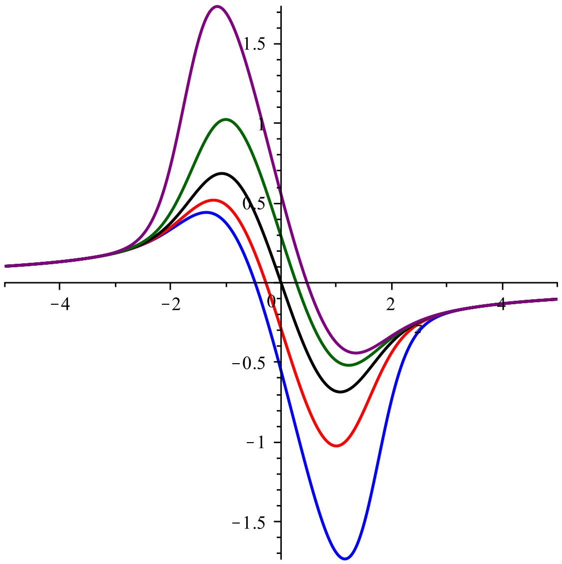


$\sigma_{10,2j}(z), \quad j = 1, 2, 3, 4$

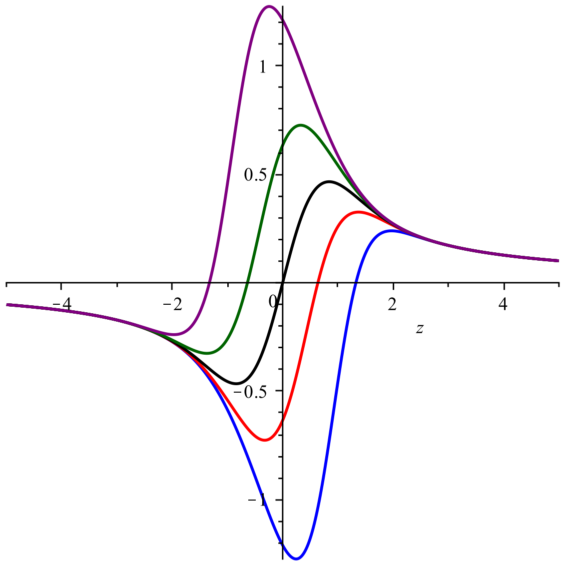
Plots of Bounded Special Function Solutions of $\mathrm{S_{IV}}$

$$\sigma_{\nu,n}(z) = -2nz + \frac{\mathrm{d}}{\mathrm{d}z}\ln \mathcal{W}\big(\psi_\nu, \psi_\nu', \ldots, \psi_\nu^{(n-1)}\big)$$

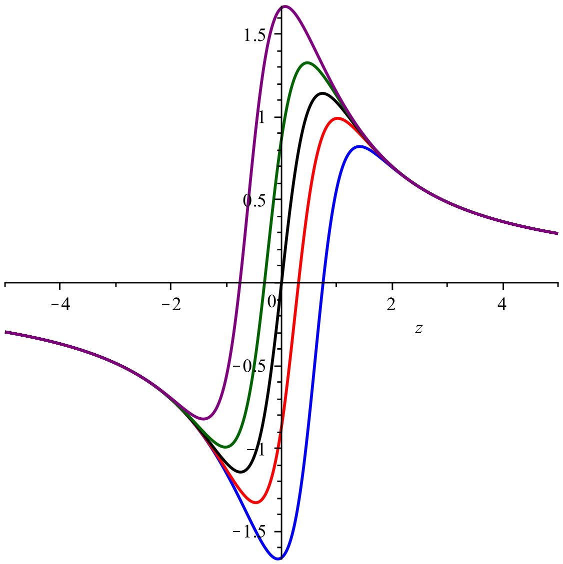
$$\psi_\nu(z) = \Big\{C_1 D_{-\nu}\big(\sqrt{2}\,z\big) + C_2 D_{-\nu}\big(-\sqrt{2}\,z\big)\Big\}\exp\left(\tfrac{1}{2}z^2\right)$$



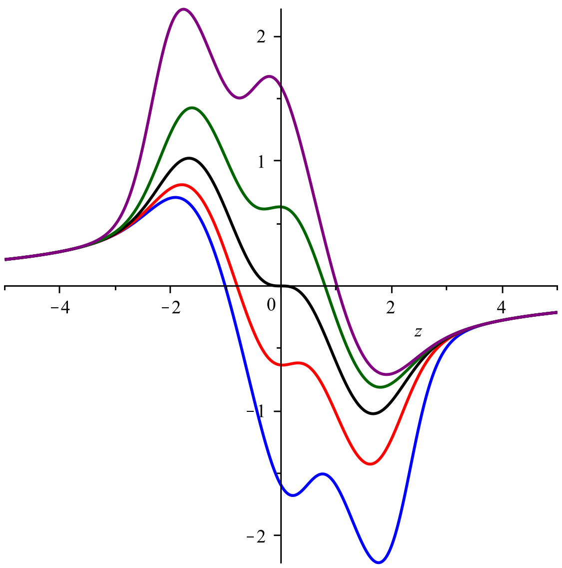
$\sigma_{1/2,1}(z)$



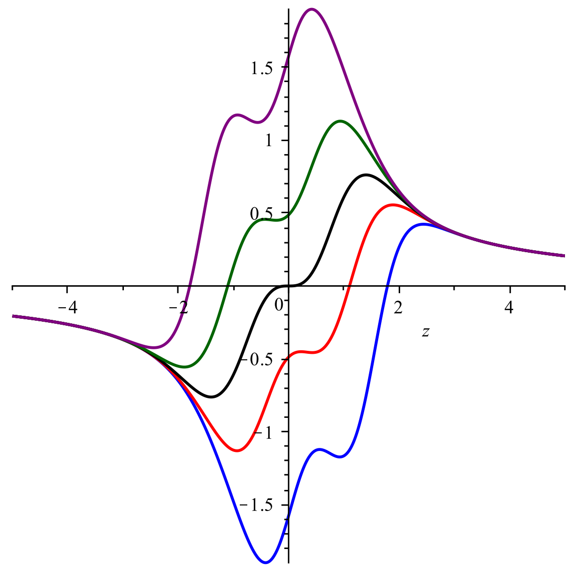
$\sigma_{3/2,1}(z)$



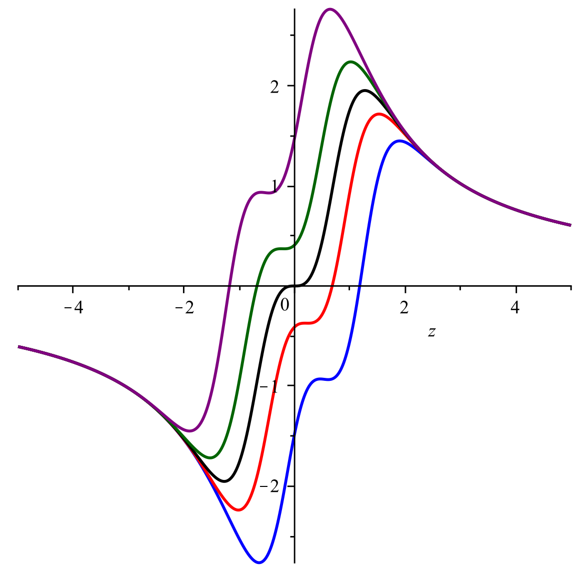
$\sigma_{5/2,1}(z)$



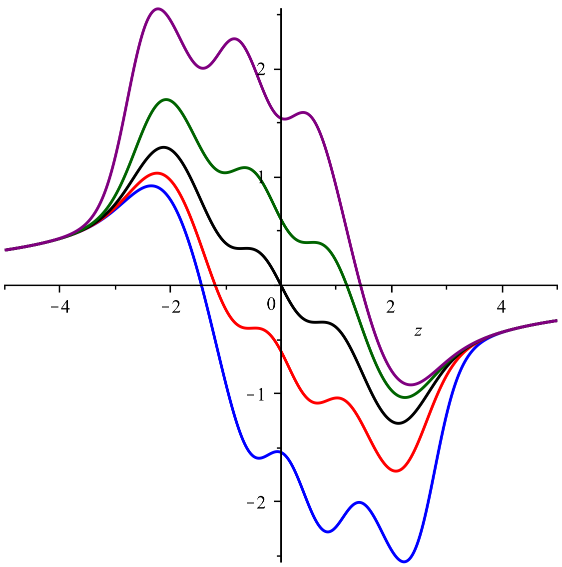
$\sigma_{1/2,2}(z)$



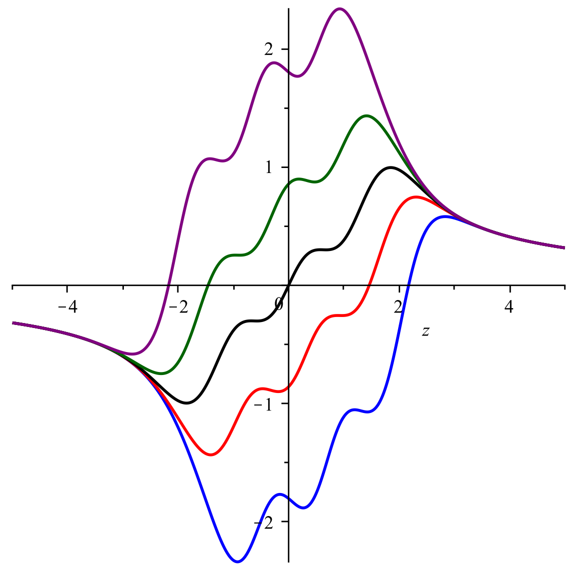
$\sigma_{3/2,2}(z)$



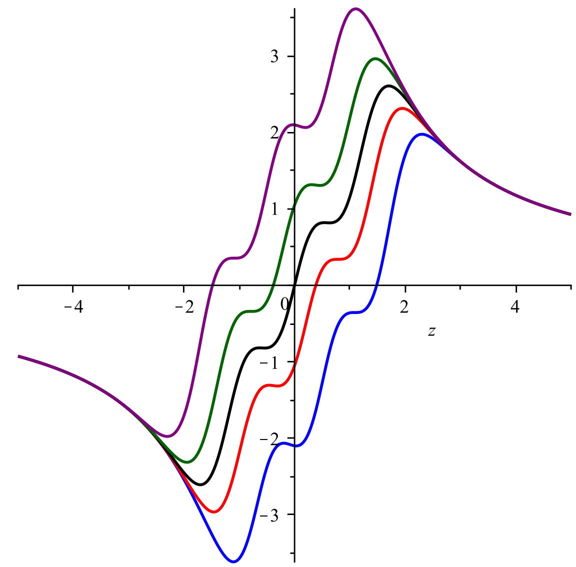
$\sigma_{5/2,2}(z)$



$\sigma_{1/2,3}(z)$



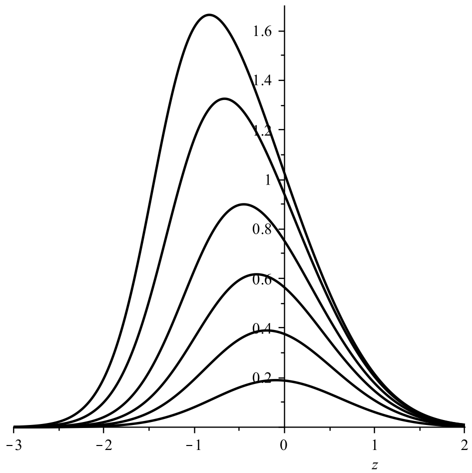
$\sigma_{3/2,3}(z)$



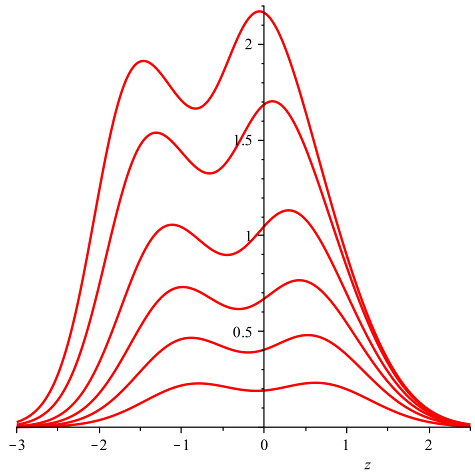
$\sigma_{5/2,3}(z)$

Plots of Error Function Solutions of S_{IV}

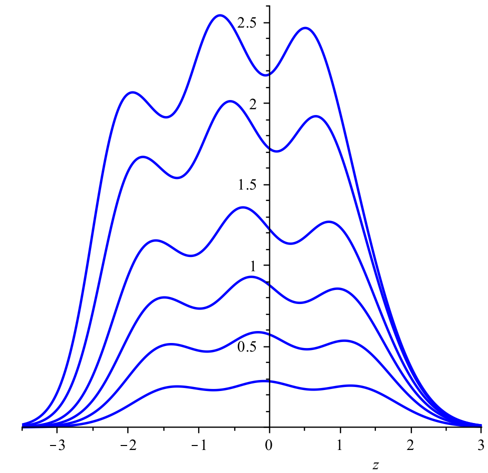
$$\sigma_{m,n} = \frac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{W}\big(\psi_m, \psi_m', \ldots, \psi_m^{(n-1)}\big), \qquad \psi_m = \exp(-z^2)\frac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{C_1 + C_2\,\mathrm{erfc}(z)\right\}\exp(z^2)$$



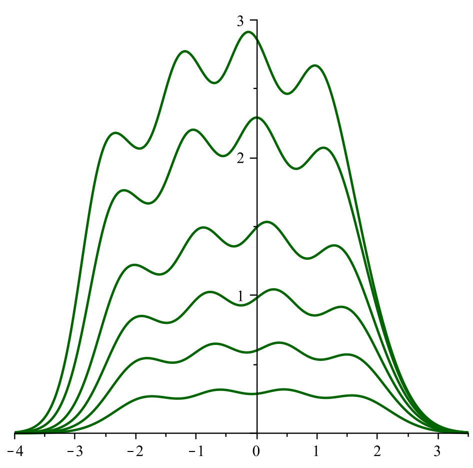
$\sigma_{1,0}(z)$



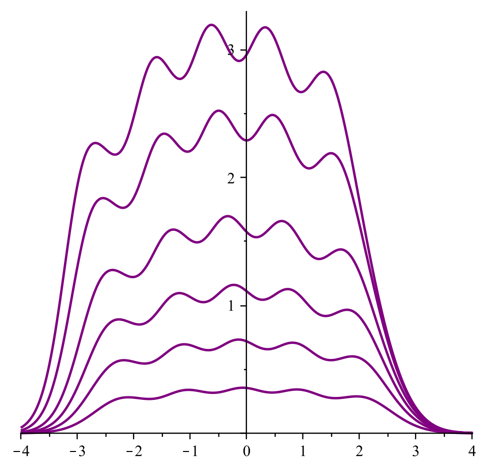
$\sigma_{2,1}(z)$



$\sigma_{3,2}(z)$



$\sigma_{4,3}(z)$



$\sigma_{5,4}(z)$

Application of $\mathrm{P_{IV}}$ to Orthogonal Polynomials

Semi-classical Laguerre Weight

$$\omega(x;t) = x^{\lambda}\exp(-x^2+tx), \qquad x\in\mathbb{R}^+, \qquad \lambda>-1$$

- P A Clarkson, "The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", preprint (2013)

Some History

- The relationship between semi-classical orthogonal polynomials and integrable equations dates back to the work of **Shohat [1939]** and later **Freud [1976]**.
- **Fokas, Its & Kitaev [1991, 1992]** identified these equations as **discrete Painlevé equations**.
- **Magnus [1995]** considered the **Freud weight**

$$\omega(x;t) = \exp\left(-\tfrac{1}{4}x^4 - tx^2\right), \qquad x, t \in \mathbb{R},$$

and showed that the coefficients in the three-term recurrence relation can be expressed in terms of solutions of

$$w_n(w_{n-1} + w_n + w_{n+1}) + 2tw_n = n$$

which is discrete $\mathrm{P_I}$ ($\mathrm{dP_I}$), and

$$\frac{\mathrm{d}^2 w_n}{\mathrm{d}z^2} = \frac{1}{2w_n}\left(\frac{\mathrm{d}w_n}{\mathrm{d}z}\right)^2 + \frac{3}{2}w_n^3 + 4zw_n^2 + 2(z^2 + \tfrac{1}{2}n)w_n - \frac{n^2}{2w_n}$$

which is $\mathrm{P_{IV}}$ with $\alpha = -\tfrac{1}{2}n$ and $\beta = -\tfrac{1}{2}n^2$.

- **Filipuk, van Assche & Zhang [2012]** comment:

  "*We note that for classical orthogonal polynomials (Hermite, Laguerre, Jacobi) one knows these recurrence coefficients explicitly in contrast to non-classical weights*".

Monic Orthogonal Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \ldots$, be the **monic orthogonal polynomials** of degree n in x, with respect to the positive weight $\omega(x)$ on the interval $[a, b]$ (which may be infinite), such that

$$\int_a^b P_m(x)P_n(x)\,\omega(x)\,\mathrm{d}x = h_n\delta_{m,n}, \quad h_n > 0, \qquad m, n = 0, 1, 2, \ldots$$

Monic orthogonal polynomials satisfy the **three-term recurrence relation**

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

where the coefficients are given by

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

with

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix}, \qquad \widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

and μ_k, the **moments** of the weight $\omega(x)$ given by

$$\mu_k = \int_a^b x^k\,\omega(x)\,\mathrm{d}x, \qquad k = 0, 1, 2, \ldots$$

Pearson Equation

Consider the **Pearson equation** satisfied by the weight $\omega(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sigma(x)\omega(x)] = \tau(x)\omega(x)$$

- **Classical orthogonal polynomials**: $\sigma(x)$ is a monic polynomial with $\deg(\sigma) \leq 2$ and $\tau(x)$ a polynomials with $\deg(\tau) = 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Associated Laguerre	$x^\lambda \exp(-x)$	x	$1+\lambda-x$

- **Semi-classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
semi-classical Laguerre	$x^\lambda \exp(-x^2+tx)$	x	$1+\lambda+tx-2x^2$
Freud	$\exp(-\frac{1}{4}x^4-tx^2)$	1	$-2tx-x^3$

Suppose the weight has the form $\omega(x;t) = \omega_0(x)\exp(tx)$, where $\omega_0(x)$ is a classical weight with finite moments, i.e.

$$\int_{-\infty}^{\infty} x^k \omega_0(x)\exp(tx)\,\mathrm{d}x < \infty, \qquad k = 0, 1, 2, \ldots$$

Then the kth moment is given by

$$\mu_k(t) = \int_{-\infty}^{\infty} x^k \omega_0(x)\exp(tx)\,\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} \omega_0(x)\exp(tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k \mu_0}{\mathrm{d}t^k}$$

and so $\Delta_n(t)$ and $\widetilde{\Delta}_n(t)$ can be expressed as Wronskians

$$\Delta_n(t) = \begin{vmatrix} \mu_0(t) & \mu_1(t) & \ldots & \mu_{n-1}(t) \\ \mu_1(t) & \mu_2(t) & \ldots & \mu_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(t) & \mu_n(t) & \ldots & \mu_{2n-2}(t) \end{vmatrix} = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

$$\widetilde{\Delta}_n(t) = \begin{vmatrix} \mu_0(t) & \mu_1(t) & \ldots & \mu_{n-2}(t) & \mu_n(t) \\ \mu_1(t) & \mu_2(t) & \ldots & \mu_{n-1}(t) & \mu_{n+1}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1}(t) & \mu_n(t) & \ldots & \mu_{2n-3}(t) & \mu_{2n-1}(t) \end{vmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

$$\Rightarrow \qquad \boxed{\frac{\widetilde{\Delta}_n(t)}{\Delta_n(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\ln \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)}$$

Semi-classical Laguerre weight

Consider monic orthogonal polynomials with respect to the **semi-classical Laguerre weight**

$$\omega(x;t) = x^{\lambda}\exp(-x^2+tx), \qquad x\in\mathbb{R}^+, \qquad \lambda>-1 \tag{1}$$

which satisfy the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t)P_n(x;t) + \beta_n(t)P_{n-1}(x;t) \tag{2}$$

Theorem **(Filipuk, van Assche & Zhang [2012])**

The coefficient $\alpha_n(t)$ in the recurrence relation (2) associated with the semi-classical Laguerre weight (1) is given by

$$\alpha_n(t) = \tfrac{1}{2}w_n(z) + \tfrac{1}{2}t, \qquad z = \tfrac{1}{2}t$$

where $w_n(z)$ satisfies

$$\frac{\mathrm{d}^2w_n}{\mathrm{d}z^2} = \frac{1}{2w_n}\left(\frac{\mathrm{d}w_n}{\mathrm{d}z}\right)^2 + \tfrac{3}{2}w_n^3 + 4zw_n^2 + 2(z^2-2n-1-\lambda)w_n - \frac{2\lambda^2}{w_n} \tag{3}$$

which is $\mathrm{P_{IV}}$ with parameters

$$(\alpha,\beta) = \left(2n+1+\lambda, -2\lambda^2\right) \tag{4}$$

- **Filipuk, van Assche & Zhang [2012]** do **not** specify the specific solution of (3).
- The parameters (4) satisfy the condition for $\mathrm{P_{IV}}$ to have solutions expressible in terms of **parabolic cylinder functions**.

Lemma (PAC [2013])

For the semi-classical Laguerre weight $\omega(x;t) = x^\lambda \exp(-x^2+tx)$, *the moment* $\mu_0(t;\lambda)$ *is given by*

$$\mu_0(t;\lambda) = \begin{cases} \dfrac{\Gamma(\lambda+1)\exp(\frac{1}{8}t^2)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right), & \text{if} \quad \lambda \notin \mathbb{N} \\ \tfrac{1}{2}\sqrt{\pi}\,\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\exp\left(\tfrac{1}{4}t^2\right)\left[1+\operatorname{erf}(\tfrac{1}{2}t)\right]\right\}, & \text{if} \quad \lambda = n \in \mathbb{N} \end{cases}$$

with $D_\nu(\zeta)$ *the* ***parabolic cylinder function*** *and* $\operatorname{erf}(z)$ *the* ***error function***.

Proof. The parabolic cylinder function $D_\nu(\zeta)$ has the integral representation

$$D_\nu(\zeta) = \frac{\exp(-\frac{1}{4}\zeta^2)}{\Gamma(-\nu)} \int_0^\infty s^{-\nu-1}\exp(-\tfrac{1}{2}s^2 - \zeta s)\,\mathrm{d}s$$

If $\lambda \notin \mathbb{N}$, then

$$\mu_0(t;\lambda) = \int_0^\infty x^\lambda \exp(-x^2+tx)\,\mathrm{d}x = \frac{\Gamma(\lambda+1)\exp\left(\frac{1}{8}t^2\right)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)$$

If $\lambda = n \in \mathbb{N}$, then

$$D_{-n-1}(\zeta) = \sqrt{\frac{\pi}{2}}\frac{(-1)^n}{n!}\exp(-\tfrac{1}{4}\zeta^2)\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left\{\exp(\tfrac{1}{2}\zeta^2)\operatorname{erfc}\left(\tfrac{1}{2}\sqrt{2}\,\zeta\right)\right\},$$

with $\operatorname{erfc}(z)$ the complementary error function. Since $\operatorname{erfc}(-z) = 1+\operatorname{erf}(z)$, then

$$\mu_0(t;n) = \tfrac{1}{2}\sqrt{\pi}\,\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\exp\left(\tfrac{1}{4}t^2\right)\left[1+\operatorname{erf}(\tfrac{1}{2}t)\right]\right\}$$

Theorem (PAC [2013])

Suppose that $\Delta_n(t)$ is the Hankel determinant given by

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right), \qquad n \geq 1$$

$\Delta_0(t) = 1$, *where*

$$\mu_0(t;\lambda) = \begin{cases} \dfrac{\Gamma(\lambda+1)\exp(\frac{1}{8}t^2)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right), & \textit{if} \quad \lambda \notin \mathbb{N} \\ \tfrac{1}{2}\sqrt{\pi}\,\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\exp\left(\tfrac{1}{4}t^2\right)\left[1+\operatorname{erf}(\tfrac{1}{2}t)\right]\right\}, & \textit{if} \quad \lambda = n \in \mathbb{N} \end{cases}$$

*with $D_\nu(\zeta)$ the **parabolic cylinder function** and $\operatorname{erf}(z)$ the **error function**. Then the coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence relation*

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t)P_n(x;t) + \beta_n(t)P_{n-1}(x;t)$$

associated with the semi-classical Laguerre weight

$$\omega(x;t) = x^\lambda \exp(-x^2 + tx), \qquad x \in \mathbb{R}^+, \qquad \lambda > -1$$

are given by

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \qquad \beta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t), \qquad n \geq 0$$

Theorem (PAC [2013])

Suppose that $\Delta_n(t)$ *is the Hankel determinant given by*

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right), \qquad n \geq 1$$

$\Delta_0(t) = 1$, *where*

$$\mu_0(t;\lambda) = \begin{cases} \dfrac{\Gamma(\lambda+1)\exp(\frac{1}{8}t^2)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}\,t\right), & \textit{if} \quad \lambda \notin \mathbb{N} \\ \frac{1}{2}\sqrt{\pi}\,\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\exp\left(\frac{1}{4}t^2\right)\left[1+\operatorname{erf}(\frac{1}{2}t)\right]\right\}, & \textit{if} \quad \lambda = n \in \mathbb{N} \end{cases}$$

with $D_\nu(\zeta)$ *the* ***parabolic cylinder function*** *and* $\operatorname{erf}(z)$ *the* ***error function****. Then* $S_n(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\ln\Delta_n(t)$ *satisfies*

$$\left(\frac{\mathrm{d}^2S_n}{\mathrm{d}t^2}\right)^2 - \frac{1}{4}\left(t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n\right)^2 + \frac{\mathrm{d}S_n}{\mathrm{d}t}\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n\right)\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n - \lambda\right) = 0$$

which is equivalent to $\mathrm{S_{IV}}$, *the* $\mathrm{P_{IV}}$ σ*-equation, through the transformation*

$$S_n(t) = \tfrac{1}{2}\sigma(z), \qquad z = 2t$$

Hence the recurrence coefficients $\alpha_n(t)$ *and* $\beta_n(t)$ *are given by*

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)} = S_{n+1}(t) - S_n(t), \qquad \beta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t) = \frac{\mathrm{d}S_n}{\mathrm{d}t}$$

Lemma

- *As $t \to \infty$, $\mu_0(t)$, $\Delta_n(t)$ and $S_n(t)$ have the respective asymptotic expansions*

$$\mu_0(t) \sim \sqrt{\pi}\left(\tfrac{1}{2}t\right)^\lambda \exp\left(\tfrac{1}{4}t^2\right)\left\{1 + \frac{\lambda(\lambda-1)}{2t^2} + \mathcal{O}\left(t^{-4}\right)\right\}$$

$$\Delta_n(t) = c_n t^{n\lambda} \exp\left(\tfrac{1}{4}nt^2\right)\left\{1 - \frac{n\lambda(n-\lambda)}{t^2} + \mathcal{O}\left(t^{-4}\right)\right\}$$

$$S_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\Delta_n(t) = \frac{nt}{2} + \frac{n\lambda}{t} + \frac{2n\lambda(n-\lambda)}{t^3} + \mathcal{O}\left(t^{-5}\right)$$

- *As $t \to \infty$, the recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ have the asymptotic expansions*

$$\alpha_n(t) = S_{n+1}(t) - S_n(t) = \frac{t}{2} + \frac{\lambda}{t} + \mathcal{O}\left(t^{-3}\right) \qquad \Rightarrow \qquad \lim_{t=\infty}\alpha_n(t) = \tfrac{1}{2}t$$

$$\beta_n(t) = \frac{\mathrm{d}S_n}{\mathrm{d}t} = \frac{n}{2} - \frac{n\lambda}{t^2} + \mathcal{O}\left(t^{-4}\right) \qquad \Rightarrow \qquad \lim_{t=\infty}\beta_n(t) = \tfrac{1}{2}n$$

Remark The three-term recurrence relation

$$xQ_n(x;t) = Q_{n+1}(x;t) + \tfrac{1}{2}tQ_n(x;t) + \tfrac{1}{2}nQ_{n-1}(x;t)$$

with $Q_{-1} = 0$ and $Q_0 = 1$, generates the monic polynomials

$$Q_n(x;t) = \left(\tfrac{1}{2}\right)^n H_n\left(x - \tfrac{1}{2}t\right)$$

with $H_n(y)$ the **Hermite polynomial**.

Special function solutions of Painlevé equations

	Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial	Number of parameters
P_{I}	0	—			
P_{II}	1	**Airy** $\mathrm{Ai}(z), \mathrm{Bi}(z)$	0	—	
P_{III}	2	**Bessel** $J_\nu(z), Y_\nu(z), J_\nu(z), K_\nu(z)$	1	—	
P_{IV}	2	**Parabolic cylinder** $D_\nu(z)$	1	**Hermite** $H_n(z)$	0
P_{V}	3	**Kummer** $M(a,b,z), U(a,b,z)$ **Whittaker** $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$	2	**Associated Laguerre** $L_n^{(k)}(z)$	1
P_{VI}	4	**hypergeometric** ${}_2F_1(a,b;c;z)$	3	**Jacobi** $P_n^{(\alpha,\beta)}(z)$	2

Further Examples

- Semi-classical weight with recurrence coefficients expressible in terms of solutions of S_{III}, the P_{III} σ-equation

$\omega(x;t)$		$\mu_0(t)=\int \omega(x;t)\,\mathrm{d}x$
$x^{\nu-1}\exp\left(-x-t/x\right)$	$x\in\mathbb{R}^+$	$2t^{\nu/2}K_\nu\left(2\sqrt{t}\right)$

with $K_\nu(z)$ the **modified Bessel function** (**Chen & Its [2010]**).

- Semi-classical weights with recurrence coefficients expressible in terms of solutions of S_{V}, the P_{V} σ-equation

$\omega(x;t)$		$\mu_0(t)=\int \omega(x;t)\,\mathrm{d}x$
$x^{\alpha-1}(1-x)^{\beta-1}\mathrm{e}^{-tx}$	$x\in[0,1]$	$\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\mathrm{e}^{-t}M(\alpha,\alpha+\beta,t)$
$x^{\alpha-1}(1-x)^{\beta-1}\mathrm{e}^{-x/t}$	$x\in[0,1]$	$\Gamma(\beta)\mathrm{e}^{-t}U(\beta,1-\alpha,t)$
$x^{\alpha-1}(x+t)^{\beta-1}\mathrm{e}^{-x}$	$x\in\mathbb{R}^+$	$\Gamma(\alpha)\,t^{\alpha+\beta-1}U(\alpha,\alpha+\beta,t)$

with $U(a,b,t)$ and $M(a,b,z)$ the **Kummer functions** (**Basor, Chen & Ehrhardt [2010]; Chen & Dai [2010]; Forrester & Witte [2007]**).

Application of $\mathrm{P_{IV}}$ to Vortex Dynamics

- The equations of motion for n point vortices with circulations Γ_j at positions z_j, in a background flow $w(z)$ are

$$\frac{\mathrm{d}z_j^*}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}} \sum_{k=1}^{n}{}' \frac{\Gamma_k}{z_j - z_k} + \frac{w^*(z_j)}{2\pi\mathrm{i}}, \qquad j = 1, 2, \ldots, n$$

- P A Clarkson, “Vortices and polynomials", *Stud. Appl. Math.*, **123** (2009) 37–62

Vortex Dynamics

The equations of motion for n point vortices with circulations Γ_j at positions z_j, are

$$\frac{\mathrm{d}z_j^*}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}} \sum_{k=1}^{n}{}' \frac{\Gamma_k}{z_j - z_k}, \qquad j = 1, 2, \ldots, n$$

If a vortex configuration rotates as a rigid body with angular velocity Ω then

$$\frac{\mathrm{d}z_j^*}{\mathrm{d}t} = -\mathrm{i}\Omega z_j^*, \qquad j = 1, 2, \ldots, n$$

and so

$$\lambda z_j^* = \sum_{k=1}^{n}{}' \frac{\Gamma_k}{z_j - z_k}, \qquad j = 1, 2, \ldots, n \tag{1}$$

where $\lambda = 2\pi\Omega$. Suppose that z_j is real, so $z_j = z_j^* = x_j$, and all the Γ_j are equal, so $\Gamma_j = \Gamma$ for $j = 1, 2, \ldots, n$, then set $\lambda = 1$ (by rescaling x_j, if necessary) and so we obtain

$$x_j = \sum_{k=1}^{n}{}' \frac{1}{x_j - x_k}, \qquad j = 1, 2, \ldots, n \tag{2}$$

which are known as **Stieltjes relations** (**Stieltjes [1885]**).

Question: **What are the solutions $x_1, x_2, \ldots, x_n$ of equation (2)?**

Answer: **They are the roots of the n^{th} Hermite polynomial $H_n(x)$.**

Quadrupole Background Flow

Lemma (Kadtke & Campbell [1987])

The equations of motion for $m+n$ point vortices with circulations Γ_j at positions z_j in a background flow $w(z)$ are

$$\frac{\mathrm{d}z_j^*}{\mathrm{d}t} = \frac{1}{2\pi \mathrm{i}} \sum_{k=1}^{m+n}{}' \frac{\Gamma_k}{z_j - z_k} + \frac{w^*(z_j)}{2\pi \mathrm{i}}, \qquad j = 1, 2, \ldots, m+n$$

When $\dfrac{\mathrm{d}z_j^*}{\mathrm{d}t} = 0$, $w(z) = \Gamma\mu^* z^*$, *with* μ^* *a (complex) constant,* $\Gamma_k = \Gamma$ *for* $k = 1, 2, \ldots, m$ *and* $\Gamma_k = -\Gamma$ *for* $k = m+1, m+2, \ldots, m+n$, *then the polynomials*

$$P(z) = \prod_{j=1}^{m} (z - z_j), \qquad Q(z) = \prod_{j=1}^{n} (z - z_{j+m})$$

satisfy

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2} Q - 2 \frac{\mathrm{d}P}{\mathrm{d}z} \frac{\mathrm{d}Q}{\mathrm{d}z} + P \frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} + 2\mu z \left(\frac{\mathrm{d}P}{\mathrm{d}z} Q - P \frac{\mathrm{d}Q}{\mathrm{d}z} \right) = 2\mu(m-n) PQ$$

Remark: If $Q = 1$ and $\mu = -1$ then P satisfies

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2} - 2z \frac{\mathrm{d}P}{\mathrm{d}z} + 2mP = 0$$

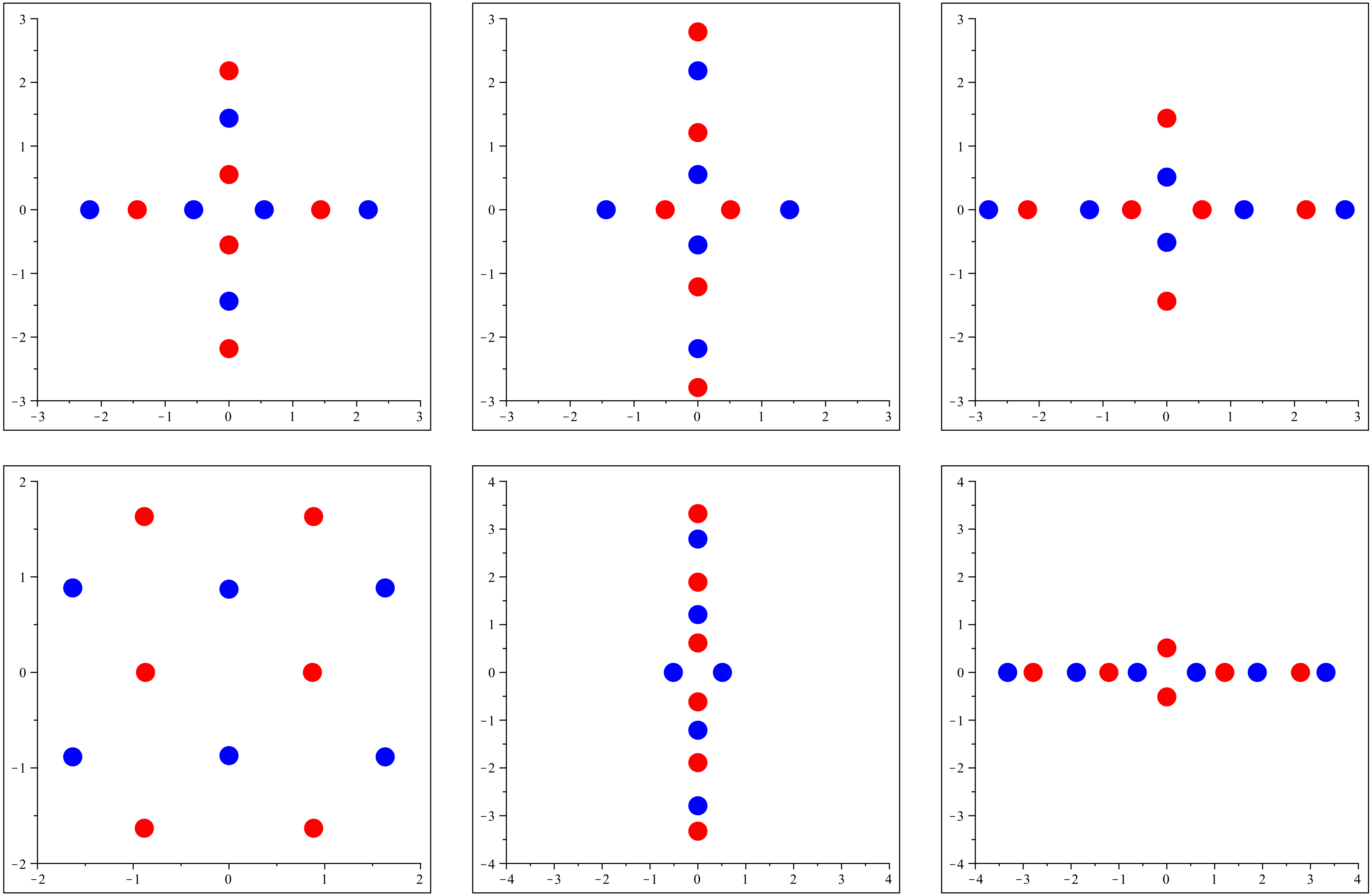
which is the equation for the m^{th} Hermite polynomial $H_m(z)$.

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\mu(m-n)PQ$$

Kadtke & Campbell [1987] obtained some polynomial solutions of this equation when $m = n$, though they claimed that there were no solutions when $m = n = 6$. However, using MAPLE, it can be shown that there are solutions when $m = n = 6$.

Solutions for $\mu = -\frac{1}{2}$ and $m = n$

	$P(z)$	$Q(z)$
$m = n = 2$	$z^2 + 1$	$z^2 - 1$
$m = n = 4$	$z^4 + 6z^2 + 3$	$z^4 + 2z^2 - 1$
	$z^4 + 2z^2 - 1$	$z^4 - 2z^2 - 1$
	$z^4 - 2z^2 - 1$	$z^4 - 6z^2 + 3$
$m = n = 6$	$z^6 + 15z^4 + 45z^2 + 15$	$z^6 + 9z^4 + 9z^2 - 3$
	$z^6 + 9z^4 + 9z^2 - 3$	$z^6 + 3z^4 - 9z^2 - 3$
	$z^6 + 3z^4 - 9z^2 - 3$	$z^6 - 3z^4 - 9z^2 + 3$
	$z^6 - 3z^4 - 9z^2 + 3$	$z^6 - 9z^4 + 9z^2 + 3$
	$z^6 - 9z^4 + 9z^2 + 3$	$z^6 - 15z^4 + 45z^2 - 15$
	$z^6 + 3z^4 + 9z^2 - 9$	$z^6 - 3z^4 + 9z^2 + 9$



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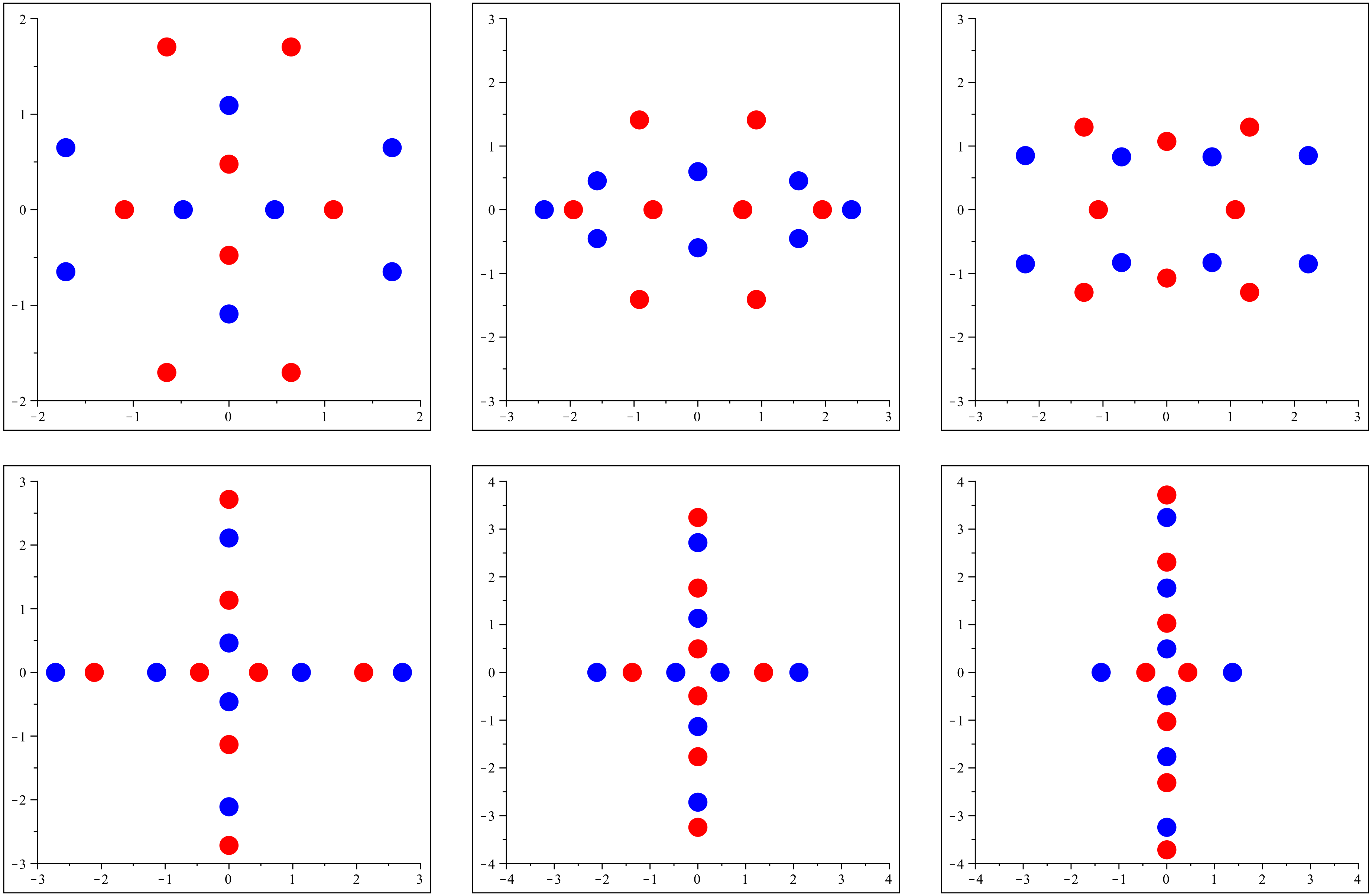
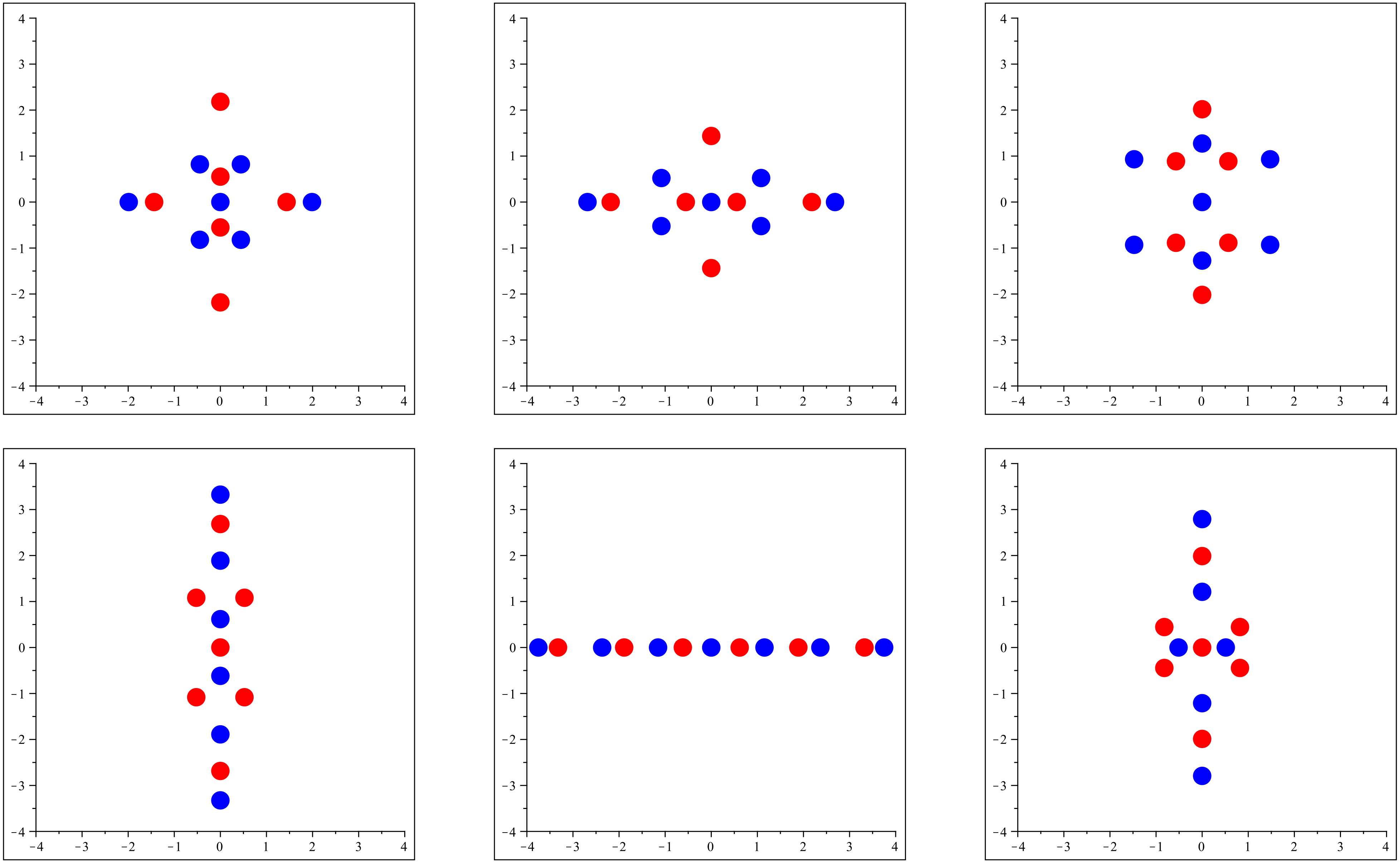
−4 −3 −2 −1 0 1 2 3 4

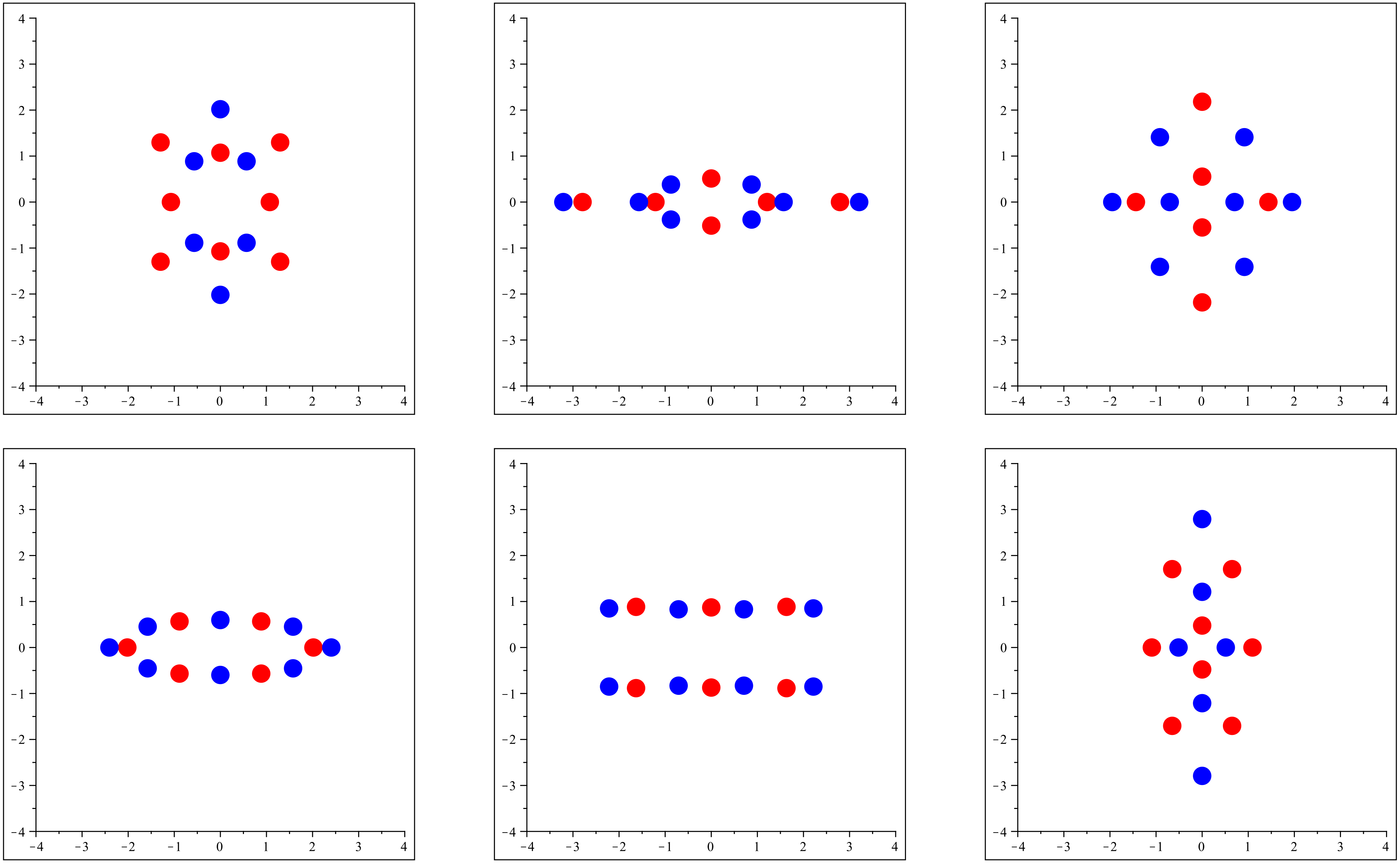
$$m = n = 6$$



$$m = n = 8$$



$$m = 7, n = 6$$



$m = 8, n = 6$

Question **What is the form of polynomial solutions of the bilinear equation**

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\mu(m-n)PQ$$

Theorem (**Crum [1955]**; also **Oblomkov [1999]**, **Veselov [2001]**)

The Schrödinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + u\psi = \lambda\psi \qquad (*)$$

with potential

$$u = z^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln \mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_\ell}\right)$$

where $H_k(z)$ *is the* k^{th} *Hermite polynomial,* $\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_\ell)$ *is the Wronskian and* $k_1, k_2, \ldots, k_\ell$ *are a sequence of **distinct** positive integers, has the solutions*

$$\psi(z) = \frac{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_\ell}, H_{k_{\ell+1}}\right)}{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_\ell}\right)}\exp\left(-\tfrac{1}{2}z^2\right)$$

$$\psi(z) = \frac{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_{\ell-1}}\right)}{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_\ell}\right)}\exp\left(\tfrac{1}{2}z^2\right)$$

with k_{n+1} *another different positive integer for the eigenvalues* $\lambda = 1 + 2(k_{\ell+1} - \ell)$ *and* $\lambda = 2(\ell - k_{\ell-1}) - 1$*, respectively.*

Remark: Substituting $u = z^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q$ and $\psi = \frac{P}{Q}\exp(-\frac{1}{2}z^2)$ into $(*)$ yields

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} - 2z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) + (\lambda - 1)PQ = 0$$

Theorem

The bilinear equation

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} - 2z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) + 2(m-n)PQ = 0$$

with $m, n \in \mathbb{Z}^+$, has polynomial solutions in the form

$$\begin{aligned} P(z) &= \mathcal{W}\left(H_{k_1}(z), H_{k_2}(z), \ldots, H_{k_\ell}(z), H_{k_{\ell+1}}(z)\right) \\ Q(z) &= \mathcal{W}\left(H_{k_1}(z), H_{k_2}(z), \ldots, H_{k_\ell}(z)\right) \end{aligned} \qquad (*)$$

where $H_k(z)$ is the k^{th} Hermite polynomial, $\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_n)$ is the Wronskian and $k_1, k_2, \ldots, k_\ell, k_{\ell+1}$ are a sequence of ***distinct*** *positive integers. The degrees of the polynomials $P(z)$ and $Q(z)$, respectively m and n, are given by*

$$m = \sum_{j=1}^{\ell+1} k_j - \tfrac{1}{2}\ell(\ell+1), \qquad n = \sum_{j=1}^{\ell} k_j - \tfrac{1}{2}\ell(\ell-1)$$

$$\Rightarrow \qquad m - n = k_{\ell+1} - \ell$$

However there are additional solutions of the equation

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} - 2z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) + 2(m-n)PQ = 0 \qquad (1)$$

in terms of the **generalized Hermite polynomials $H_{m,n}(z)$** and the **generalized Okamoto polynomials $Q_{m,n}(z)$**.

Example 1 A set of solutions of (1) is given by

$$\begin{aligned} P(z) &= H_{k_1,k_2}(z) = \mathcal{W}\left(H_{k_1}, H_{k_1+1}, \ldots, H_{k_1+k_2-1}\right) \\ Q(z) &= H_{k_1+1,k_2}(z) = \mathcal{W}\left(H_{k_1+1}, H_{k_1+2}, \ldots, H_{k_1+k_2}\right) \end{aligned}$$

where the Wronskians defining $P(z)$ and $Q(z)$ have the **same** number of Hermite polynomials.

Example 2 Another set of solutions of (1) is given by

$$\begin{aligned} P(z) &= Q_{k_1,k_2}(z) = \mathcal{W}(H_1, H_4, \ldots, H_{3k_1+3k_2-5}, H_2, H_5, \ldots, H_{3k_2-4}) \\ Q(z) &= Q_{k_1,k_2+1}(z) = \mathcal{W}(H_1, H_4, \ldots, H_{3k_1+3k_2-2}, H_2, H_5, \ldots, H_{3k_2-1}) \end{aligned}$$

where the Wronskian defining $P(z)$ has **two** fewer Hermite polynomials than that defining $Q(z)$.

Symmetric Form of $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}z} + \varphi_0(\varphi_1 - \varphi_2) + 2\mu_0 = 0$$
$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}z} + \varphi_1(\varphi_2 - \varphi_0) + 2\mu_1 = 0$$
$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}z} + \varphi_2(\varphi_0 - \varphi_1) + 2\mu_2 = 0$$

where μ_0, μ_1 and μ_2 are constants, with constraints

$$\mu_0 + \mu_1 + \mu_2 = 1$$
$$\varphi_0 + \varphi_1 + \varphi_2 = -2z$$

Symmetric Form of P_{IV}

(**Bureau [1980], Veselov & Shabat [1993], Adler [1994], Noumi & Yamada [1998]**)

Consider the **symmetric** P_{IV} **system**

$$\begin{aligned} \frac{\mathrm{d}\varphi_0}{\mathrm{d}z} + \varphi_0(\varphi_1 - \varphi_2) + 2\mu_0 = 0 \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}z} + \varphi_1(\varphi_2 - \varphi_0) + 2\mu_1 = 0 \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}z} + \varphi_2(\varphi_0 - \varphi_1) + 2\mu_2 = 0 \end{aligned} \tag{1}$$

where μ_0, μ_1 and μ_2 are constants, with constraints

$$\mu_0 + \mu_1 + \mu_2 = 1, \qquad \varphi_0 + \varphi_1 + \varphi_2 = -2z \tag{2}$$

Eliminating φ_1 and φ_2, then $\varphi = \varphi_0$ satisfies P_{IV}

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2} = \frac{1}{2\varphi}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right)^2 + \tfrac{3}{2}\varphi^3 + 4z\varphi^2 + 2[z^2 + (\mu_1 - \mu_2)]\varphi - \frac{2\mu_0^2}{\varphi}$$

The system (1) is associated with the **affine Weyl group** $A_2^{(1)}$ and has the simple rational solutions

$$\begin{array}{lll} \text{(i)} & (\varphi_0, \varphi_1, \varphi_2) = (-2z, 0, 0), & (\mu_0, \mu_1, \mu_2) = (1, 0, 0) \\ \text{(ii)} & (\varphi_0, \varphi_1, \varphi_2) = (-\tfrac{2}{3}z, -\tfrac{2}{3}z, -\tfrac{2}{3}z), & (\mu_0, \mu_1, \mu_2) = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) \end{array}$$

Rational solutions arising from (i) are expressed in terms of **generalized Hermite polynomials** $H_{m,n}(z)$ and from (ii) in terms of **generalized Okamoto polynomials** $Q_{m,n}(z)$.

Theorem (**Noumi & Yamada [1998]**)

Rational solutions of the symmetric P_{IV} *system*

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}z} + \varphi_0(\varphi_1 - \varphi_2) + 2\mu_0 = 0$$
$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}z} + \varphi_1(\varphi_2 - \varphi_0) + 2\mu_1 = 0$$
$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}z} + \varphi_2(\varphi_0 - \varphi_1) + 2\mu_2 = 0$$

either have the form

$$(\varphi_0, \varphi_1, \varphi_2) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{m+1,n}}{H_{m,n}}, \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{m,n}}{H_{m,n+1}}, -2z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{m,n+1}}{H_{m+1,n}}\right)$$

for parameters

$$(\mu_0, \mu_1, \mu_2) = (n, -m-n, m+1)$$

or

$$(\varphi_0, \varphi_1, \varphi_2) = \left(-\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m+1,n}}{Q_{m,n}}, -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n}}{Q_{m,n+1}}, -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n+1}}{Q_{m+1,n}}\right)$$

for parameters

$$(\mu_0, \mu_1, \mu_2) = (n - \tfrac{1}{3}, -m-n+\tfrac{2}{3}, m+\tfrac{2}{3})$$

with $H_{m,n}(z)$ *the* ***generalized Hermite polynomials*** and $Q_{m,n}(z)$ ***generalized Okamoto polynomials****.*

Symmetric P_{IV} Hierarchy

(Noumi & Yamada [1998])

The symmetric hierarchy of P_{IV} associated with the affine Weyl group of type $A_{2n}^{(1)}$ is

$$\frac{d\varphi_j}{dz} + \varphi_j \sum_{r=1}^{n} (\varphi_{j+2r-1} - \varphi_{j+2r}) + 2\mu_j = 0, \qquad j = 0, 1, \ldots, 2n \tag{1}$$

with constraints

$$\sum_{j=0}^{2n} \varphi_j = -2z, \qquad \sum_{j=0}^{2n} \mu_j = 1 \tag{2}$$

where μ_j are complex constants. The system (1) has the simple rational solutions

$$\varphi_0 = \varphi_1 = \ldots = \varphi_{2k} = -z/(2k+1), \quad \varphi_{2k+1} = \ldots = \varphi_{2n} = 0, \quad k = 0, 1, \ldots, n$$

with

$$\mu_0 = \mu_1 = \ldots = \mu_{2k} = 1/(2k+1), \quad \mu_{2k+1} = \ldots = \mu_{2n} = 0, \quad k = 0, 1, \ldots, n$$

Special cases are

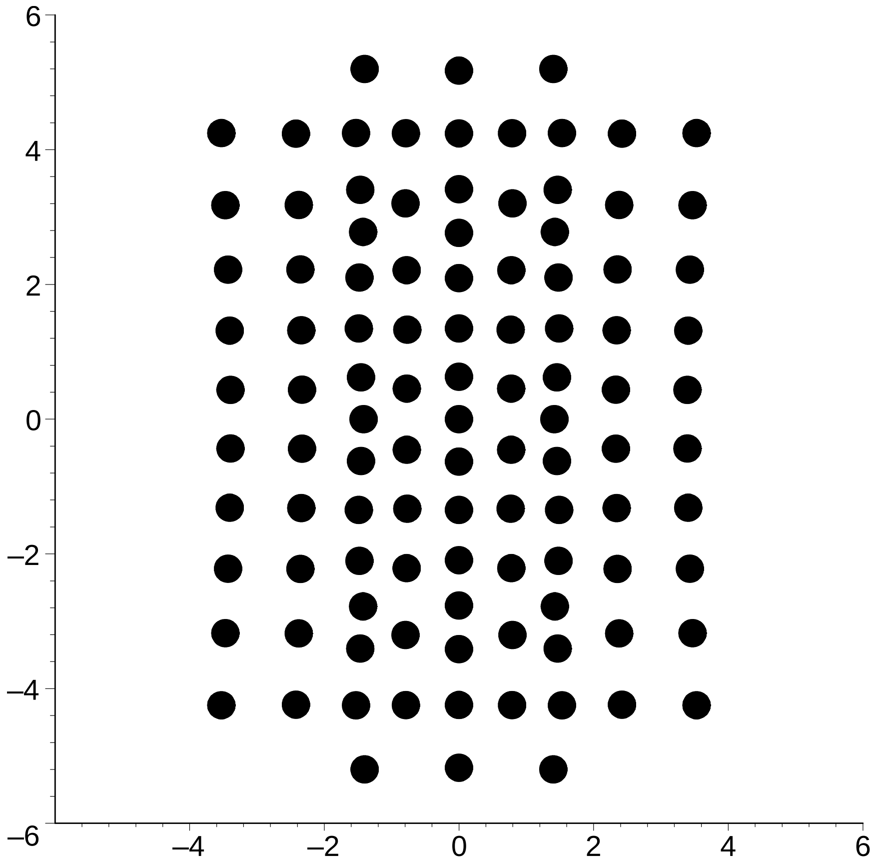
(i) $\varphi_0 = -2z, \quad \varphi_1 = \ldots = \varphi_{2n} = 0, \qquad \mu_0 = 1, \quad \mu_1 = \ldots = \mu_{2n} = 0$

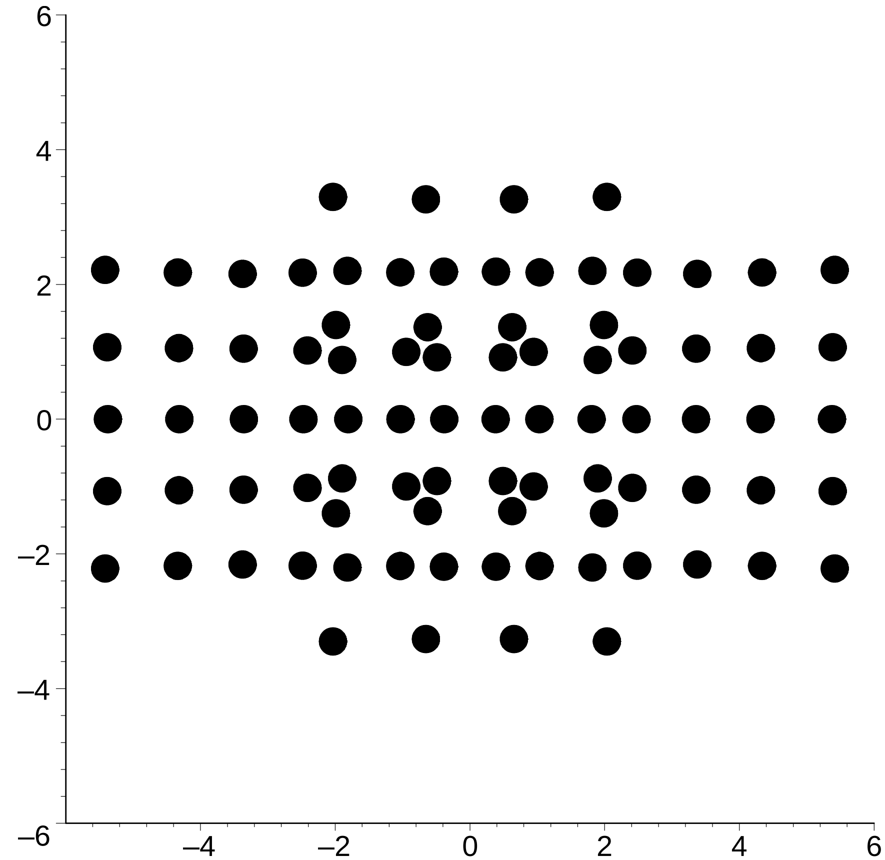
(ii) $\varphi_0 = \varphi_1 = \ldots = \varphi_{2n} = -2z/(2n+1), \qquad \mu_0 = \mu_1 = \ldots = \mu_{2n} = 1/(2n+1)$

Rational solutions arising from (i) are expressed in terms of **symmetric Hermite polynomials** and from (ii) in terms of **symmetric Okamoto polynomials**.

Roots of symmetric Hermite polynomials associated with $A_4^{(1)}$

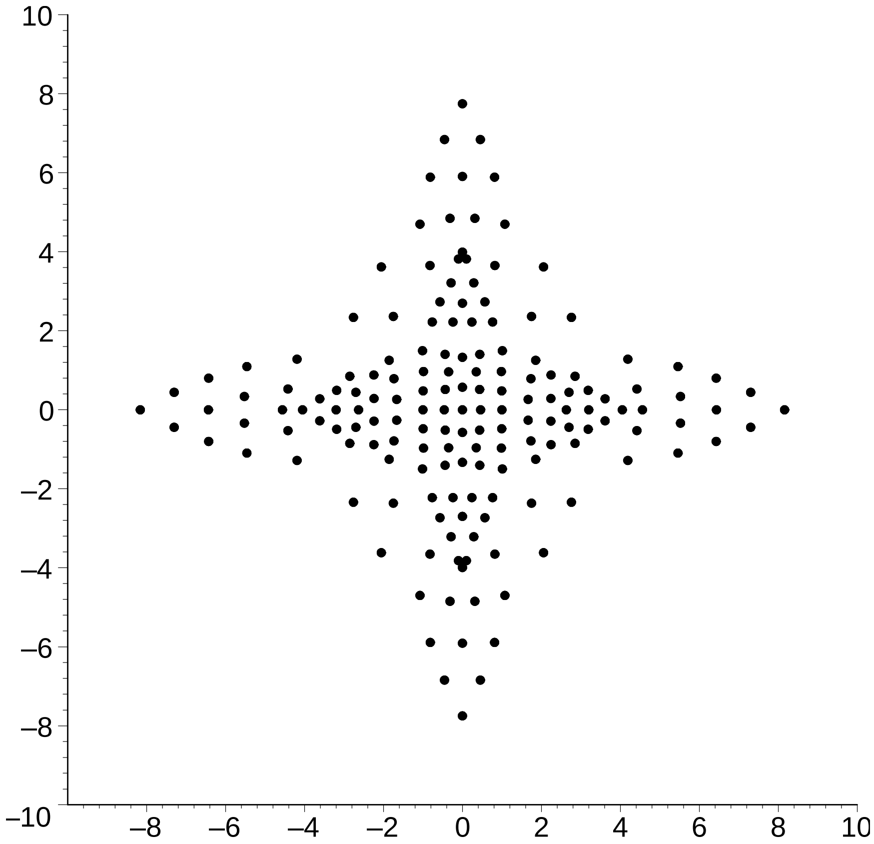
(PAC & Filipuk [2008])

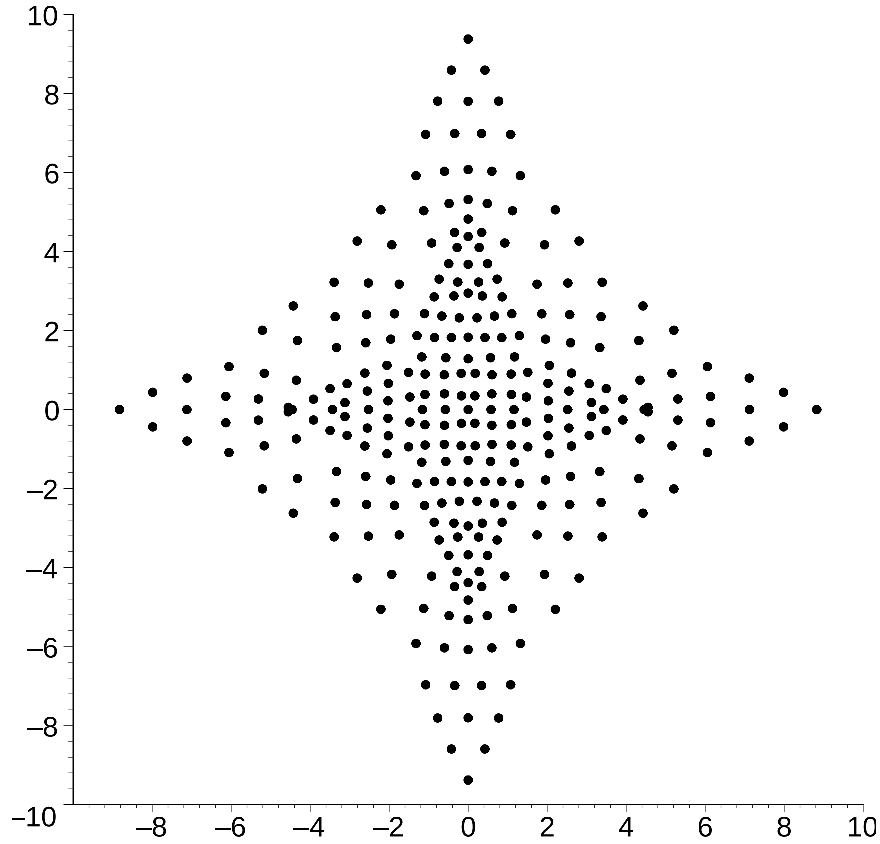




Roots of symmetric Okamoto polynomials associated with $A_4^{(1)}$

(PAC & Filipuk [2008])





Numerics and Asymptotics for the Fourth Painlevé Equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

- In the special case when $\alpha = 2\nu + 1$ and $\beta = 0$, we make the transformation

  $$w(z) = 2\sqrt{2}\, u^2(x), \qquad x = \sqrt{2}\, z$$

  This yields

  $$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 3u^5 + 2xu^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})u$$

  which is a **nonlinear harmonic oscillator**.

The **parabolic cylinder function** $D_\nu(x)$ satisfies

$$\frac{\mathrm{d}^2 D_\nu}{\mathrm{d}x^2} = (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})D_\nu$$

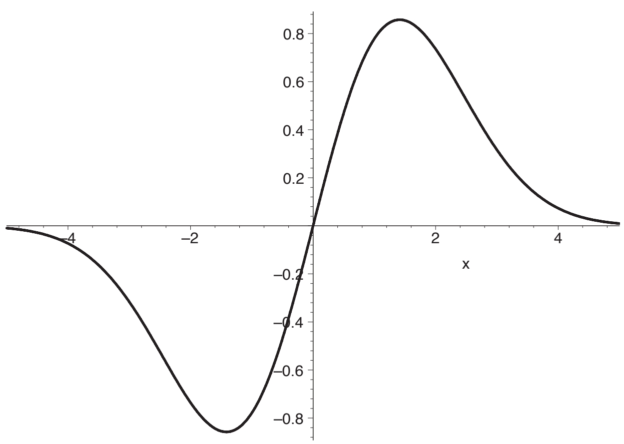
$$D_\nu(x) \sim x^\nu \exp\left(-\tfrac{1}{4}x^2\right), \qquad \text{as} \quad x \to +\infty$$

When $\nu = n \in \mathbb{N}$,

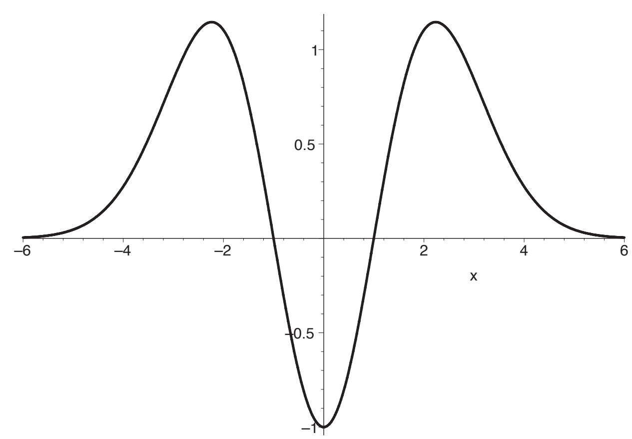
$$D_n(x) = \mathrm{He}_n(x)\,\exp\left(-\tfrac{1}{4}x^2\right)$$

which are **bound state solutions** that decay exponentially as $x \to \pm\infty$, where $\mathrm{He}_n(x)$ is the **Hermite polynomial** defined by

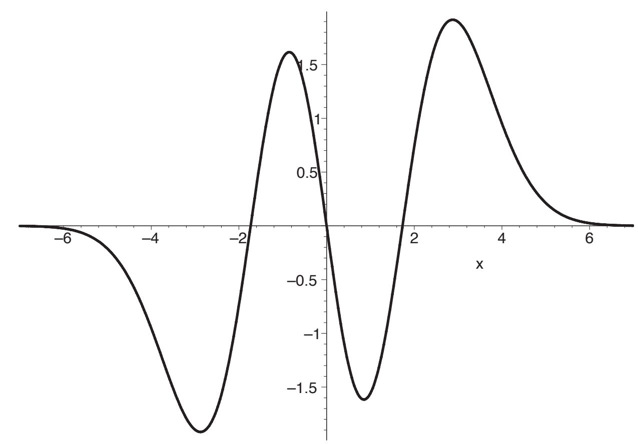
$$\mathrm{He}_n(x) = (-1)^n \exp(\tfrac{1}{2}x^2)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{\exp(-\tfrac{1}{2}x^2)\right\}$$



$D_1(x)$



$D_2(x)$



$D_3(x)$

Asymptotics of P_{IV} — Nonlinear Harmonic Oscillator

Consider the special case of P_{IV} where $w(z) = 2\sqrt{2}\, u^2(x)$ and $x = \sqrt{2}\, z$, with $\alpha = 2\nu+1$ and $\beta = 0$, i.e.

$$\frac{d^2u}{dx^2} = 3u^5 + 2xu^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})u \qquad (1)$$

and the boundary condition

$$u(x) \to 0, \qquad \text{as} \quad x \to +\infty \qquad (2)$$

This equation has solutions have exponential decay as $x \to \pm\infty$ and so are nonlinear analogues of **bound state solutions** for the **linear harmonic oscillator**.

Let $u_k(x)$ denote the unique solution of (1) which is asymptotic to $kD_\nu(x)$, i.e.

$$\frac{d^2u_k}{dx^2} = 3u_k^5 + 2xu_k^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})u_k$$

with boundary condition

$$u_k(x) \sim kD_\nu(x), \qquad \text{as} \quad x \to +\infty$$

where $D_\nu(x)$ is the **parabolic cylinder function** which satisfies

$$\frac{d^2D_\nu}{dx^2} = (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})D_\nu$$

with boundary condition

$$D_\nu(x) \sim x^\nu \exp\left(-\tfrac{1}{4}x^2\right), \qquad \text{as} \quad x \to +\infty$$

Theorem (Bassom, PAC, Hicks & McLeod [1992])

Let $u_k(x)$ *be the unique solution of*

$$\frac{\mathrm{d}^2u_k}{\mathrm{d}x^2} = 3u_k^5 + 2xu_k^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})u_k, \qquad u_k(x) \sim kD_\nu(x), \quad \textit{as} \quad x \to +\infty$$

- *The solution* $u_k(x)$ *exists for all real* x *provided that* $0 \le k < k_*$, *where*

$$k_*^2 = \frac{1}{2\sqrt{2\pi}\,\Gamma(\nu+1)}$$

In this case, if $\nu = n \in \mathbb{N}$, *then as* $x \to -\infty$

$$u_k(x) \sim \frac{k\exp(-\tfrac{1}{4}x^2)\,H_n\big(\tfrac{1}{2}\sqrt{2}\,x\big)}{2^{n/2}\sqrt{1-2\sqrt{2\pi}\,n!\,k^2}}$$

whilst if $\nu \notin \mathbb{N}$, *then for some* d *and* $\theta_0 \in \mathbb{R}$, *as* $x \to -\infty$

$$u_k(x) = (-1)^{[\nu+1]}\big(-\tfrac{1}{6}x\big)^{1/2} + d|x|^{-1/2}\sin\left(\frac{x^2}{2\sqrt{3}} - \frac{4d^2}{\sqrt{3}}\ln|x| - \theta_0\right) + \mathcal{O}\left(|x|^{-3/2}\right)$$

- *If* $k = k_*$, *then as* $x \to -\infty$

$$u_{k_*}(x) \sim \big(-\tfrac{1}{2}x\big)^{1/2}$$

- *If* $k > k_*$ *then* $u_k(x)$ *has a pole at a finite* x_0, *depending on* k, *so as* $x \downarrow x_0$

$$u_k(x) \sim (x - x_0)^{-1/2}$$

Theorem (Its & Kapaev [1998], Wong & Zhang [2009])

Let $u_k(x)$ *be the unique solution of*

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}x^2} = 3u_k^5 + 2xu_k^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})u_k$$

with boundary condition

$$u_k(x) \sim kD_\nu(x), \quad \textit{as} \quad x \to +\infty$$

If $0 \le k < k_*$, *with*

$$k_*^2 = \frac{1}{2\sqrt{2\pi}\,\Gamma(\nu+1)}$$

and $\nu \notin \mathbb{N}$, *then as* $x \to -\infty$

$$u_k(x) = (-1)^{[\nu+1]}\left(-\tfrac{1}{6}x\right)^{1/2} + d|x|^{-1/2}\sin\left(\frac{x^2}{2\sqrt{3}} - \frac{4d^2}{\sqrt{3}}\ln|x| - \theta_0\right) + \mathcal{O}\left(|x|^{-3/2}\right)$$

where the connection formulae are

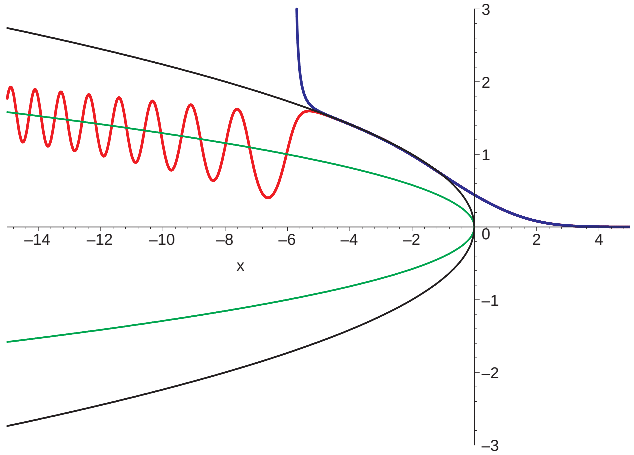
$$d^2(k;\nu) = -\tfrac{1}{4}\sqrt{3}\,\pi^{-1}\,\ln(1-|\mu|^2)$$

$$\theta_0(k;\nu) = \tfrac{1}{3}\sqrt{3}\,d^2\ln 3 + \arg\left\{\Gamma\left(-\tfrac{2}{3}\mathrm{i}\sqrt{3}\,d^2\right)\right\} + (\tfrac{2}{3}\nu + \tfrac{7}{12})\pi + \arg(\mu)$$

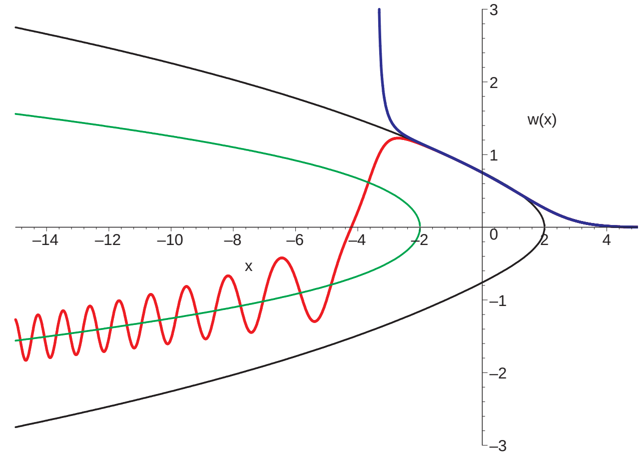
with

$$\mu(k;\nu) = 1 + \frac{2\mathrm{i}k\pi^{3/2}\exp(-\mathrm{i}\pi\nu)}{\Gamma(-\nu)}.$$

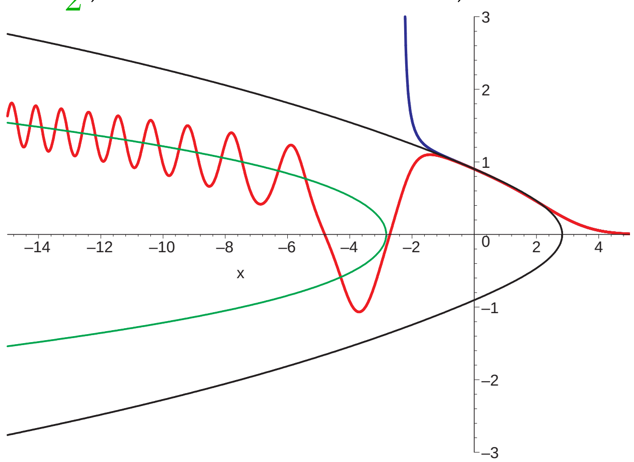
$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}x^2} = 3u_k^5 + 2xu_k^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})u_k, \qquad u_k(x) \sim kD_\nu(x), \quad \text{as} \quad x \to +\infty$$



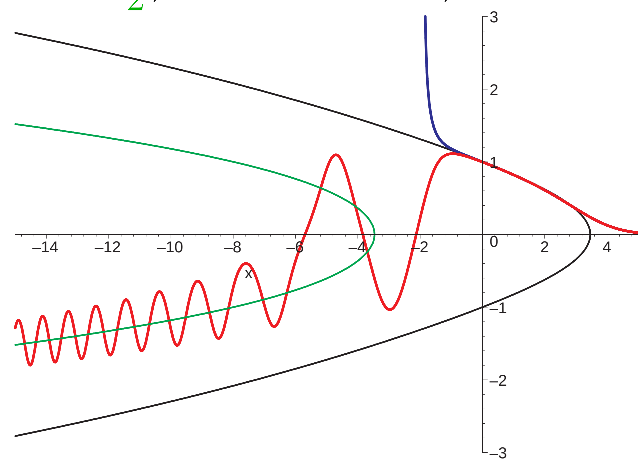
$\nu = -\frac{1}{2}$, $k = 0.33554691, 0.33554692$



$\nu = \frac{1}{2}$, $k = 0.47442, 0.47443$

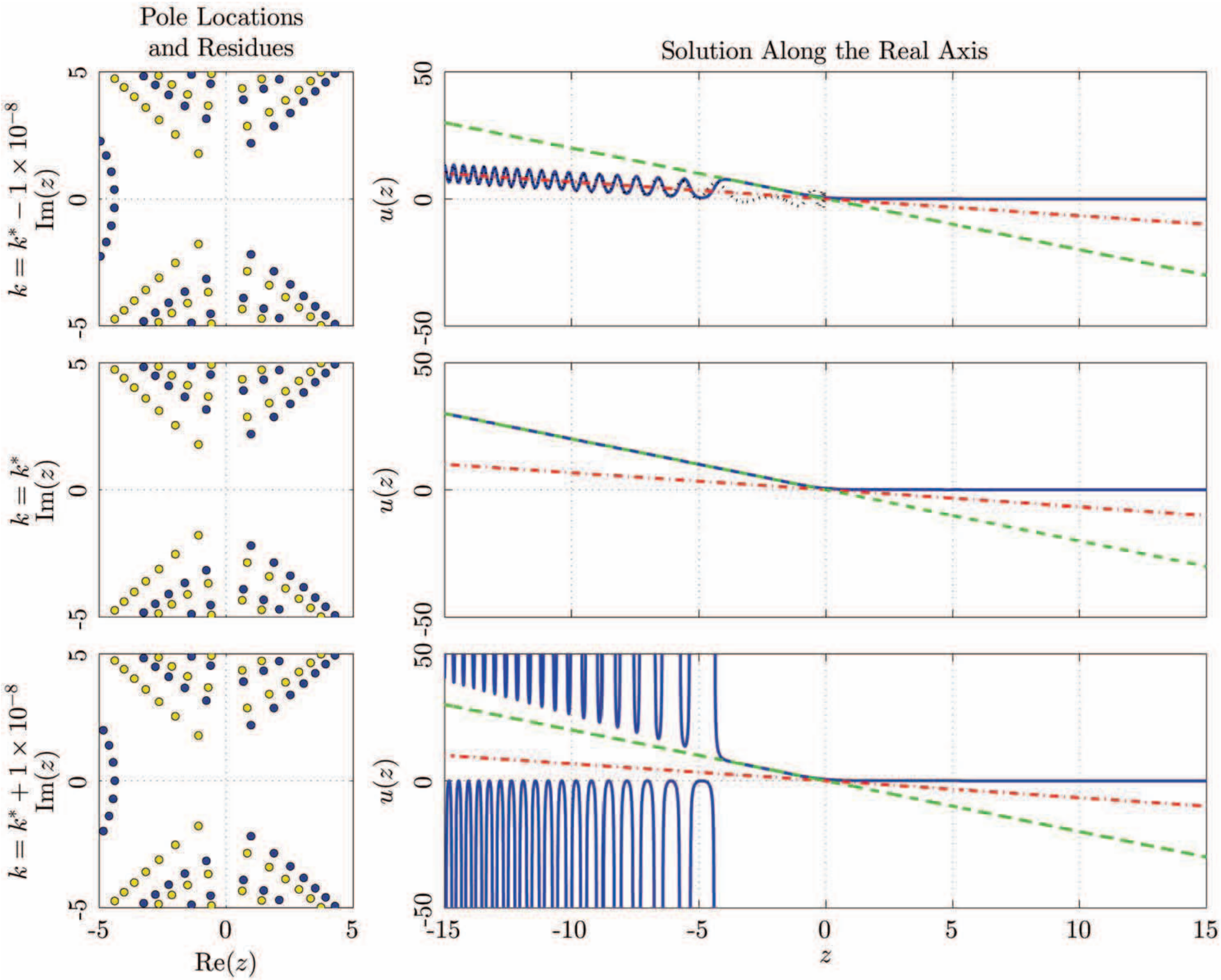


$\nu = \frac{3}{2}$, $k = 0.38736, 0.38737$



$\nu = \frac{5}{2}$, $k = 0.244992, 0.244993$

Numerical Solutions of P_{IV} (Reeger & Fornberg [2012])

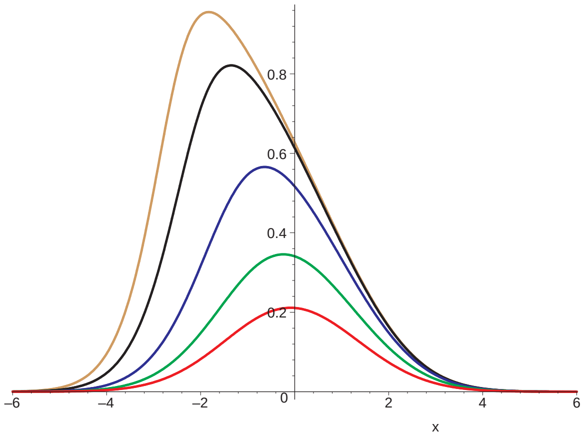


Solid lines: numerical solutions, **dashed-dotted** lines: $-\frac{2}{3}z$ and **dashed** lines: $-2z$.

The first two **bound state solutions** are

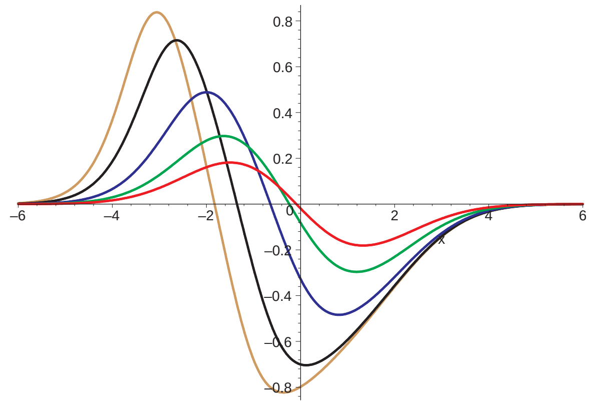
$$u_k(x;0) = \frac{k\exp(-\frac{1}{4}x^2)}{\sqrt{1-k^2\psi(x)}} \equiv \Psi_k(x), \qquad u_k(x;1) = \frac{\left[x + 2\Psi_k^2(x)\right]\Psi_k(x)}{\sqrt{1-2x\Psi_k^2(x)-4\Psi_k^4(x)}}$$

where $\psi(x) = \sqrt{2\pi}\ \mathrm{erfc}\left(\frac{1}{2}\sqrt{2}\,x\right)$ [note that $\psi(\infty)=0$ and $\psi(-\infty)=2\sqrt{2\pi}$].



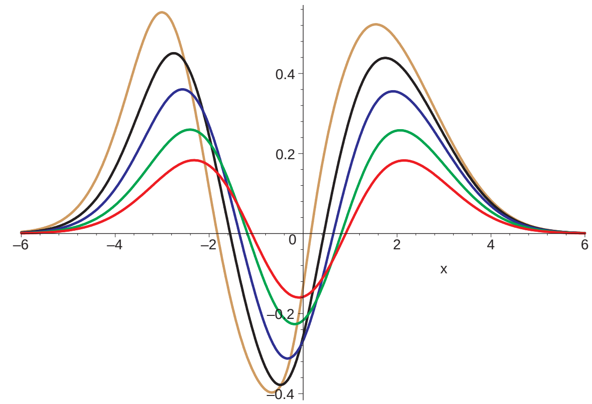
$u_k(x;0)$

0.2, 0.3, 0.4, 0.44, 0.445



$u_k(x;1)$

0.2, 0.3, 0.4, 0.44, 0.445



$u_k(x;2)$

0.15, 0.2, 0.25, 0.28, 0.3

For $n \in \mathbb{Z}^+$, $u_k(x;n)$ exists for all x, has n distinct zeros and decays exponentially to zero as $x \to \pm\infty$ with asymptotic behaviour

$$u_k(x;n) \sim \begin{cases} k\exp(-\frac{1}{4}x^2), & \text{as} \quad x \to \infty \\ \dfrac{k\exp(-\frac{1}{4}x^2)}{\sqrt{1-2\sqrt{2\pi}\,n!\,k^2}}, & \text{as} \quad x \to -\infty \end{cases} \qquad k^2 < \frac{1}{2\sqrt{2\pi}\,n!}$$

The corresponding solutions of $\mathrm{P_{IV}}$

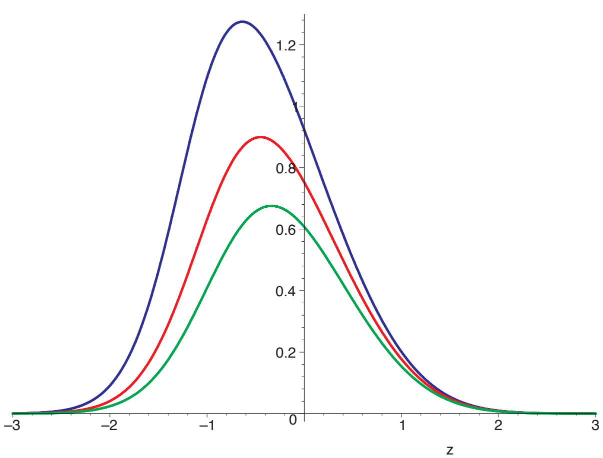
$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}$$

have the form

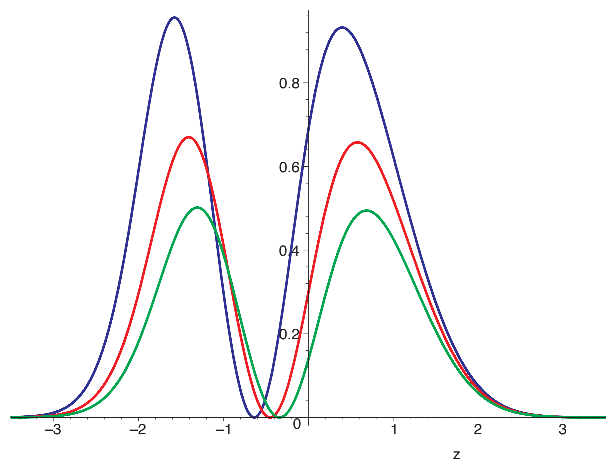
$$w(z;1,0) = \Psi(z;\xi) = \frac{2\xi\exp(-z^2)}{\sqrt{\pi}[1-\xi\,\mathrm{erfc}(z)]}$$

$$w(z;3,0) = -\frac{\Psi(\Psi+2z)^2}{\Psi^2+2z\Psi-2}$$

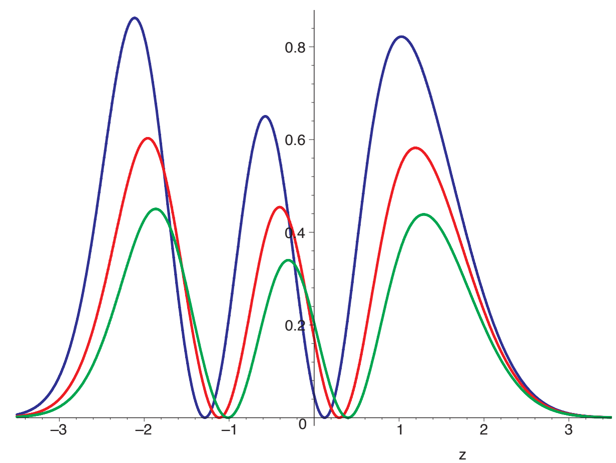
$$w(z;5,0) = \frac{4\Psi(\Psi^2+3z\Psi+2z^2-1)}{(\Psi^2+2z\Psi-2)[z\Psi^3+(4z^2+3)\Psi^2+2z(2z^2+3)\Psi-4]}$$



$w(z;1,0)$
$\xi = 0.7, 0.8, 0.9$



$w(z;3,0)$
$\xi = 0.7, 0.8, 0.9$



$w(z;5,0)$
$\xi = 0.7, 0.8, 0.9$

Some Open Problems

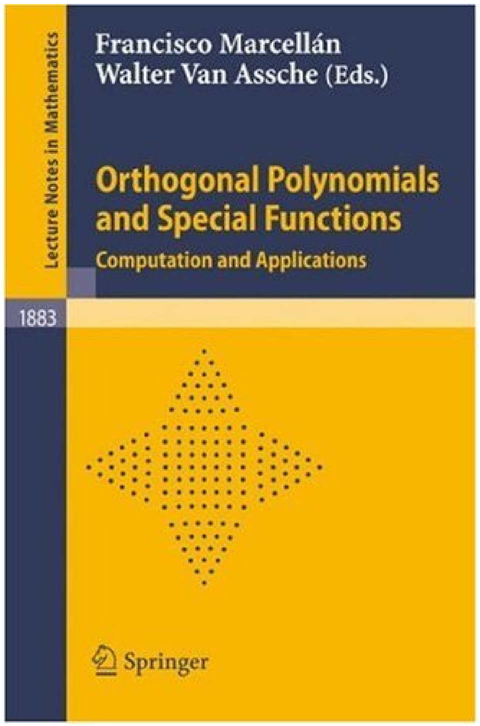
- Study exact (rational, algebraic and special function) solutions of Painlevé equations. This is particularly important with regard to applications of the Painlevé equations, as illustrated by semi-classical orthogonal polynomials and vortex dynamics.
- Is there an analytical explanation and interpretation of the computational results for the special polynomials associated with rational solutions of the Painlevé equations?
  - Is there an interlacing property for the roots in the complex plane?
  - Do these special polynomials have applications, e.g. in numerical analysis?
- Study asymptotics and connection formulae for the Painlevé equations using the isomonodromy method, for example the construction of uniform asymptotics around a nonlinear Stokes ray.

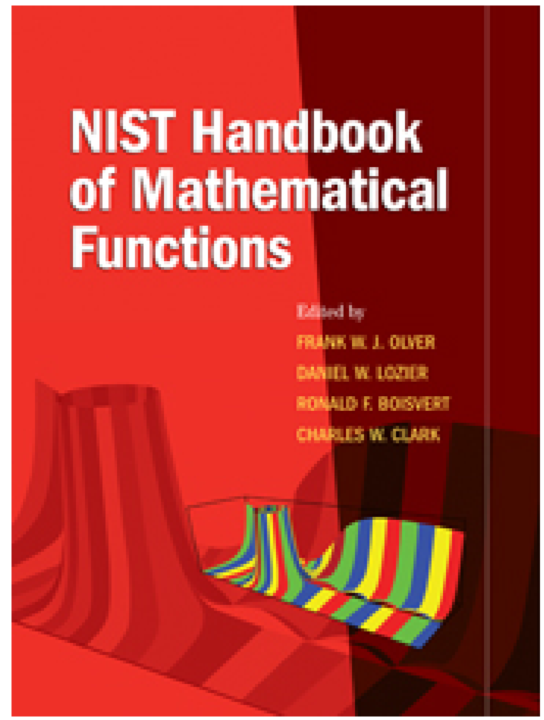
Objectives

- To provide a complete classification and unified structure of the special properties which the Painlevé equations (and Painlevé σ-equations) possess — the presently known results are rather fragmentary and non-systematic.
- Develop algorithmic procedures for the classification of equations with the Painlevé property; this is straightforward for linear equations but significantly more difficult for nonlinear equations.
- Develop software for numerically studying the Painlevé equations which utilizes the fact that they are integrable equations solvable using isomonodromy methods.

References

P A Clarkson, Painlevé equations — nonlinear special functions, in "*Orthogonal Polynomials and Special Functions: Computation and Application*" [Editors F Marcellàn and W van Assche], *Lect. Notes Math.*, **1883**, Springer, Berlin (2006) pp 331–411





Some Books on Painlevé Equations