On Nonlinear Partial Differential Equations of Mixed Type

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Three of the Basic Types: Representatives

- Elliptic PDE:

  Laplace's Equation: $\Delta_{\mathbf{x}} u = 0$

- Parabolic PDE:

  Heat Equation: $\partial_t u - \Delta_{\mathbf{x}} u = 0$

- Hyperbolic PDE:

  Wave Equation: $\partial_{tt} u - \Delta_{\mathbf{x}} u = 0$

  Transport Equation: $\partial_t u + \mathbf{b} \cdot \nabla_{\mathbf{x}} u = 0$

$\mathbf{x} = (x_1, \cdots, x_n), \qquad \Delta_{\mathbf{x}} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$

- **Mixed Hyperbolic-Parabolic Type**

$$\partial_t u + \mathbf{b}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} u - \nabla_{\mathbf{x}} \cdot (\mathbf{A}(\mathbf{x}) \nabla_{\mathbf{x}} u) = 0, \qquad \mathbf{A}(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{d\times d} \geq 0$$

- **Mixed Hyperbolic-Elliptic Type**

Lavrentyev Equation:

$$\partial_{xx} u + \mathsf{sign}(x) \partial_{yy} u = 0$$

Tricomi Equation (hyperbolic degeneracy at $x = 0$):

$$\partial_{xx} u + x \partial_{yy} u = 0$$

Keldysh Equation (parabolic degeneracy at $x = 0$):

$$x \partial_{xx} u + \partial_{yy} u = 0$$

Outline

1. Nonlinear PDEs of Mixed Hyperbolic-Parabolic Type
2. Nonlinear PDEs of Mixed Hyperbolic-Elliptic Type in Fluid Mechanics
3. Nonlinear PDEs of Mixed Hyperbolic-Elliptic Type in Differential Geometry
4. Nonlinear PDEs of No Type in Differential Geometry

Nonlinear PDEs of Mixed Hyperbolic-Parabolic Type

Anisotropic Degenerate Diffusion-Convection Equations

$$\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \nabla_{\mathbf{x}} \cdot (\mathbf{A}(u)\nabla_{\mathbf{x}} u) \qquad u \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d$$

for the unknown function $u\colon \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$, where

- $\mathbf{f} \in \mathrm{Lip}_{loc}(\mathbb{R}; \mathbb{R}^d)$ is the convection flux function
- $\mathbf{A}(u) = (a_{ij}(u)) \geq 0$ is the symmetric diffusion matrix so that

$$\mathbf{A}(u) = \sigma(u)\sigma(u)^\top, \qquad \sigma(u) = (\sigma_{jk}(u))$$

where $\sigma_k(u) = (\sigma_{1k}(u), \ldots, \sigma_{dk}(u))^\top \in L^\infty_{loc}(\mathbb{R}; \mathbb{R}^d),\ k = 1, \ldots, d$

Applications

- Viscous-inviscid two phase flows, $\cdots\cdots$
- Sedimentation-consolidation processes, fluids in porous media $\cdots\cdots$

Example:

Anisotropic Degenerate Diffusion-Convection Equations

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Example: $\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \Delta_{\mathbf{x}} u_+$

- $\{u > 0\}$: Viscous phase
- $\{u < 0\}$: Inviscid phase
- $\{u = 0\}$: Free boundary interface separating the two phases

Diffusion Dominated Case: Degenerate Parabolic Eqs.

$$\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \nabla_{\mathbf{x}} \cdot (\mathbf{A}(u)\nabla_{\mathbf{x}} u) \qquad u \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d$$

Degenerate points in u are isolated:

The set $\{u\ :\ \text{rank}(\mathbf{A}(u)) < d\}$ contains only isolated points

Well-Posedness

- Caffarelli-Friedman, Brézis-Crandall, Vázquez, $\cdots\cdots$
- Bénilan, DeBenedetto, Giding, Jäger, $\cdots\cdots$
- $\cdots\cdots$

Similarity Solutions, $\cdots$

- Barenblatt, $\cdots\cdots$

Parabolic Approaches

Diffusion Vanished Case: Hyperbolic Conservation Laws

In this case, the equation becomes a scalar hyperbolic conservation law

$$\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = 0, \qquad u|_{t=0} = u_0(\mathbf{x})$$

Linear Case: E.g. Transport equation: $\partial_t u + a\,\partial_x u = 0$, $a \neq 0$ const.

$$u(t,x) = u_0(x - at).$$

- Well-posed in any norms
- If $u_0(x+P) = u_0(x)$, then $u(t,x)$ is oscillatory as $t \to \infty$: No Limit

Nonlinear Case: E.g. Burgers equation: $\partial_t u + \partial_x(\frac{u^2}{2}) = 0$

- Well-posed in BV, L^∞, L^1: Oleinik, Lax, Volpert, Kruzkov, $\cdots$
  Lions-Perthame-Tadmor, $\cdots$
- Decay of period solutions in L^∞: Lax ($d = 1$)
  Engquist-E ($d = 2$), Chen-Frid($d \geq 2$)

If $\mathcal{L}\big(\{\xi \,:\, \tau + \mathbf{f}'(\xi) \cdot \boldsymbol{\kappa} = 0\}\big) = 0$ for any $|\tau|^2 + |\boldsymbol{\kappa}|^2 = 1$,
then $u \longrightarrow \bar{u} := \frac{1}{|P|} \int_P u_0(\mathbf{x}) d\mathbf{x}$ in L^1

Hyperbolic Approaches

Isotropic Degenerate Parabolic-Hyperbolic Equations

Isotropic Case ($\mathbf{A}(u) = \beta'(u)\mathbf{I}$): $\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \Delta_{\mathbf{x}} \beta(u)$
Solutions $u \in L^\infty$

One-dimensional case $d = 1$:

- Wu-Yin 1989
- Bénilan-Touré 1995

Multidimensional case $d \geq 1$:

- Regularity: $\beta(u(t, \mathbf{x})) \in C$ DiBenedetto 1982
- $u \in L^\infty$: Carrillo 1999 $\longrightarrow$ Karlsen-Risebro 2001
  Mascia-Porretta-Terracina 2002
- $u \in L^\infty_{loc}$: Chen-DiBenedetto: SIAM J. Math. Anal. 2001

General Case & Solutions $u \in L^1$

Goal: Identify and/or develop Unified Approaches to deal with both parabolic and hyperbolic phases

Chen-Perthame: Kinetic Approach

Ann. I. H. Poincaré–Anal. Non Linéaire, **20** (2003), 645–668
Proc. Amer. Math. Soc. **137** (2009), 3003–3011

Motivations/Connections:

- Boltzmann Equation, Large Particle Systems
  $\Rightarrow$ Euler Equations, Navier-Stokes Equations

$$\partial_t \chi + \xi \cdot \nabla_{\mathbf{x}} \chi = Q(\chi, \chi)$$

- Earlier Works on Numerical Methods for Conservation Laws:
  Brenier, Giga-Miyawake, Croisille-Delorme, Deshpande, Kanel, Perthame, Xu-Prendergast, $\cdots$
- Earlier Works on Kinetic Formulation, esp. for Hyperbolic Conservation Laws: Lions-Perthame-Tadmor 1991, 1994....

Kinetic Function

Cauchy Problem

$$\begin{cases} \partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \nabla_{\mathbf{x}} \cdot (\mathbf{A}(u) \nabla_{\mathbf{x}} u) \qquad u \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d, \\ u|_{t=0} = u_0(\mathbf{x}) \in L^1(\mathbb{R}^d) \end{cases}$$

Introduce the kinetic function: Quasi-Maxwellian χ on $\mathbb{R}^2$:

$$\chi(\xi; u) = \begin{cases} +1 & \text{for } \ 0 < \xi < u, \\ -1 & \text{for } \ u < \xi < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$u \in L^\infty([0,\infty); L^1(\mathbb{R}^d)) \Rightarrow \chi \in L^\infty([0,\infty); L^1(\mathbb{R}^{d+1}))$$

Kinetic Solutions $u \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$

(i) Kinetic Equation:

$$\partial_t \chi(\xi;u) + \mathbf{f}'(\xi)\cdot\nabla_{\mathbf{x}}\chi(\xi;u) - \nabla_{\mathbf{x}}\cdot(\mathbf{A}(\xi)\nabla_{\mathbf{x}}\chi(\xi;u))$$
$$= \partial_\xi(m+n)(t,\mathbf{x};\xi) \qquad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^{d+1})$$

holds with initial data: $\chi(\xi;u)|_{t=0} = \chi(\xi;u_0)$,

for some measures $m(t,\mathbf{x};\xi)\ge 0$ and $n(t,\mathbf{x};\xi)\ge 0$:

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holds with initial data: $\chi(\xi; u)|_{t=0} = \chi(\xi; u_0)$,
for some measures $m(t, \mathbf{x}; \xi) \geq 0$ and $n(t, \mathbf{x}; \xi) \geq 0$:

$\int_{\mathbb{R}} \psi(\xi)\, n(t, \mathbf{x}; \xi)\, d\xi = \sum_{k=1}^d \left(\nabla_{\mathbf{x}} \cdot \beta_k^\psi(u(t, \mathbf{x}))\right)^2 \in L^2([0,\infty) \times \mathbb{R}^d)$

with $\beta_k^\psi(u) = \int^u \sqrt{\psi(v)} \sigma_k(v) dv$ for any $\psi \in C_0^\infty(\mathbb{R})$ with $\psi \geq 0$;

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(ii) The following inequality is satisfied:

$$\int_0^\infty \int_{\mathbb{R}^d} (m+n)(t,\mathbf{x};\xi)\, dt\, d\mathbf{x} \to 0 \qquad \text{as} \quad |\xi|\to\infty;$$

Kinetic Solutions $u \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$

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$$\begin{aligned} \partial_t\, \chi(\xi; u) + \mathbf{f}'(\xi) \cdot \nabla_{\mathbf{x}} \chi(\xi; u) - \nabla_{\mathbf{x}} \cdot (\mathbf{A}(\xi) \nabla_{\mathbf{x}} \chi(\xi; u)) \\ = \partial_\xi (m+n)(t, \mathbf{x}; \xi) \qquad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^{d+1}) \end{aligned}$$

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$$\int_0^\infty \int_{\mathbb{R}^d} (m+n)(t, \mathbf{x}; \xi)\, dt\, d\mathbf{x} \to 0 \qquad \text{as} \quad |\xi| \to \infty;$$

(iii) For any nonnegative $\psi_1,\ \psi_2 \in C_0^\infty(\mathbb{R})$,

$$\sqrt{\psi_1(u(t, \mathbf{x}))}\, \nabla_{\mathbf{x}} \cdot \beta_k^{\psi_2}(u(t, \mathbf{x})) = \nabla_{\mathbf{x}} \cdot \beta_k^{\psi_1 \psi_2}(u(t, \mathbf{x})) \qquad a.e.$$

Remarks on Kinetic Solutions

- L^1: Well-posed space for kinetic solutions and well-defined space for the kinetic equation, although $\mathbf{f}(u), \mathbf{A}(u) \notin L^1$ generally.
- If $u \in L^\infty$, then u is an entropy solution:
  For any $\eta \in C^2, \eta''(u) \geq 0$, multiplying $\eta'(\xi)$ both sides of the kinetic equation and then integrating in $\xi \in \mathbb{R}$ yields
  $$\partial_t \eta(u) + \nabla \cdot (\mathbf{q}(u) - \mathbf{A}(u)\nabla\eta(u)) = -\int_{\mathbb{R}} \eta''(\xi)(m+n)(t,\mathbf{x};\xi)d\xi \leq 0.$$
  In particular, take $\eta(u) = \pm u$ to yield the PDE.

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- Existence of a kinetic solution when $u_0 \in W^{2,1} \cap H^1 \cap L^\infty(\mathbb{R}^d)$ can be achieved by the vanishing viscosity method.

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- Condition (iii) implies that $m+n$ has no support at $u = \infty$.

L^1–Stability Theorem and Remarks

Theorem (Chen-Perthame: Ann. I. H. Poincaré–AN 2003)

Let $u, v \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$ *be kinetic solutions with initial data* $u_0, v_0 \in L^1(\mathbb{R}^d)$ *respectively. Then*

- $\|u(t,\cdot) - u_0(\cdot)\|_{L^1(\mathbb{R}^d)} \to 0$ *as* $t \to 0$*;*
- $\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}$.

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- $\|u(t,\cdot) - u_0(\cdot)\|_{L^1(\mathbb{R}^d)} \to 0$ *as* $t \to 0$;
- $\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}$.

Remarks

- Existence for $u_0 \in L^1$: There exists a sequence $u_0^\varepsilon \in W^{2,1} \cap H^1 \cap L^\infty(\mathbb{R}^d)$ such that $\|u_0^\varepsilon - u_0\|_{L^1(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. Then there exists a corresponding sequence of kinetic solutions $u^\varepsilon \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$. The L^1-stability theorem implies that $\{u^\varepsilon\}$ is a Cauchy sequence so that there exists $u \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$ satisfying that $u^\varepsilon(t,\mathbf{x}) \to u(t,\mathbf{x})$ in L^1 when $\varepsilon \to 0$.

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- When $u_0 \in L^\infty(\mathbb{R}^d)$, then the kinetic solution $u \in L^\infty$ is the unique entropy solution.

Ideas of Proof for the L^1-Stability—Formal Proof

$u(t,\mathbf{x}) \sim m(t,\mathbf{x};\xi) \geq 0$

$n(t,\mathbf{x};\xi) = \delta(\xi - u(t,\mathbf{x}))\sum_{k=1}^{K}(\nabla_{\mathbf{x}} \cdot \beta_k(u(t,\mathbf{x})))^2$

$v(t,\mathbf{x}) \sim p(t,\mathbf{x};\xi) \geq 0$

$q(t,\mathbf{x};\xi) = \delta(\xi - v(t,\mathbf{x}))\sum_{k=1}^{K}(\nabla_{\mathbf{x}} \cdot \beta_k(v(t,\mathbf{x})))^2$

Step 1. Microscopic Contraction Function:

$Q(t,\mathbf{x};\xi) = |\chi(\xi;u)| + |\chi(\xi;v)| - 2\chi(\xi;u)\chi(\xi;v) \geq 0$

Then

$$\int_{\mathbb{R}} Q(t,\mathbf{x};\xi)d\xi = |u(t,\mathbf{x}) - v(t,\mathbf{x})|$$

Step 2. Multiplying the Kinetic Equation by $sign(\xi)$:

$$\partial_t|\chi(\xi;u)| + \mathbf{f}'(\xi)\cdot\nabla_{\mathbf{x}}|\chi(\xi;u)| - \nabla_{\mathbf{x}}\cdot(\mathbf{A}(\xi)\nabla\chi(\xi;u)) \\ = sign(\xi)\partial_\xi(m+n)(t,\mathbf{x};\xi).$$

Then

$$\frac{d}{dt}\int_{\mathbb{R}^{d+1}}|\chi(\xi;u)|d\mathbf{x}d\xi = -2\int_{\mathbb{R}^d}(m+n)(t,\mathbf{x};0)d\mathbf{x},$$

$$\frac{d}{dt}\int_{\mathbb{R}^{d+1}}|\chi(\xi;v)|d\mathbf{x}d\xi = -2\int_{\mathbb{R}^d}(p+q)(t,\mathbf{x};0)d\mathbf{x}.$$

Ideas of Proof: Conti.

Step 3. Multiplying the Eq. for u by $\chi(\xi; v)$ and the Eq. for v by $\chi(\xi; u)$, and then adding together yield

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d+1}}(-2\chi(\xi;u)\chi(\xi;v))d\mathbf{x}d\xi \\
&= 4\sum_{i,j}\int_{\mathbb{R}^{d+1}} a_{ij}(\xi)\partial_{x_i}\chi(\xi;u)\partial_{x_j}\chi(\xi;v)d\mathbf{x}d\xi \\
&\quad -2\int_{\mathbb{R}^{d+1}}\{(m+n)(t,\mathbf{x};\xi)(\delta(\xi-v)-\delta(\xi)) \\
&\qquad\qquad +(p+q)(t,\mathbf{x};\xi)(\delta(\xi-u)-\delta(\xi))\}d\mathbf{x}d\xi
\end{aligned}$$

Step 4. Combining Step 2 with Step 3 yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,\mathbf{x}) - v(t,\mathbf{x})| d\mathbf{x} \\
&= \frac{d}{dt}\int_{\mathbb{R}^{d+1}} Q(t,\mathbf{x};\xi) d\mathbf{x} d\xi \\
&= 4\sum_{i,j}\int_{\mathbb{R}^{d+1}} a_{ij}(\xi)\partial_{x_i}\chi(\xi;u)\,\partial_{x_j}\chi(\xi;v) d\mathbf{x} d\xi \\
&\quad -2\int_{\mathbb{R}^{d+1}} \{(m+n)(t,\mathbf{x};\xi)\,\delta(\xi - v) \\
&\qquad\qquad +(p+q)(t,\mathbf{x};\xi)\,\delta(\xi - u)\} d\mathbf{x} d\xi \\
&\le 4\sum_{i,j}\int_{\mathbb{R}^{d+1}} a_{ij}(\xi)\partial_{x_i}u\,\partial_{x_j}v\,\delta(\xi-u)\delta(\xi-v) d\mathbf{x} d\xi \\
&\quad -2\sum_{k}\int_{\mathbb{R}^{d+1}} \delta(\xi-u)\delta(\xi-v)\Big((\nabla_{\mathbf{x}}\cdot\beta_k(u))^2 + (\nabla_{\mathbf{x}}\cdot\beta_k(v))^2\Big) d\mathbf{x} d\xi \\
&\le 4\sum_{i,j}\int_{\mathbb{R}^{d+1}} a_{ij}(\xi)\partial_{x_i}u\,\partial_{x_j}v\,\delta(\xi-u)\delta(\xi-v) d\mathbf{x} d\xi \\
&\quad -4\sum_{k,i,j}\int_{\mathbb{R}^{d+1}} \delta(\xi-u)\delta(\xi-v)\sigma_{ki}(u)\sigma_{kj}(v)\partial_{x_i}u\,\partial_{x_j}v\, d\mathbf{x} d\xi \\
&= 0.
\end{aligned}
$$

Rigorous Proof: Regularization, Truncation, and Limits

(i) $\varepsilon = (\varepsilon_1, \varepsilon_2)$: $\varphi_\varepsilon(t, \mathbf{x}) = \frac{1}{\varepsilon_1}\varphi_1(\frac{t}{\varepsilon_1})\,\frac{1}{\varepsilon_2^d}\varphi_2(\frac{\mathbf{x}}{\varepsilon_2})$
for $\varphi_j \geq 0, \int \varphi_j = 1, \mathrm{supp}(\varphi_1) \subset (-1, 0)$
Set

$$\chi_\varepsilon := \chi_\varepsilon(\xi; u(t,\mathbf{x})) = (\chi *_{(t,\mathbf{x})} \varphi_\varepsilon)(t, \mathbf{x}; \xi),$$
$$m_\varepsilon := m *_{(t,\mathbf{x})} \varphi_\varepsilon, \qquad n_\varepsilon := n *_{(t,\mathbf{x})} \varphi_\varepsilon.$$

(ii) $\psi_\delta(\xi) = \frac{1}{\delta}\psi(\frac{\xi}{\delta})$: Set $\chi_{\varepsilon,\delta} := \chi_\varepsilon * \psi_\delta$.

(iii) ξ-Truncation: $K_R(\xi) = K(\frac{\xi}{R}) \to 1$ as $R \to \infty$
for $0 \leq K(\xi) \leq 1; K(\xi) = 1$ as $|\xi| \leq 1/2; K(\xi) = 0$ as $|\xi| \geq 1$.

Sending $\delta \to 0$ first and $R \to \infty$ second lead to

$$\frac{d}{dt}\int_{\mathbb{R}^{d+1}} Q_\varepsilon(t, \mathbf{x}; \xi)\, d\mathbf{x}\, d\xi \leq 0 \quad \text{for any } \varepsilon,$$

where $Q_\varepsilon(t, \mathbf{x}; \xi) = |\chi_\varepsilon(\xi; u)| + |\chi_\varepsilon(\xi; v)| - 2\chi_\varepsilon(\xi; u)\chi_\varepsilon(\xi; v)$.

Theorem (Chen-Perthame: PAMS 2009)

Let $u_0(\mathbf{x} + P_i e_i) = u_0(\mathbf{x})$ *be periodic with period* $\mathbb{T}_P := \Pi_{i=1}^d [0, P_i] \subset \mathbb{R}^d$ *with* $P_i > 0$ *and* $(e_i)_{1 \le i \le d}$ *the basis of* $\mathbb{R}^d$*. Let the flux function* $\mathbf{f}(u)$ *and the diffusion matrix* $\mathbf{A}(u)$ *satisfy the "nonlinearity-diffusivity" condition: For any* $\delta > 0$,

$$\sup_{|\tau|+|\boldsymbol{\kappa}|\ge\delta} \int_{|\xi|\le\|u^0\|_\infty} \frac{\lambda\, d\xi}{\lambda + |\tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}|^2 + (\boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa})^2} := \omega_\delta(\lambda) \underset{\lambda\to 0}{\longrightarrow} 0.$$

Then the kinetic solution $u(t, \mathbf{x}) \in L^\infty$ *is asymptotically decay:*

$$\int_{\mathbb{T}_P} \left| u(t,\mathbf{x}) - \tfrac{1}{|\mathbb{T}_P|}\int_{\mathbb{T}_P} u_0(\mathbf{x}) d\mathbf{x} \right| d\mathbf{x} \longrightarrow 0 \qquad \text{when } t \to \infty.$$

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For smooth coefficients, the "nonlinearity-diffusivity" condition is equivalent to

$$\mathcal{L}\Big(\{\xi\in\mathbb{R} : \tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa} = 0, \textstyle\sum_{i,j=1}^d a_{ij}(\xi)\kappa_i\kappa_j = 0\}\Big) = 0, \quad \forall\ \tau^2 + |\boldsymbol{\kappa}|^2 = 1$$

Theorem (Chen-Perthame: PAMS 2009)

Let $u_0(\mathbf{x}+P_i e_i)=u_0(\mathbf{x})$ be periodic with period $\mathbb{T}_P:=\Pi_{i=1}^d[0,P_i]\subset\mathbb{R}^d$ with $P_i>0$ and $(e_i)_{1\le i\le d}$ the basis of $\mathbb{R}^d$. Let the flux function $\mathbf{f}(u)$ and the diffusion matrix $\mathbf{A}(u)$ satisfy the "nonlinearity-diffusivity" condition: For any $\delta>0$,

$$\sup_{|\tau|+|\boldsymbol{\kappa}|\ge\delta}\int_{|\xi|\le\|u^0\|_\infty}\frac{\lambda\,d\xi}{\lambda+|\tau+\mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}|^2+(\boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa})^2}:=\omega_\delta(\lambda)\xrightarrow[\lambda\to0]{}0.$$

Then the kinetic solution $u(t,\mathbf{x})\in L^\infty$ is asymptotically decay:

$$\int_{\mathbb{T}_P}\Big|u(t,\mathbf{x})-\tfrac{1}{|\mathbb{T}_P|}\int_{\mathbb{T}_P}u_0(\mathbf{x})d\mathbf{x}\Big|\,d\mathbf{x}\longrightarrow0\qquad\textit{when } t\to\infty.$$

For smooth coefficients, the "nonlinearity-diffusivity" condition is equivalent to

$$\mathcal{L}\Big(\{\xi\in\mathbb{R}\,:\,\tau+\mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}=0,\textstyle\sum_{i,j=1}^d a_{ij}(\xi)\kappa_i\kappa_j=0\}\Big)=0,\quad\forall\ \tau^2+|\boldsymbol{\kappa}|^2=1$$

The well-posedness theory $\Longrightarrow$ There exists a unique entropy solution $u\in L^\infty([0,\infty)\times\mathbb{R}^d)$ such that $u(t,\mathbf{x})$ is $\mathbb{T}_P$-periodic a.e. and $\|u(t,\cdot)\|_{L^\infty}\le\|u_0\|_{L^\infty}$

Ideas of Proof—I: W.O.L.G., assume $\int_{\mathbb{T}_P} u_0(\mathbf{x})d\mathbf{x} = 0$

Step 1. The well-posedness theory and the averaging compactness result $\Longrightarrow$ The unique periodic entropy solution $u(t,x) \in L^\infty$ satisfies

(i) For any $t_2 > t_1 > 0$,

$$\int_{\mathbb{T}_P} |u(t_2,\mathbf{x})|\, d\mathbf{x} \le \int_{\mathbb{T}_P} |u(t_1,\mathbf{x})|\, d\mathbf{x},$$

$$\int_{\mathbb{T}_P} |u(t_2,\mathbf{x})|^2\, d\mathbf{x} \le \int_{\mathbb{T}} |u(t_1,\mathbf{x})|^2\, d\mathbf{x} \qquad \text{(energy estimate).}$$

$\Longrightarrow$ The function $I(t) := \int_{\mathbb{T}_P} |u(t,\mathbf{x})|^2\, dx$ is a non-increasing, bounded function, which implies that the following limit exists:

$$\lim_{t\to\infty} I(t) = I(\infty) =: I_\infty \in [0,\infty).$$

Question: $I_\infty = 0$??

(ii) When the flux function $f(u)$ and the diffusion matrix $A(u)$ satisfy the nonlinearity-diffusivity condition, then the solution operator $u(t,\cdot) = S_t u_0(\cdot) \, : \, L^\infty \to L^1_{loc}$ is locally compact for $t > 0$.

Averaging Compactness: Lions-Perthame-Tadmor 1994; Also see

Ideas of Proof for the Decay—II

Step 2. Translations. Set $v_k(t,\mathbf{x}) := u(t+k,\mathbf{x})$.
Then we find that, for $t \geq -k$,

(i) $\|v_k(t,\cdot)\|_{L^\infty} = \|u(t+k,\cdot)\|_{L^\infty} \leq \|u_0\|_{L^\infty}$;

(ii) $v_k(t,\mathbf{x})$ is also a periodic entropy solution;

(iii) $\int_{\mathbb{T}_P} v_k(t,\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{T}_P} u_0(\mathbf{x})\,d\mathbf{x} = 0$;

(iv) for each $k > 0$, $\chi(\xi; v^k(t,\mathbf{x}))$ satisfies

$$\partial_t\chi(\xi;v^k) + \mathbf{f}'(\xi)\cdot\nabla_{\mathbf{x}}\chi(\xi;v^k) - \nabla_{\mathbf{x}}\cdot\big(A(\xi)\nabla_{\mathbf{x}}\chi(\xi;v^k)\big)$$
$$= \partial_\xi(m^k+n^k)(t,\mathbf{x},\xi) \qquad \text{in } \mathcal{D}'((-k,\infty)\times\mathbb{R}^{d+1})$$

Step 1 (ii) applied to v^k,

Step 2. Translations. Set $v_k(t,\mathbf{x}) := u(t+k,\mathbf{x})$.
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(ii) $v_k(t,\mathbf{x})$ is also a periodic entropy solution;

(iii) $\int_{\mathbb{T}_P} v_k(t,\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{T}_P} u_0(\mathbf{x})\,d\mathbf{x} = 0$;

(iv) for each $k > 0$, $\chi(\xi; v^k(t,\mathbf{x}))$ satisfies

$$\partial_t \chi(\xi; v^k) + \mathbf{f}'(\xi)\cdot\nabla_{\mathbf{x}}\chi(\xi; v^k) - \nabla_{\mathbf{x}}\cdot\big(A(\xi)\nabla_{\mathbf{x}}\chi(\xi; v^k)\big)$$
$$= \partial_\xi(m^k + n^k)(t,\mathbf{x},\xi) \qquad \text{in } \mathcal{D}'((-k,\infty)\times\mathbb{R}^{d+1})$$

Step 1 (ii) applied to v^k, there exists a subsequence v^{k_j} and $v(t,\mathbf{x}) \in L^\infty(\mathbb{R}^{d+1})$, with $\int_{\mathbb{T}} v(t,\mathbf{x})d\mathbf{x} = 0$, such that

$$v^{k_j}(t,\mathbf{x}) \to v(t,\mathbf{x}) \qquad \text{a.e. } (t,\mathbf{x}) \in \mathbb{R}^{d+1} \qquad \text{as } j \to \infty.$$
$$\chi(\xi; v^{k_j}(t,\mathbf{x})) \to \chi(\xi; v(t,\mathbf{x})) \qquad \text{a.e. } (t,\mathbf{x},\xi) \qquad \text{as } j \to \infty.$$

Ideas of Proof for the Decay—III

Step 3. The kinetic equation $\Longrightarrow$

$$\begin{aligned}\int_{-T}^{T}\int_{\mathbb{T}_P}(m^k+n^k)(t,\mathbf{x},\xi)\,dtdxd\xi &\le \frac{1}{2}\big(I(k-T)-I(k+T)\big)\\ &\le \frac{1}{2}|\mathbb{T}_P|\|u_0\|_{L^\infty}^2\end{aligned}$$

$\Longrightarrow$ There exists a subsequence k_j and a measure $M(t,\mathbf{x},\xi)$ such that

$(m^{k_j}+n^{k_j})(t,\mathbf{x},\xi) \rightharpoonup M(t,\mathbf{x},\xi) \ge 0$ weakly in $\mathcal{M}$ as $j\to\infty$.

Ideas of Proof for the Decay—III

Step 3. The kinetic equation $\Longrightarrow$

$$\int_{-T}^{T}\int_{\mathbb{T}_P}(m^k+n^k)(t,\mathbf{x},\xi)\,dtdxd\xi \leq \frac{1}{2}\big(I(k-T)-I(k+T)\big) \leq \frac{1}{2}|\mathbb{T}_P|\|u_0\|_{L^\infty}^2$$

$\Longrightarrow$ There exists a subsequence k_j and a measure $M(t,\mathbf{x},\xi)$ such that

$$(m^{k_j}+n^{k_j})(t,\mathbf{x},\xi)\rightharpoonup M(t,\mathbf{x},\xi)\geq 0 \qquad \text{weakly in } \mathcal{M} \quad \text{as } j\to\infty.$$

Since $I(t)$ converges, we have $I(k-T)-I(T+k)\to 0$ as $k\to\infty$, which implies $\quad M(\mathbb{R}^{d+2})=0.$

Ideas of Proof for the Decay—III

Step 3. The kinetic equation $\Longrightarrow$

$$\int_{-T}^{T}\int_{\mathbb{T}_P}(m^k+n^k)(t,\mathbf{x},\xi)\,dtdxd\xi \le \frac{1}{2}\big(I(k-T)-I(k+T)\big) \le \frac{1}{2}|\mathbb{T}_P|\|u_0\|_{L^\infty}^2$$

$\Longrightarrow$ There exists a subsequence k_j and a measure $M(t,\mathbf{x},\xi)$ such that

$$(m^{k_j}+n^{k_j})(t,\mathbf{x},\xi)\rightharpoonup M(t,\mathbf{x},\xi)\ge 0 \qquad \text{weakly in } \mathcal{M} \quad \text{as } j\to\infty.$$

Since $I(t)$ converges, we have $I(k-T)-I(T+k)\to 0$ as $k\to\infty$, which implies $M(\mathbb{R}^{d+2})=0$. Letting $k\to\infty$ in the kinetic equation, then $\chi(\xi;v)$ is a $\mathbb{T}_P$-periodic solution in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^{d+1})$ of

$$\partial_t\chi+\mathbf{f}'(\xi)\cdot\nabla_{\mathbf{x}}\chi-\nabla_{\mathbf{x}}\cdot\big(A(\xi)\nabla_{\mathbf{x}}\chi\big)=0.$$

Multiplying the above equation by ξ and then integrating $d\mathbf{x}d\xi$, we have

Step 3. The kinetic equation $\Longrightarrow$

$$\int_{-T}^{T}\int_{\mathbb{T}_P}(m^k+n^k)(t,\mathbf{x},\xi)\,dtdxd\xi \;\le\; \frac{1}{2}\big(I(k-T)-I(k+T)\big) \;\le\; \frac{1}{2}|\mathbb{T}_P|\|u_0\|_{L^\infty}^2$$

$\Longrightarrow$ There exists a subsequence k_j and a measure $M(t,\mathbf{x},\xi)$ such that

$$(m^{k_j}+n^{k_j})(t,\mathbf{x},\xi)\rightharpoonup M(t,\mathbf{x},\xi)\ge 0 \qquad \text{weakly in } \mathcal{M} \quad \text{as } j\to\infty.$$

Since $I(t)$ converges, we have $I(k-T)-I(T+k)\to 0$ as $k\to\infty$, which implies $M(\mathbb{R}^{d+2})=0$. Letting $k\to\infty$ in the kinetic equation, then $\chi(\xi;v)$ is a $\mathbb{T}_P$-periodic solution in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^{d+1})$ of

$$\partial_t\chi+\mathbf{f}'(\xi)\cdot\nabla_{\mathbf{x}}\chi-\nabla_{\mathbf{x}}\cdot\big(A(\xi)\nabla_{\mathbf{x}}\chi\big)=0.$$

Multiplying the above equation by ξ and then integrating $d\mathbf{x}d\xi$, we have

$$\int_{\mathbb{T}_P}|v(t,\mathbf{x})|^2dx=I_\infty\in[0,\infty), \qquad \forall\, t\in\mathbb{R},$$

where $I_\infty=I(\infty)$ is a constant, independent of t.

Ideas of Proof for the Decay—IV

Step 4. The rest of the proof consists in showing that

- such a function $\chi(\xi; v)$ is very particular and is in fact constant;
- $v(t, \mathbf{x}) \equiv 0$ a.e. for $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$.

$\Longrightarrow \quad I_\infty = 0$

$\Longrightarrow$ The proof is complete.

Ideas of Proof for the Decay—V

Step 5. Claim: $v(t, \mathbf{x}) \equiv 0$ a.e. for $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$.

For a time-truncation function $\phi(t), 0 \le \phi(t) \le 1$, so that $\phi\chi$ belongs to $L^2(\mathbb{R} \times \mathbb{T}_P \times \mathbb{R})$, we have

$$\partial_t(\phi\chi) + \mathbf{f}'(\xi) \cdot \nabla_x(\phi\chi) - \nabla_x \cdot \big(A(\xi)\nabla_x(\phi\chi)\big) = \chi\,\partial_t\phi \quad \text{in } \mathcal{D}'(\mathbb{R}^{d+2}).$$

Step 5. Claim: $v(t,\mathbf{x}) \equiv 0$ a.e. for $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$.

For a time-truncation function $\phi(t), 0 \le \phi(t) \le 1$, so that $\phi\chi$ belongs to $L^2(\mathbb{R}\times\mathbb{T}_P\times\mathbb{R})$, we have

$$\partial_t(\phi\chi) + \mathbf{f}'(\xi)\cdot\nabla_x(\phi\chi) - \nabla_x\cdot\big(A(\xi)\nabla_x(\phi\chi)\big) = \chi\,\partial_t\phi \quad \text{in } \mathcal{D}'(\mathbb{R}^{d+2}).$$

Since $\phi\chi$ and $\chi\phi_t$ are periodic in $\mathbf{x}$, we take the global Fourier transform in $t\in\mathbb{R}$ and the local Fourier transform in $\mathbf{x}\in\mathbb{T}_P$ to obtain

$$\hat{g}(\tau,\boldsymbol{\kappa};\xi) \text{ for } (\phi\chi)(t,\mathbf{x},\xi) \quad \text{and} \quad \hat{h}(\tau,\boldsymbol{\kappa};\xi) \text{ for } (\chi\,\partial_t\phi)(t,\mathbf{x},\xi) \text{ in } L^2,$$

where the frequencies $\boldsymbol{\kappa} = (\kappa_1,\cdots,\kappa_d)$ are discrete:

$$\kappa_i = n\tfrac{2\pi}{P_i}, \qquad n = 0, \pm1, \pm2, \cdots.$$

That is, for example, $\hat{g}(\tau,\boldsymbol{\kappa};\xi) = \frac{1}{|\mathbb{T}|}\int_{\mathbb{R}}\int_{\mathbb{T}}(\phi\chi)(t,\mathbf{x},\xi)\,e^{-i(\tau t+\boldsymbol{\kappa}\cdot\mathbf{x})}dtdx$

so that $\quad (\phi\chi)(t,\mathbf{x},\xi) = \frac{1}{2\pi}\sum_{\kappa}\int_{\mathbb{R}}\hat{g}(\tau,\boldsymbol{\kappa};\xi)\,e^{i(\tau t+\boldsymbol{\kappa}\cdot\mathbf{x})}d\tau.$

Ideas of Proof for the Decay—V

Step 5. Claim: $v(t, \mathbf{x}) \equiv 0$ a.e. for $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$.

For a time-truncation function $\phi(t), 0 \le \phi(t) \le 1$, so that $\phi\chi$ belongs to $L^2(\mathbb{R} \times \mathbb{T}_P \times \mathbb{R})$, we have

$$\partial_t(\phi\chi) + \mathbf{f}'(\xi) \cdot \nabla_x(\phi\chi) - \nabla_x \cdot \big(A(\xi)\nabla_x(\phi\chi)\big) = \chi\,\partial_t\phi \quad \text{in } \mathcal{D}'(\mathbb{R}^{d+2}).$$

Since $\phi\chi$ and $\chi\phi_t$ are periodic in $\mathbf{x}$, we take the global Fourier transform in $t \in \mathbb{R}$ and the local Fourier transform in $\mathbf{x} \in \mathbb{T}_P$ to obtain

$$\hat{g}(\tau, \boldsymbol{\kappa}; \xi) \text{ for } (\phi\chi)(t, \mathbf{x}, \xi) \quad \text{and} \quad \hat{h}(\tau, \boldsymbol{\kappa}; \xi) \text{ for } (\chi\,\partial_t\phi)(t, \mathbf{x}, \xi) \text{ in } L^2,$$

where the frequencies $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_d)$ are discrete:

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That is, for example, $\hat{g}(\tau, \boldsymbol{\kappa}; \xi) = \frac{1}{|\mathbb{T}|}\int_{\mathbb{R}}\int_{\mathbb{T}}(\phi\chi)(t, \mathbf{x}, \xi)\, e^{-i(\tau t + \boldsymbol{\kappa}\cdot\mathbf{x})}dtdx$

so that $\quad (\phi\chi)(t, \mathbf{x}, \xi) = \frac{1}{2\pi}\sum_{\kappa}\int_{\mathbb{R}}\hat{g}(\tau, \boldsymbol{\kappa}; \xi)\, e^{i(\tau t + \boldsymbol{\kappa}\cdot\mathbf{x})}d\tau.$

Taking the global Fourier transform in $t \in \mathbb{R}$ and the local Fourier transform in $\mathbf{x} \in \mathbb{T}$ in the kinetic equation, we obtain

$$\Big(i\big(\tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}\Big)\hat{g} = \hat{h}.$$

Step 5.–Conti: Following usual ideas from the kinetic averaging lemmas, we may introduce a free parameter $\lambda > 0$ (to be chosen later on) and write

$$\Big(\sqrt{\lambda} + i\big(\tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}\Big)\hat{g} = \hat{h} + \sqrt{\lambda}\hat{g}.$$

This leads to

$$\hat{g} = (\hat{h} + \sqrt{\lambda}\hat{g})\ \frac{1}{\sqrt{\lambda} + i\big(\tau + f'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}}.$$

Integrating in ξ and using the Cauchy-Schwarz inequality, we find

Ideas of Proof for the Decay—VI

Step 5.–Conti: Following usual ideas from the kinetic averaging lemmas, we may introduce a free parameter $\lambda > 0$ (to be chosen later on) and write

$$\Big(\sqrt{\lambda} + i\big(\tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}\Big)\hat{g} = \hat{h} + \sqrt{\lambda}\hat{g}.$$

This leads to

$$\hat{g} = (\hat{h} + \sqrt{\lambda}\hat{g})\ \frac{1}{\sqrt{\lambda} + i\big(\tau + f'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}}.$$

Integrating in ξ and using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}&|\widehat{\phi v}|^2(\tau,\boldsymbol{\kappa})\\ &\le 2\Big(\int_{\mathbb{R}} \hat{h}^2 d\xi + \lambda\int_{\mathbb{R}} \hat{g}^2 d\xi\Big)\ \int_{\mathbb{R}} \Big|\frac{1}{\sqrt{\lambda} + i\big(\tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}}\Big|^2 d\xi.\end{aligned}$$

The nonlinearity-diffusivity condition gives that, when $\boldsymbol{\kappa} \neq 0$,

Ideas of Proof for the Decay—VI

Step 5.–Conti: Following usual ideas from the kinetic averaging lemmas, we may introduce a free parameter $\lambda > 0$ (to be chosen later on) and write

$$\Big(\sqrt{\lambda} + i\big(\tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}\Big)\hat{g} = \hat{h} + \sqrt{\lambda}\hat{g}.$$

This leads to

$$\hat{g} = (\hat{h} + \sqrt{\lambda}\hat{g})\ \frac{1}{\sqrt{\lambda} + i\big(\tau + f'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}}.$$

Integrating in ξ and using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}&|\widehat{\phi v}|^2(\tau,\boldsymbol{\kappa})\\ &\le 2\Big(\int_{\mathbb{R}} \hat{h}^2 d\xi + \lambda\int_{\mathbb{R}} \hat{g}^2 d\xi\Big)\ \int_{\mathbb{R}} \Big|\frac{1}{\sqrt{\lambda} + i\big(\tau + \mathbf{f}'(\xi)\cdot\boldsymbol{\kappa}\big) + \boldsymbol{\kappa}^\top A(\xi)\boldsymbol{\kappa}}\Big|^2 d\xi.\end{aligned}$$

The nonlinearity-diffusivity condition gives that, when $\boldsymbol{\kappa} \neq 0$,

$$|\widehat{\phi v}|^2 \le C\frac{\omega_\delta(\lambda)}{\lambda}\int_{\mathbb{R}} \hat{h}^2 d\xi + C\omega_\delta(\lambda)\int_{\mathbb{R}} |\hat{g}|^2 d\xi \qquad \text{for any } \delta \in (0,\delta_0).$$

Step 5.–Conti: Notice that the frequencies $\boldsymbol{\kappa}$ are discrete and may include $\boldsymbol{\kappa} = 0$; When $\boldsymbol{\kappa} \neq 0$, there exists $\delta_0 > 0$ such that $|\boldsymbol{\kappa}| \geq \delta_0$. Since $v(t, \mathbf{x})$ has mean zero in $\mathbf{x}$ over $\mathbb{T}_P$, we have $\widehat{\phi v}(\tau, 0) = 0.$

Step 5.–Conti: Notice that the frequencies $\boldsymbol{\kappa}$ are discrete and may include $\boldsymbol{\kappa} = 0$; When $\boldsymbol{\kappa} \neq 0$, there exists $\delta_0 > 0$ such that $|\boldsymbol{\kappa}| \geq \delta_0$. Since $v(t, \mathbf{x})$ has mean zero in $\mathbf{x}$ over $\mathbb{T}_P$, we have $\widehat{\phi v}(\tau, 0) = 0$. Thus,

$$\sum_{\boldsymbol{\kappa}\neq 0} \int_{\mathbb{R}} |\widehat{\phi v}|^2 d\tau \leq C\frac{\omega_\delta(\lambda)}{\lambda} \sum_{\boldsymbol{\kappa}\neq 0} \int_{\mathbb{R}^2} \hat{h}^2 d\xi d\tau + C\omega_\delta(\lambda) \sum_{\boldsymbol{\kappa}\neq 0} \int_{\mathbb{R}^2} |\hat{g}|^2 d\xi d\tau$$

$$\leq C\frac{\omega_\delta(\lambda)}{\lambda} \int_{\mathbb{R}\times\mathbb{T}\times\mathbb{R}} (\chi\phi_t)^2 dt dx d\xi + C\omega_\delta(\lambda) \int_{\mathbb{R}\times\mathbb{T}\times\mathbb{R}} |\phi\chi|^2 dt dx d\xi.$$

Step 5.–Conti: Notice that the frequencies $\boldsymbol{\kappa}$ are discrete and may include $\boldsymbol{\kappa}=0$; When $\boldsymbol{\kappa}\neq 0$, there exists $\delta_0>0$ such that $|\boldsymbol{\kappa}|\geq\delta_0$. Since $v(t,\mathbf{x})$ has mean zero in $\mathbf{x}$ over $\mathbb{T}_P$, we have $\widehat{\phi v}(\tau,0)=0$. Thus,

$$\sum_{\boldsymbol{\kappa}\neq 0}\int_{\mathbb{R}}|\widehat{\phi v}|^2 d\tau \leq C\frac{\omega_\delta(\lambda)}{\lambda}\sum_{\boldsymbol{\kappa}\neq 0}\int_{\mathbb{R}^2}\hat{h}^2 d\xi d\tau + C\omega_\delta(\lambda)\sum_{\boldsymbol{\kappa}\neq 0}\int_{\mathbb{R}^2}|\hat{g}|^2 d\xi d\tau$$

$$\leq C\frac{\omega_\delta(\lambda)}{\lambda}\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{R}}(\chi\phi_t)^2 dtdxd\xi + C\omega_\delta(\lambda)\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{R}}|\phi\chi|^2 dtdxd\xi.$$

$\Longrightarrow$

$$\int_{\mathbb{R}\times\mathbb{T}_P}|\phi v|^2 dtd\mathbf{x} \leq C\frac{\omega_\delta(\lambda)}{\lambda}\int_{\mathbb{R}\times\mathbb{T}}|\phi_t|^2|v|dtd\mathbf{x}+C\omega_\delta(\lambda)\int_{\mathbb{R}\times\mathbb{T}_P}|\phi|^2|v|dtd\mathbf{x}.$$

Ideas of Proof for the Decay—VII

Step 5.–Conti: Notice that the frequencies $\boldsymbol{\kappa}$ are discrete and may include $\boldsymbol{\kappa}=0$; When $\boldsymbol{\kappa}\neq 0$, there exists $\delta_0>0$ such that $|\boldsymbol{\kappa}|\geq\delta_0$. Since $v(t,\mathbf{x})$ has mean zero in $\mathbf{x}$ over $\mathbb{T}_P$, we have $\widehat{\phi v}(\tau,0)=0$. Thus,

$$\sum_{\boldsymbol{\kappa}\neq 0}\int_{\mathbb{R}}|\widehat{\phi v}|^2d\tau \leq C\frac{\omega_\delta(\lambda)}{\lambda}\sum_{\boldsymbol{\kappa}\neq 0}\int_{\mathbb{R}^2}\hat{h}^2d\xi d\tau + C\omega_\delta(\lambda)\sum_{\boldsymbol{\kappa}\neq 0}\int_{\mathbb{R}^2}|\hat{g}|^2d\xi d\tau$$

$$\leq C\frac{\omega_\delta(\lambda)}{\lambda}\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{R}}(\chi\phi_t)^2dtdxd\xi + C\omega_\delta(\lambda)\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{R}}|\phi\chi|^2dtdxd\xi.$$

$\Longrightarrow$

$$\int_{\mathbb{R}\times\mathbb{T}_P}|\phi v|^2dtd\mathbf{x} \leq C\frac{\omega_\delta(\lambda)}{\lambda}\int_{\mathbb{R}\times\mathbb{T}}|\phi_t|^2|v|dtd\mathbf{x}+C\omega_\delta(\lambda)\int_{\mathbb{R}\times\mathbb{T}_P}|\phi|^2|v|dtd\mathbf{x}.$$

$\Longrightarrow$

$$\begin{aligned} I_\infty\int_{\mathbb{R}}|\phi|^2dt &\leq C\omega_\delta(\lambda)\Big(\int_{\mathbb{T}}|v|^2d\mathbf{x}\Big)^{1/2}\Big(\frac{1}{\lambda}\int_{\mathbb{R}}|\phi_t|^2dt+\int_{\mathbb{R}}|\phi|^2dt\Big)\\ &\leq C\sqrt{I_\infty}\,\omega_\delta(\lambda)\Big(\frac{1}{\lambda}\int_{\mathbb{R}}|\phi_t|^2dt+\int_{\mathbb{R}}|\phi|^2dt\Big) \qquad (*)\end{aligned}$$

Step 5.–Conti:

$$I_\infty \int_{\mathbb{R}} |\phi|^2 dt \le C\sqrt{I_\infty}\,\omega_\delta(\lambda)\Big(\frac{1}{\lambda}\int_{\mathbb{R}} |\phi_t|^2 dt + \int_{\mathbb{R}} |\phi|^2 dt\Big) \qquad (*)$$

Choosing first λ small and then $\int_{\mathbb{R}} |\phi_t|^2$ small, we conclude from (*) that $I_\infty = 0$, that is, $\quad v(t,x) \equiv 0 \quad$ a.e. $(t,x) \in \mathbb{R} \times \mathbb{R}^d$.

On the contrary, if $I_\infty > 0$,

Step 5.–Conti:

$$I_\infty \int_{\mathbb{R}} |\phi|^2 dt \le C\sqrt{I_\infty}\omega_\delta(\lambda)\Big(\frac{1}{\lambda}\int_{\mathbb{R}} |\phi_t|^2 dt + \int_{\mathbb{R}} |\phi|^2 dt\Big) \quad (*)$$

Choosing first λ small and then $\int_{\mathbb{R}} |\phi_t|^2$ small, we conclude from (*) that $I_\infty = 0$, that is, $v(t,x) \equiv 0$ a.e. $(t,x) \in \mathbb{R} \times \mathbb{R}^d$.

On the contrary, if $I_\infty > 0$, then we can choose λ small enough so that $C\omega_\delta(\lambda)/\sqrt{I_\infty} \le \frac{1}{2}$ and find from (*) that

$$\sqrt{I_\infty}\int_{\mathbb{R}} |\phi|^2 dt \le 2C\frac{\omega_\delta(\lambda)}{\lambda}\int_{\mathbb{R}} |\phi_t|^2 dt.$$

Step 5.–Conti:

$$I_\infty \int_{\mathbb{R}} |\phi|^2 dt \le C\sqrt{I_\infty}\,\omega_\delta(\lambda)\Big(\frac{1}{\lambda}\int_{\mathbb{R}} |\phi_t|^2 dt + \int_{\mathbb{R}} |\phi|^2 dt\Big) \quad (*)$$

Choosing first λ small and then $\int_{\mathbb{R}} |\phi_t|^2$ small, we conclude from (*) that $I_\infty = 0$, that is, $v(t,x) \equiv 0$ a.e. $(t,x) \in \mathbb{R} \times \mathbb{R}^d$.

On the contrary, if $I_\infty > 0$, then we can choose λ small enough so that $C\omega_\delta(\lambda)/\sqrt{I_\infty} \le \frac{1}{2}$ and find from (*) that

$$\sqrt{I_\infty}\int_{\mathbb{R}} |\phi|^2 dt \le 2C\frac{\omega_\delta(\lambda)}{\lambda}\int_{\mathbb{R}} |\phi_t|^2 dt.$$

It remains to choose a sequence of functions $\phi_B(t) = 1$ for $|t| \le B$, with B a given large number and $\phi_B'(t) = \frac{2B-|t|}{B}$ for $B \le |t| \le 2B$, and $\phi_B(t) = 0$ for $|t| \ge 2B$. In the above inequality, we find

$$\sqrt{I_\infty} \le C\frac{\omega_\delta(\lambda)}{\lambda}\frac{1}{B^2},$$

where $C > 0$ is a constant independent of B and λ. When B tends to ∞, this implies that I_∞ must vanish, which is a contradiction.

Further Results

- L^1-Error Estimates and Continuous Dependence
  Chen-Karlsen: Trans. Amer. Math. Soc. 2006
- More General Degenerate Diffusion-Convection-Reaction Equations

$$\partial_t u + \nabla \cdot \mathbf{f}(t, \mathbf{x}; u) = \nabla \cdot (\mathbf{A}(t, \mathbf{x}; u)\nabla u) + c(t, \mathbf{x}; u)$$

Chen-Karlsen: Comm. Pure Appl. Anal. 2005: Kinetic Equations:

$$\partial_t \chi(\xi; u) + \mathbf{f}_u(t, \mathbf{x}; \xi) \cdot \nabla \chi(\xi; u) - \nabla \cdot (\mathbf{A}(t, \mathbf{x}; \xi)\nabla \chi(\xi; u)) + (\textstyle\sum_j \mathbf{f}_{x_j}(t, \mathbf{x}; \xi) - c(t, \mathbf{x}; \xi))\partial_u \chi(\xi; u) = \partial_\xi (m + n)(t, \mathbf{x}; \xi).$$

- Other Related Notions and Regularity Results of Weak Solutions

  Bendahmane-Karlsen: Renormalized Solutions, SIMA 2004
  Perthame-Souganidis: Dissipative Solutions, SIMA 2005, 2006
  Tadmor-Terence Tao: Regularity of Solutions, CPAM 2008
- Initial-Boundary Value Problems
- ······

Outline

Bow Shock in Space generated by a Solar Explosion

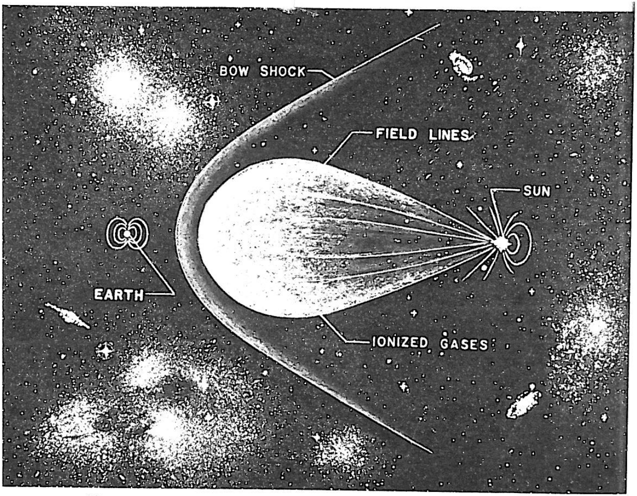


FIG. 50: SOLAR EXPLOSION

A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTIAS, after Gold, 1962).

Blast Wave from a TNT Surface Explosion

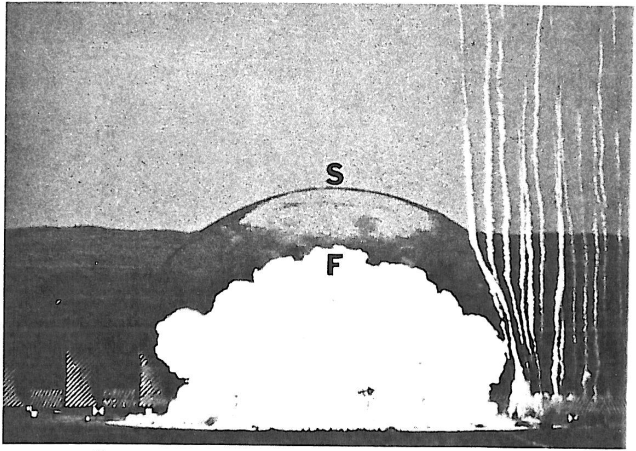
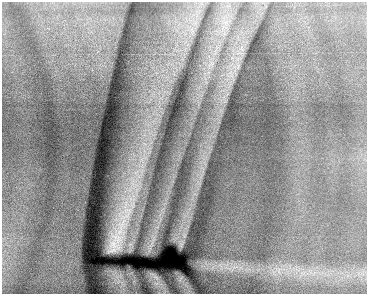


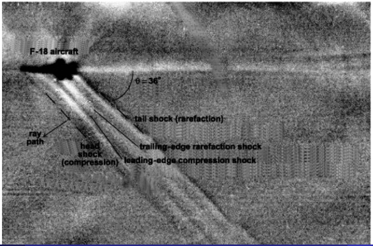
FIG. 22: EXPLOSION FROM A 20-TON HEMISPHERE OF TNT

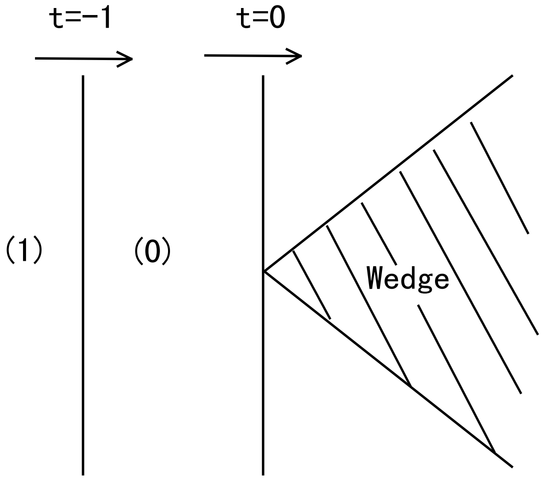
The blast wave S, and fireball F, from a 20-ton TNT surface explosion are clearly shown. The backdrops are 50 feet by 30 feet and in conjunction with the rocket smoke trails, it is possible to distinguish shock waves and particle paths and to measure their velocities. Owing to unusual daylight conditions, the hemispherical shock wave became visible. (Courtesy: Defence Research Board of Canada).

Shock Waves generated by Transonic Aircrafts

Shock Waves generated by Supersonic Aircrafts







? **Shock Wave Patterns** around a Wedge (airfoils, inclined ramps, $\cdots$)

Ernst Mach: 1878

Complexity of Reflection-Diffraction Configurations:

Über den verlauf von funkenwellenin der ebene und im raume, Sitzungsber. Akad. Wiss. Wien, **78** (1878), 819–838.

John von Neumann: 1943

1. **Oblique Reflection of Shocks**, Explos. Res. Rep. **12** (1943), Navy Dept., Bureau of Ordnance, Washington, DC., USA.
2. **Refraction, Intersection, and Reflection of Shock Waves**, NAVORD Rep. **203-45** (1945), Navy Dept., Bureau of Ordnance, Washington, DC, USA.
3. **Collected Works**, Vol. **6**, Pergamon Press, 1963.

Richard Courant and Kurt Otto Friedrichs: 1948

Supersonic Flow and Shock Waves,
Springer-Verlag: New York, 1948. xvi+464 pp.

Experimental Analysis: 1940s–

Walker Bleakney: Palmer Physical Laboratory
Princeton University, USA

Irvine Israel Glass: Institute for Aerospace Studies
University of Toronto, Canada

LeRoy Freame Henderson: School of Aerospace, Mechanical and
Mechatronic Engineering, University of Sydney, Australia

Tatiana V. Bazhenova: Joint Institute of High Temperatures
Russian Academy of Sciences, Moscow, Russia

Kazuyoshi Takayama: Institute of Fluid Science
Tohoku University, Japan

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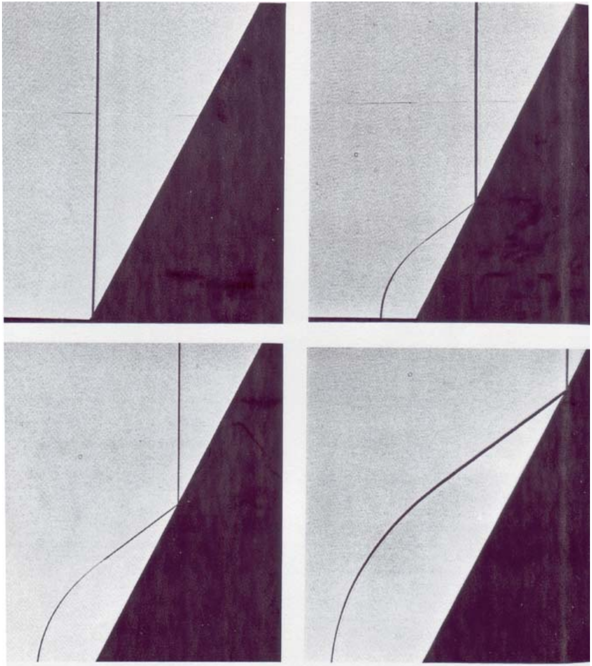
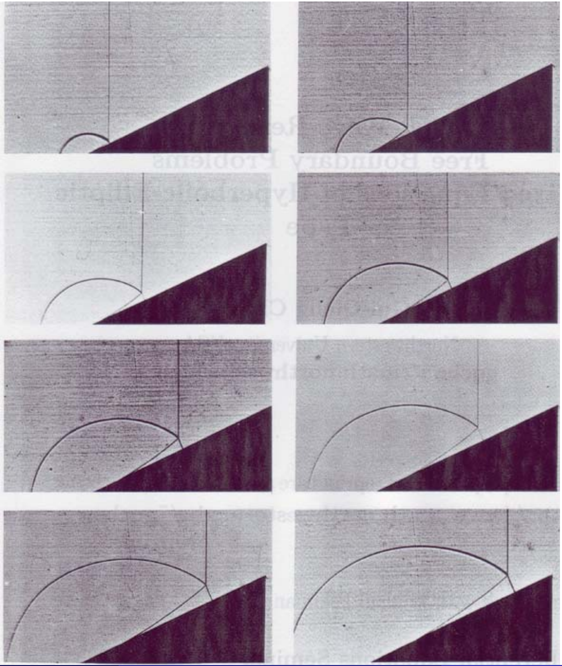
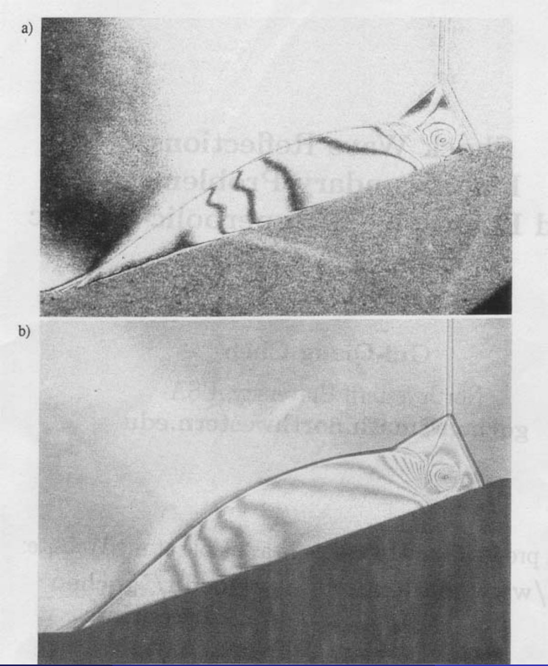
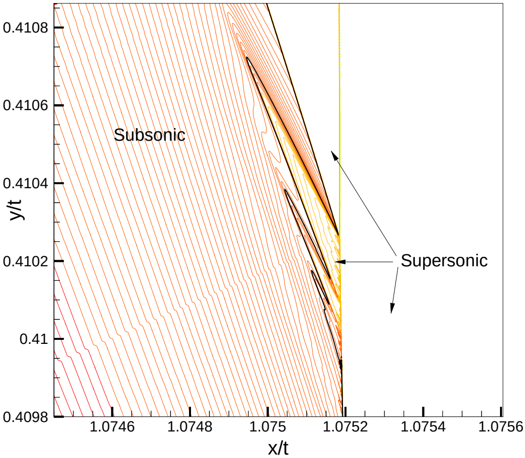





a)
b)



A New Mach Reflection-Diffraction Pattern:

A. M. Tesdall and J. K. Hunter: TSD, 2002

A. M. Tesdall, R. Sanders, and B. L. Keyfitz: NWE, 2006; Full Euler, 2008

B. Skews and J. Ashworth: J. Fluid Mech. 542 (2005), 105-114

Shock Reflection-Diffraction Patterns

- **Gabi Ben-Dor** Shock Wave Reflection Phenomena
  Springer-Verlag: New York, 307 pages, 1992.

  Experimental results before 1991
  Various proposals for transition criteria

- **Peter O. K. Krehl** History of Shock Waves, Explosions and Impact
  A Chronological and Biographical Reference
  2009, XXII, 1288 p. 1200 illus., 300 in color.

- **Milton Van Dyke** An Album of Fluid Motion
  The parabolic Press: Stanford, 176 pages, 1982.

  Various photographs of shock wave reflection phenomena

Scientific Issues

- **Structure of the Shock Reflection-Diffraction Patterns**
- **Transition Criteria among the Patterns**
- **Dependence of the Patterns on the Parameters**
  wedge angle θ_w, **adiabatic exponent $\gamma \geq 1$**
  incident-shock-wave Mach number M_s
- $\cdots\cdots$

Interdisciplinary Approaches:

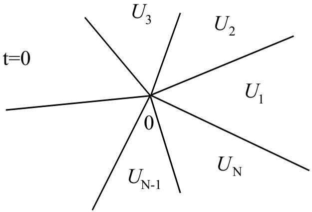
- **Experimental Data and Photographs**
- **Large or Small Scale Computing**
  Colella, Berger, Deschambault, Glass, Glaz, Woodward,....
  Anderson, Hindman, Kutler, Schneyer, Shankar, ...
  Yu. Dem'yanov, Panasenko, ...
- **Asymptotic Analysis**
  Lighthill, Keller, Majda, Hunter, Rosales, Tabak, Gamba, Harabetian...
  Morawetz: CPAM 1994
- **Rigorous Mathematical Analysis??** **(Global Solutions)**
  Existence, Stability, Regularity, Bifurcation, $\cdots\cdots$

2-D Riemann Problem for Hyperbolic Conservation Laws

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

or

$$\partial_t \mathbf{A}(U, U_t, \nabla_{\mathbf{x}} U) + \nabla_{\mathbf{x}} \cdot \mathbf{B}(U, U_t, \nabla_{\mathbf{x}} U) = 0$$



Books and Survey Articles:
Chang-Hsiao 1989, Glimm-Majda 1991, Li-Zhang-Yang 1998, Zheng 2001
Chen-Wang 2002, Serre 2005, Chen 2005, Dafermos 2010, $\cdots$

Numerical Solutions: Glimm-Klingenberg-McBryan-Plohr-Sharp-Yaniv 1985
Lax-Liu 1998, Schulz-Rinne-Collins-Glaz 1993, Chang-Chen-Yang 1995, 2000
Kurganov-Tadmor 2002, $\cdots$

Theoretical Roles: Asymptotic States and Attractors
Local Structure and Building Blocks...

Riemann Solutions I

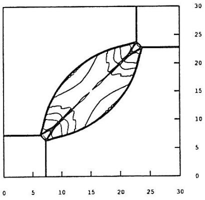


FIG. 5.5A
Density contour curves

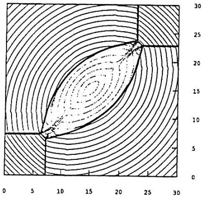


FIG. 5.5B
Self-Mach number contour curves

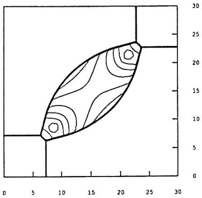


FIG. 5.5C. Pressure contour curves

Riemann Solutions II

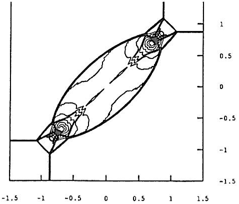


FIG. 5.6A
Density contour curves

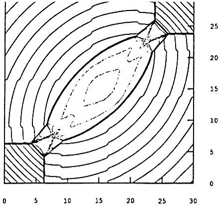


FIG. 5.6B
Self-Mach number contour curves

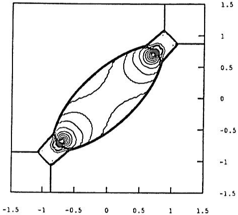


FIG. 5.6C. Pressure contour curves

Asymptotic States and Attractors

Observation (C–Frid 1998):

- Let $R(\frac{\mathbf{x}}{t})$ be the unique piecewise Lipschitz continuous Riemann solution with Riemann data: $R|_{t=0} = R_0(\frac{\mathbf{x}}{|\mathbf{x}|})$
- Let $U(t, \mathbf{x})$ be a bounded entropy solution with initial data:

$$U|_{t=0} = R_0(\frac{\mathbf{x}}{|\mathbf{x}|}) + P_0(\mathbf{x}), \qquad R_0 \in L^\infty(S^{d-1}), P_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$$

- The corresponding self-similar sequence $U^T(t, \mathbf{x}) := U(Tt, T\mathbf{x})$ is compact in $L^1_{loc}(\mathbb{R}^{d+1}_+)$

$$\Longrightarrow \quad \text{ess} \lim_{t\to\infty} \int_\Omega |U(t, t\xi) - R(\xi)|\, d\xi = 0 \qquad \text{for any } \Omega \subset \mathbb{R}^d$$

Building Blocks and Local Structure

* Local structure of entropy solutions

* Building blocks for numerical methods

Full Euler Equations: $(t, \mathbf{x}) \in \mathbb{R}^3_+ := (0, \infty) \times \mathbb{R}^2$

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0 & \text{(conservation of mass)} \\ \partial_t (\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0 & \text{(conservation of momentum)} \\ \partial_t (\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot \big((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v} \big) = 0 & \text{(conservation of energy)} \end{cases}$$

Constitutive Relations: $p = p(\rho, e)$

- ρ–density, $\mathbf{v} = (v_1, v_2)^\top$–fluid velocity, p–pressure
- e–internal energy, θ–temperature, S–entropy

For a polytropic gas: $p = (\gamma - 1)\rho e, \;\; e = c_v \theta, \qquad \gamma = 1 + \frac{R}{c_v} > 1$

$$p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \qquad e = e(\rho, S) \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v},$$

- $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas
- $c_v > 0$ is the specific heat at constant volume
- $\gamma > 1$ is the adiabatic exponent, $\kappa > 0$ is any constant under scaling

Euler Equations for Potential Flow: $\mathbf{v} = \nabla\Phi$

$$\begin{cases} \partial_t\rho + \nabla_{\mathbf{x}} \cdot (\rho\nabla_{\mathbf{x}}\Phi) = 0, & \text{(conservation of mass)} \\ \partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}, & \text{(Bernoulli's law)} \end{cases}$$

or, equivalently,

$$\partial_t\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0)\nabla_{\mathbf{x}}\Phi\big) = 0,$$

with

$$\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2)\big)^{\frac{1}{\gamma-1}}.$$

- The potential flow equations and the full Euler equations coincide in important regions of the solution and are very close each other in the other regions in the configuration of regular shock reflection-diffraction.
- Aerodynamics/Gas Dynamics: Fundamental PDE
- J. Hadamard: Leçons sur la Propagation des Ondes, Hermann: Paris 1903, ...

Discontinuities of Solutions: Entropy Solutions

$$\partial_t U + \nabla_{\mathbf{x}} \cdot F(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

An oriented surface Γ with unit normal $\mathbf{n} = (n_t, n_1, n_2) \in \mathbb{R}^3$ in the $(t, \mathbf{x})$-space is a discontinuity of a piecewise smooth entropy solution U with

$$U(t, \mathbf{x}) = \begin{cases} U_+(t, \mathbf{x}), & (t, \mathbf{x}) \cdot \mathbf{n} > 0, \\ U_-(t, \mathbf{x}), & (t, \mathbf{x}) \cdot \mathbf{n} < 0, \end{cases}$$

if the Rankine-Hugoniot Condition is satisfied

$$(U_+ - U_-, F(U_+) - F(U_-)) \cdot \mathbf{n} = 0 \qquad \text{along } \Gamma.$$

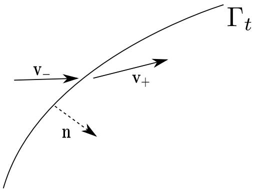
The surface $(\Gamma, \mathbf{n})$ is called a Shock Wave if the Entropy Condition (i.e., the Second Law of Thermodynamics) is satisfied:

$$(\eta(U_+) - \eta(U_-), q(U_+) - q(U_-)) \cdot \mathbf{n} \geq 0 \qquad \text{along } \Gamma,$$

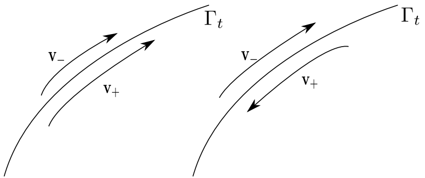
where $(\eta(U), q(U)) = (-\rho S, -\rho \mathbf{v} S)$.

Two Types of Discontinuities

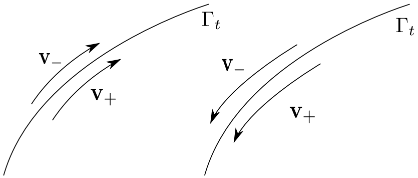
Shock Waves:



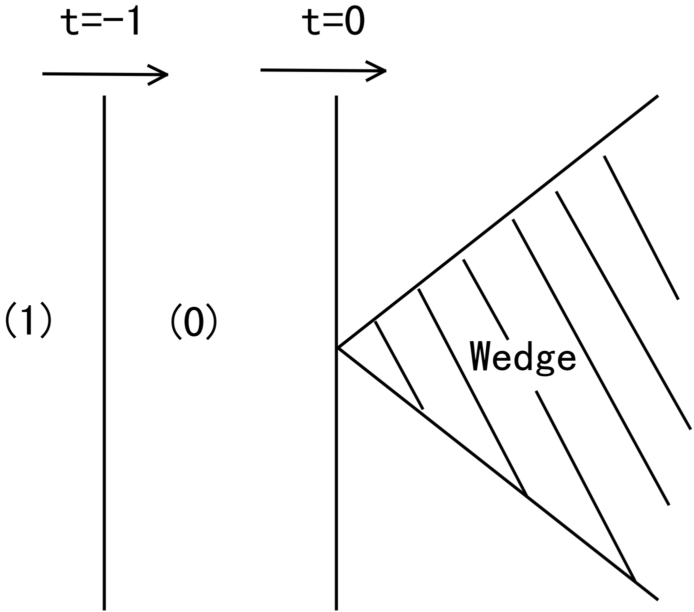
Characteristic Discontinuities: Vortex Sheets/Entropy Waves



(i) $(p_+, \rho_+) = (p_-, \rho_-)$, $\mathbf{v}_+ \neq \mathbf{v}_-$



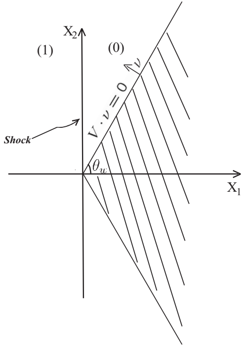
(ii) $(p_+, \mathbf{v}_+) = (p_-, \mathbf{v}_-)$, $\rho_+ \neq \rho_-$


t=-1
t=0
(1)
(0)
Wedge

Initial condition **at** $t = 0$:

$$(\mathbf{v}, p, \rho) = \begin{cases} (0, 0, p_0, \rho_0), & |x_2| > x_1 \tan\theta_w, x_1 > 0, \\ (u_1, 0, p_1, \rho_1), & x_1 < 0; \end{cases}$$

Slip boundary condition **on the wedge bdry:** $\mathbf{v} \cdot \boldsymbol{\nu} = 0.$



Invariant under the Self-Similar Scaling: $(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha \mathbf{x}), \ \ \alpha \neq 0$

Self-Similar Solutions for the Full Euler Equations

$(\mathbf{v}, p, \rho)(t, \mathbf{x}) = (\mathbf{v}, p, \rho)(\xi, \eta), \ \ (\xi, \eta) = (\frac{x_1}{t}, \frac{x_2}{t})$

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p)_\xi + (\rho U V)_\eta + 3\rho U = 0, \\ (\rho U V)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\ (U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0, \end{cases}$$

where $q = \sqrt{U^2 + V^2}$ and $(U, V) = (v_1 - \xi, v_2 - \eta)$ is the pseudo-velocity.

Eigenvalues: $\lambda_0 = \frac{V}{U}$(repeated), $\lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}$,

where $c = \sqrt{\gamma p / \rho}$ is the **sonic speed**

When the flow is pseudo-subsonic: $q < c$**,** the system consists of

- 2-transport equations: Compressible vortex sheets
- 2-nonlinear equations of mixed hyperbolic-elliptic type: Two kinds of transonic flow: Transonic shocks and sonic curves

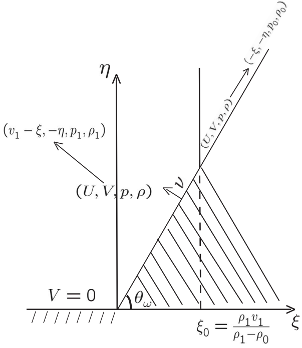
Boundary Value Problem in the Unbounded Domain

Slip boundary condition on the wedge boundary:

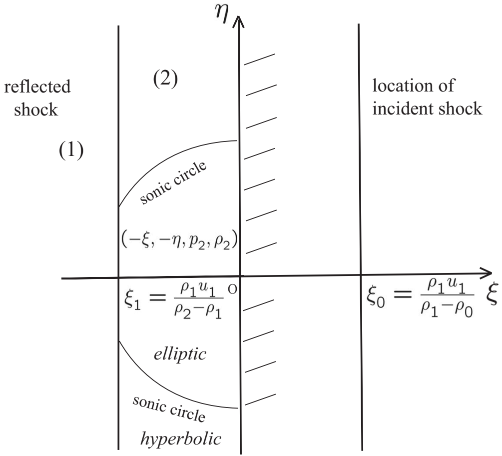
$$(U, V) \cdot \nu = 0 \qquad \textbf{on} \quad \partial D$$

Asymptotic boundary condition as $\xi^2 + \eta^2 \to \infty$:

$$(U + \xi, V + \eta, p, \rho) \to \begin{cases} (0, 0, p_0, \rho_0), & \xi > \xi_0, \eta > \xi \tan \theta_w, \\ (u_1, 0, p_1, \rho_1), & \xi < \xi_0, \eta > 0. \end{cases}$$



When $\theta_w = \frac{\pi}{2}$, the reflection becomes the normal reflection, for which the incident shock normally reflects and the reflected shock is also a plane.



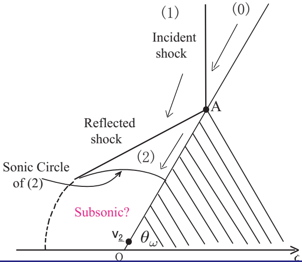
Local Theory for Regular Reflection (cf. Chang-C. 1986)

$\exists\, \theta_d = \theta_d(M_s, \gamma) \in (0, \frac{\pi}{2})$ such that, when $\theta_W \in (\theta_d, \frac{\pi}{2})$, there exist two states $(2) = (U_2^a, V_2^a, p_2^a, \rho_2^a)$ and $(U_2^b, V_2^b, p_2^b, \rho_2^b)$ such that $|(U_2^a, V_2^a)| > |(U_2^b, V_2^b)|$ and $|(U_2^b, V_2^b)| < c_2^b$.

Stability as $\theta_W \to \frac{\pi}{2}$ (C-Feldman 2005): Choose $(2) = (U_2^a, V_2^a, p_2^a, \rho_2^a)$

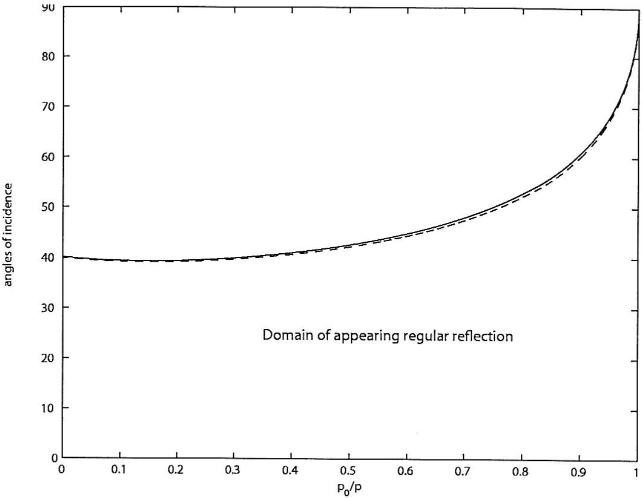
Sonic Conjecture: There exists a Regular Reflection Configuration when $\theta_W \in (\theta_s, \frac{\pi}{2})$, for $\theta_s > \theta_d$ such that $|(U_2^a, V_2^a)| > c_2^a$ at A.

Detachment Conjecture: There exists a Regular Reflection Configuration when the wedge angle $\theta_W \in (\theta_d, \frac{\pi}{2})$.

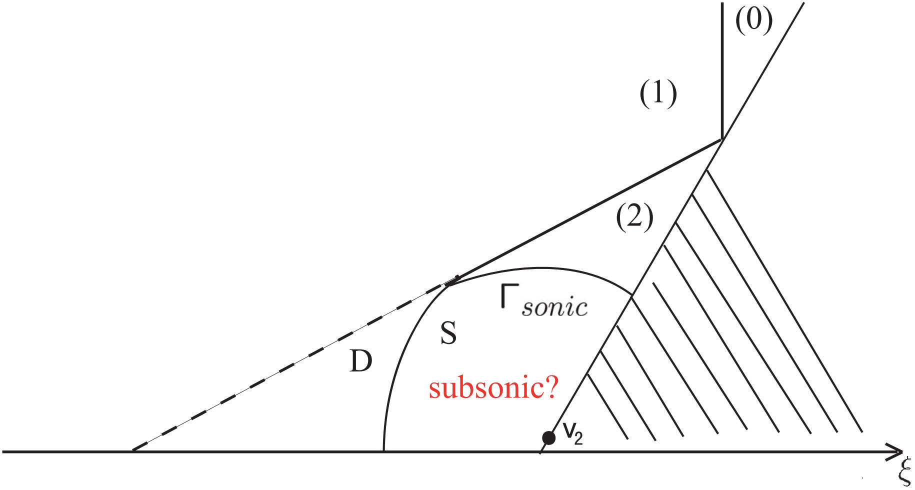


Detachment Criterion vs Sonic Criterion $\theta_c > \theta_s$: $\gamma = 1.4$

Courtesy of W. Sheng and G. Yin: ZAMP, 2008



Global Theory?



$$
\begin{cases}
\nabla\cdot(\rho\nabla\varphi)+2\rho+\nabla\cdot(\rho\nabla W)=0,\\
\nabla(\frac{1}{2}|\nabla\varphi|^2+\varphi)+\frac{1}{\rho}\nabla p=(\nabla\varphi+W)\cdot\nabla W+(\nabla^2\varphi+I)W,\\
\nabla\cdot((\nabla\varphi+W)\omega)+\omega+\nabla\times(\frac{1}{\rho}\nabla p)=0,\\
(\nabla\varphi+W)\cdot\nabla S=0.
\end{cases}
$$

$S=c_v\ln(p\rho^{-\gamma})$–**Entropy**; $\omega=\mathrm{curl}\,W=\mathrm{curl}(U,V)$–**Vorticity**

When $S=const.$, $W=0$, and $\omega=0$ on a curve transverse to the fluid direction, then, in the region of the fluid trajectories past the curve,

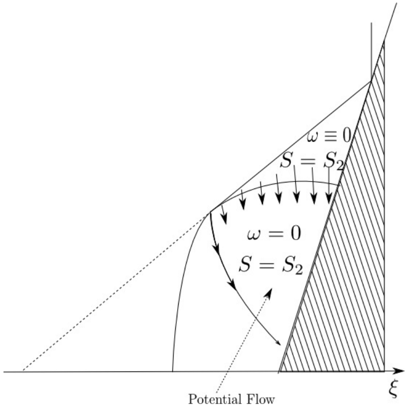
$$W=0,\ S=const. \quad\Rightarrow\quad W=0,\ p=const.\,\rho^{\gamma}$$

Then we obtain the Potential Flow Equation (by scaling):

$$
\begin{cases}
\nabla\cdot(\rho\nabla\varphi)+2\rho=0,\\
\frac{1}{2}(|\nabla\varphi|^2+\varphi)+\frac{\rho^{\gamma-1}}{\gamma-1}=const.>0.
\end{cases}
$$

Potential Flow Dominates the Regular Reflection, provided that $\varphi \in C^{1,1}$ across the Sonic Circle

Potential Flow Equation :
$$\begin{cases} \nabla\cdot(\rho\nabla\varphi)+2\rho=0, \\ \frac{1}{2}|\nabla\varphi|^2+\varphi+\frac{\rho^{\gamma-1}}{\gamma-1}=\frac{\rho_0^{\gamma-1}}{\gamma-1}, \qquad \gamma>1 \end{cases}$$



Potential Flow Equation

$$\nabla \cdot (\rho(\nabla\varphi, \varphi, \rho_0)\nabla\varphi) + 2\rho(\nabla\varphi, \varphi, \rho_0) = 0$$

- Incompressible: $\rho = const. \Longrightarrow \Delta\varphi + 2 = 0$
- Elliptic: $|\nabla\varphi| < c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$
- Hyperbolic: $|\nabla\varphi| > c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$

Second-order nonlinear equations of mixed hyperbolic-elliptic type

Simpler Nonlinear Models and Equations of Mixed Type

- **Transonic Small Disturbance Equation**:

$$\left((u-x)u_x + \frac{u}{2}\right)_x + u_{yy} = 0$$

or, for $v = u - x$, $\quad v\, v_{xx} + v_{yy} + \text{l.o.t.} = 0$

Analysis: Morawetz, Hunter, Canic-Keyfitz-Lieberman-Kim, $\cdots$

- **Pressure-Gradient Equations, Nonlinear Wave Equations**

Y. Zheng, Canic-Keyfitz-Kim-Jegdic, C-Deng-Xiang (2011), $\cdots$

- **Steady Potential Flow Equation of Aerodynamics**

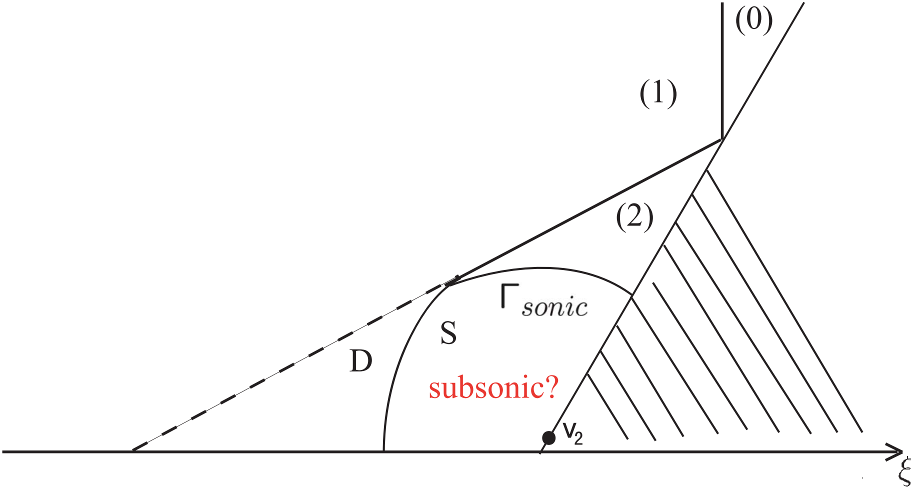
$$\nabla \cdot (\rho(\nabla\varphi, \rho_0)\nabla\varphi) = 0$$

- **Elliptic:** $|\nabla\varphi| < c_*(\rho_0) := \sqrt{\frac{2}{\gamma+1}\rho_0^{\gamma-1}}$
- **Hyperbolic:** $|\nabla\varphi| > c_*(\rho_0)$

$$\nabla \cdot (\rho(\nabla\varphi, \rho_0)\nabla\varphi) = 0$$

- **Pure Elliptic Case: Subsonic Flow past an Obstacle**
  Shiffman, L. Bers, Finn-Gilbarg, G. Dong, $\cdots$
- **Degenerate Elliptic Case: Subsonic-Sonic Flows**
  Shiffman, Chen-Dafermos-Slemrod-Wang, Elling-Liu, Xin, $\cdots$
- **Pure Hyperbolic Case (even 2-D Full Euler Eqs.)**:
  Gu, Li, Schaeffer, S. Chen, Xin-Yin, Y. Zheng, $\cdots$
  T.-P. Liu-Lien, S. Chen-Zhang-Wang, Chen-Zhang-Zhu, $\cdots$
- **Elliptic-Hyperbolic Mixed Case**
  Transonic Nozzles: Chen-Feldman, S. Chen, Xin-Yin, J. Chen, Yuan...
  Wedge or Conical Body: S. Chen, B. Fang, Chen-Fang, $\cdots$
  Transonic Flow past an Obstacle: Morawetz, Chen-Slemrod-Wang,...

Global Theory?



Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- div $(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2))) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \boldsymbol{\nu}|_{wedge} = 0$
- $\psi|_{\Gamma_{sonic}} = 0 \Longrightarrow \psi_{\boldsymbol{\nu}}|_{\Gamma_{sonic}} = 0$
- Rankine-Hugoniot Conditions on Shock S:

$$[\psi]_S = 0$$

$$[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2)) \cdot \boldsymbol{\nu}]_S = 0 \qquad (**)$$

Free Boundary Problem:

- div $(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2))) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \boldsymbol{\nu}|_{wedge} = 0$
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Free Boundary Problem:

- $\exists\, S = \{\xi = f(\eta)\}$ such that $f \in C^{1,\alpha}, f'(0) = 0$ and

$$\Omega_+ = \{\xi > f(\eta)\} \cap D = \{\psi < \varphi_1 - \varphi_2\} \cap D,$$

$$S = \{\psi = \varphi_1 - \varphi_2\} \cap D$$ (free boundary as a level set)

Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

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- $\psi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+)$ $\begin{cases} \text{solves (*) in } \Omega_+, \\ \text{is subsonic in } \Omega_+ \end{cases}$

  with $(\psi, \psi_{\boldsymbol{\nu}})|_{\Gamma_{sonic}} = 0, \quad \nabla\psi \cdot \boldsymbol{\nu}|_{wedge} = 0$

Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- div $(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2))) + l.o.t. = 0 \qquad (*)$
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- $\psi|_{\Gamma_{sonic}} = 0 \Longrightarrow \psi_{\boldsymbol{\nu}}|_{\Gamma_{sonic}} = 0$
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$$[\psi]_S = 0$$

$$[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2)) \cdot \boldsymbol{\nu}]_S = 0 \qquad (**)$$

Free Boundary Problem:

- $\exists\, S = \{\xi = f(\eta)\}$ such that $f \in C^{1,\alpha}$, $f'(0) = 0$ and

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- $\psi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+)$ $\begin{cases} \text{solves (*) in } \Omega_+, \\ \text{is subsonic in } \Omega_+ \end{cases}$

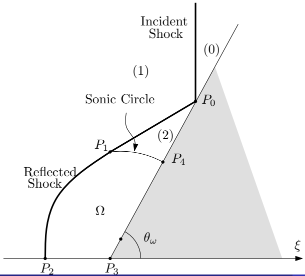
  with $(\psi, \psi_{\boldsymbol{\nu}})|_{\Gamma_{sonic}} = 0$, $\quad \nabla\psi \cdot \boldsymbol{\nu}|_{wedge} = 0$
- (ψ, f) satisfy the R-H Condition: Free Boundary Condition (**)

Theorem (Global Theory for Shock Reflection-Diffraction C.–Feldman: PNAS 2005; Annals of Math. 2010)

$\exists\ \theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ *such that, when* $\theta_W \in (\theta_c, \frac{\pi}{2})$, *there exist* (φ, f) *satisfying*

- $\varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ *and* $f \in C^\infty(P_1P_2) \cap C^2(\{\overline{P_1P_2}\})$;
- $\varphi \in C^{1,1}$ *across the sonic circle* P_1P_4;
- $\varphi \longrightarrow \varphi_{NR}$ *in* $W^{1,1}_{loc}$ *as* $\theta_W \to \frac{\pi}{2}$.

$\Rightarrow \Phi(t, \mathbf{x}) = t\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t}$, $\rho(t, \mathbf{x}) = \left(\rho_0^{\gamma-1} - (\gamma-1)(\Phi_t + \frac{1}{2}|\nabla\Phi|^2)\right)^{\frac{1}{\gamma-1}}$ *form a solution of the IBVP.*



Approach for the Large Wedge-Angle Case

- **Cutoff Techniques by Shiffmanization**

  $\Rightarrow$ Elliptic Free-Boundary Problem with Elliptic Degeneracy on $\mathbf{\Gamma}_{sonic}$

- **Iteration Scheme for the Free Boundary Problem**

  Chen-Feldman: J. Amer. Math. Soc. 2003

- **Domain Decomposition**

  Near $\mathbf{\Gamma}_{sonic}$; Away from $\mathbf{\Gamma}_{sonic}$

- $C^{1,1}$ **Parabolic Estimates near the Degenerate Elliptic Curve** $\mathbf{\Gamma}_{sonic}$

- **Corner Singularity Estimates**

  In particular, when the Elliptic Degenerate Curve $\mathbf{\Gamma}_{sonic}$ Meets the Free Boundary, i.e., the Transonic Shock

- **Removal of the Cutoff**

  Require the Elliptic-Parabolic Estimates
  with respect to the Large Wedge-Angle

Near Γ_{sonic} away from P_1: Mixed Elliptic-Hyperbolic Type

- **Linear:** $2x\psi_{xx} + \frac{1}{c_2^2}\psi_{yy} - \psi_x \sim 0$

$$\psi \sim Ax^{3/2} + h.o.t. \qquad \text{when } x \sim 0$$

- **Nonlinear:** $(2x - (\gamma+1)\psi_x)\,\psi_{xx} + \frac{1}{c_2^2}\psi_{yy} - \psi_x \sim o(x^2)$

Ellipticity: $\psi_x \le \frac{2x}{\gamma+1}$

Apriori Estimate: $|\psi_x| \le \frac{4x}{3(\gamma+1)}$

More precisely, for $\Omega' = \Omega \cap \{x < \epsilon\}$ with small $\varepsilon > 0$,

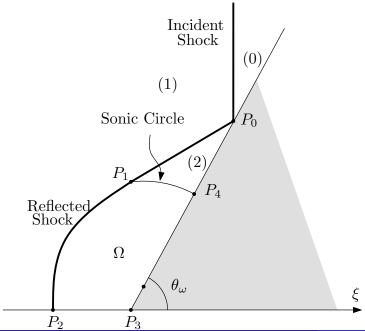
$$\sum_{0\le k+l\le 2} \sup_{z\in\Omega'} \left(x^{k+l/2-2} |\partial_x^k \partial_y^l \psi(z)| \right)$$

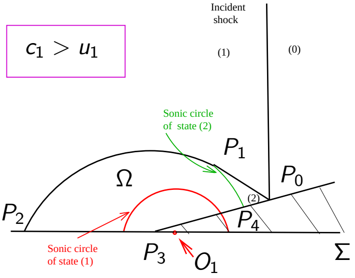
$$+ \sum_{k+l=2} \sup_{z,\tilde{z}\in\Omega', z\neq\tilde{z}} \left(\min(x,\tilde{x})^{\alpha-l/2} \frac{|\partial_x^k\partial_y^l\psi(z) - \partial_x^k\partial_y^l\psi(\tilde{z})|}{\delta_\alpha^{(par)}(z,\tilde{z})} \right) \le C.$$

Asymptotics: $\psi \quad \text{``}\sim\text{''} \quad \frac{x^2}{2(\gamma+1)} + h.o.t. \qquad \text{when } x \approx 0$

Theorem (Optimal Regularity; Bae-Chen-Feldman: Invent. Math. 2009):
$\varphi \in C^{1,1}$ but NOT in C^2 across P_1P_4;
$\varphi \in C^\infty(\Omega \backslash (\overline{P_1P_4} \cup \{P_3\})) \cap C^{2,\alpha}(\overline{\Omega} \backslash \{P_1, P_3\}) \cap C^{1,1}(\overline{\Omega} \backslash \{P_3\}) \cup C^{1,\alpha}(\overline{\Omega})$
$f \in C^\infty(P_1P_2) \cap C^2(\overline{P_1P_2})$.

$\Longrightarrow$ **C-Feldman 2011**: The global existence and the optimal regularity hold up to the sonic wedge-angle θ_s for any $\gamma \geq 1$ for $u_1 < c_1$; $u_1 \geq c_1$. (the von Neumann's sonic conjecture)





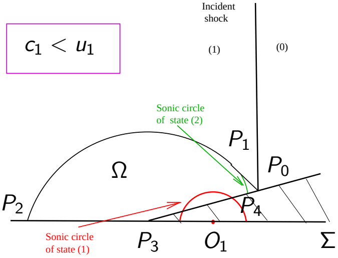
Issues: As the wedge angle becomes smaller, prove the shock does not hit

(i) Wedge boundary,

(ii) Symmetry line Σ,

(iii) Sonic circle $\partial B_{c_1}(O_1)$ of state (1), where $O_1 = (u_1, 0)$,

(iv) Vertex point P_3.



Issues: As the wedge angle becomes smaller, prove the shock does not hit

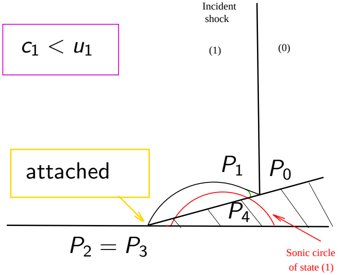
(i) Wedge boundary,

(ii) Symmetry line Σ,

(iii) Sonic circle $\partial B_{c_1}(O_1)$ of state (1), where $O_1 = (u_1, 0)$,

(iv) Vertex point P_3. This is unclear in the case $c_1 < u_1$.

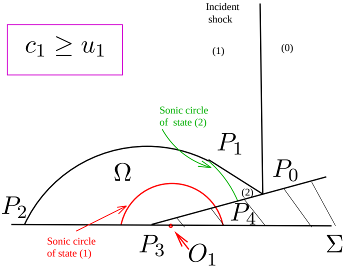
Existence for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$



Is attached case possible for regular reflection?

For irregular Mach reflection attached case appears to be possible, see Fig. 238 (page 144) of
M. Van Dyke, *An Album of Fluid Motion*,
The Parabolic Press: Stanford, 1982.

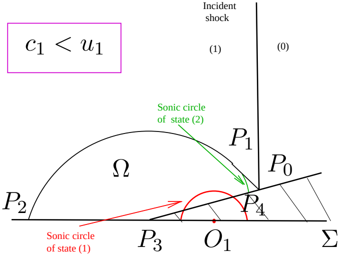
$c_1 \geq u_1$



Theorem (C-Feldman). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 \leq c_1$, then a regular reflection solution φ as our Theorem (2005) exists for all wedge angles $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$.

The solution satisfies all properties stated in our Theorem (2005). In particular, φ is $C^{1,1}$ near and across the sonic arc P_1P_4, and the shock is a C^2 curve, and $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω.

Existence for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$

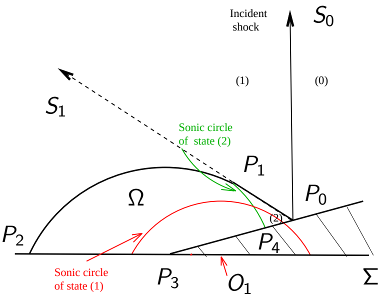


Theorem (C-Feldman). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 > c_1$, then a regular reflection solution φ as in our Theorem (2005) exists for all wedge angles $\theta_w \in (\theta_c, \frac{\pi}{2})$, where

-either $\theta_c = \theta_{sonic}$,

-or $\theta_c > \theta_{sonic}$ and for $\theta_w = \theta_c$ there exists an attached weak solution of regular reflection-diffraction problem.

Large Angle $\Longrightarrow$ Sonic Angle θ_{sonic}: Admissible Solutions



The solution φ is called an admissible solution if

1. $\varphi \in C^1(P_0P_1P_2P_3P_4)$, and $P_0P_1P_2$ is C^1 curve,
2. Equation is (strictly) elliptic in $\overline{\Omega} \setminus \overline{P_1P_4}$.
3. $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω.
4. $\varphi_1 - \varphi$ in Ω monotonically non-increases in directions S_0 and S_1.

Large Wedge Angle $\Longrightarrow$ Sonic Angle θ_{sonic}

- **Class of Admissible Solutions including the Global Solutions Constructed for the Large Wedge-Angle Case**
- **Apriori Estimates**

  Separation of the Diffracted Shock from the Wedge, the Symmetric Line, the Sonic Circle

  Boundedness of the Diffracted Shock

  Compactness

  ......
- ......
- **Continuity Method/Degree Theory**

$\Longrightarrow$ **Existence of Admissible Solutions for the Wedge Angle Up to the Sonic Angle or the Attached Angle**

$\Longrightarrow$ **von Neumann's Sonic Conjecture: Chen-Feldman 2011**

$\Longrightarrow$ **von Neumann's Detachment Conjecture: C-Feldman, Nov.2012**

Large Angles $\Longrightarrow$ Sonic Angle θ_{sonic}
Approach: Apriori Estimates and Compactness

(a) Establish the strict inequalities in (iii) and the strict monotonicities in (iv) (thus $\varphi_1 - \varphi$ strictly decreases for a cone of directions, thus the shocks are Lipschitz graphs with uniform Lip estimates)

(b) Establish uniform bounds on diam(Ω), $\|\varphi\|_{C^{0,1}(\Omega)}$, the monotonicity of $\varphi - \varphi_2$ near the sonic arc;

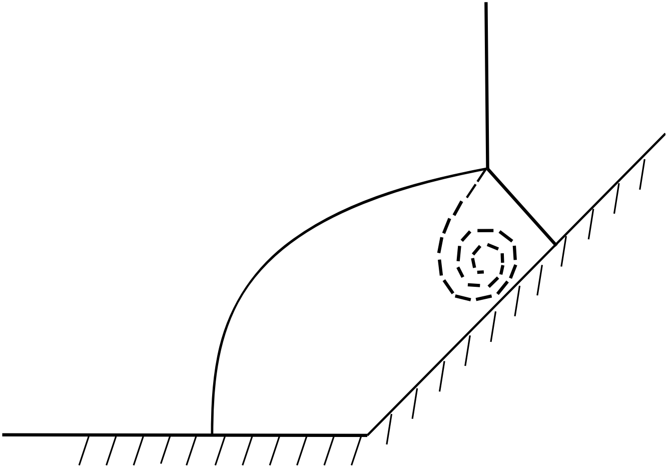
(c) Establish a uniform positive lower bound for the distance from the shock to the wedge, the sonic circle of state (1), and the uniform separation of the shock and the symmetry line;

(d) Make uniform regularity estimates for the solution and its shock in weighted/scaled Hölder norms (including near the sonic arc, which imply C^1 across the sonic arc);

(e) Prove that the uniform limit of admissible solutions is an admissible solution, and the uniform limit of the sequence of shocks is a shock.

Continuity Method/Degree Theory$\Longrightarrow$Existence of Admissible Solutions for Large Wedge-Angle: $\Longrightarrow$ von Neumann's Sonic Conjecture

Mach Reflection: Full Euler Equations



? Right space for vorticity ω?

? Chord-arc $z(s) = z_0 + \int_0^s e^{ib(s)} ds$, $b \in BMO$?

Self-Similar Solutions for the Full Euler Equations

$(\mathbf{v}, p, \rho)(t, \mathbf{x}) = (\mathbf{v}, p, \rho)(\xi, \eta), \ \ (\xi, \eta) = (\frac{x_1}{t}, \frac{x_2}{t})$

$$
\begin{cases}
(\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\
(\rho U^2 + p)_\xi + (\rho U V)_\eta + 3\rho U = 0, \\
(\rho U V)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\
(U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0,
\end{cases}
$$

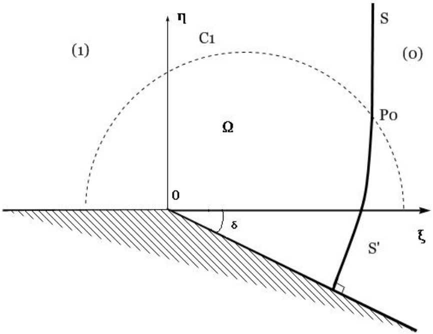
where $q = \sqrt{U^2 + V^2}$ and $(U, V) = (v_1 - \xi, v_2 - \eta)$ is the pseudo-velocity.

Eigenvalues: $\lambda_0 = \frac{V}{U}$(repeated), $\lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}$,
where $c = \sqrt{\gamma p / \rho}$ is the **sonic speed**

When the flow is pseudo-subsonic: $q < c$**,** the system consists of

- 2-transport equations: Compressible vortex sheets
- 2-nonlinear equations of mixed hyperbolic-elliptic type: Two kinds of transonic flow: Transonic shocks and sonic curves

Further Problem 1: Shock Diffraction by the Wedge Corner



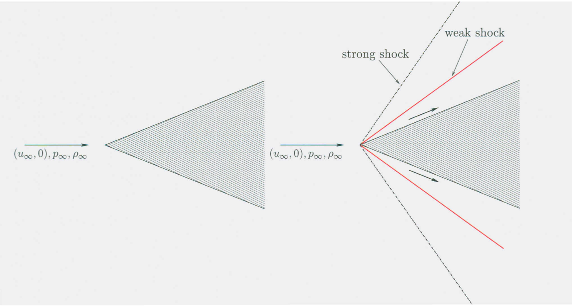
Experimental/Asymptotic Results: Bargman (1945), Lighthill (1949), Fletcher (1951), $\cdots$

Rigorous Results: Nonlinear Wave System: Chen-Deng-Xiang 2011

Potential Flow Equation: Chen-Xiang 2012

Further Problem 2: Supersonic Flow onto a Solid Wedge

? Two Steady Solutions with Shocks around the Solid Wedge

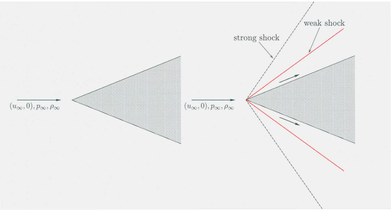


Courant-Friedrichs (1948): "The question arises which of the two actually occurs. It has frequently been stated that the strong one is unstable and that, therefore, only the weak one could occur.
A convincing proof of this instability has apparently never been given".

von Neumann's celebrated panel discussions (Aug. 17, 1949) on
the existence and uniqueness of multiplicity of solutions of the aerodynamical equations: von Neumann, Burgers, Heisenberg, Liepmann, von Karman......

Shock Problem: Supersonic Flow onto a Solid Wedge

? Two Steady Solutions with Shocks around the Solid Wedge



von Neumann's celebrated panel discussions (Aug. 17, 1949):

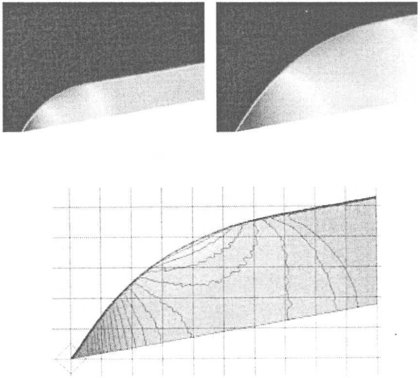
Chairman: Dr. J. von Neumann, Discussion on the existence and uniqueness or multiplicity of solutions of the aerodynamical equations, *Bull. Amer. Math. Soc.* **47** (2010), 145–154.

Members: von Neumann, Burgers, Heisenberg, Liepmann, von Karman...

Serre, D.: von Neumann's comments about existence and uniqueness for the initial-boundary value problem in gas dynamics, *Bull. Amer. Math. Soc.* **47** (2010), 139–144.

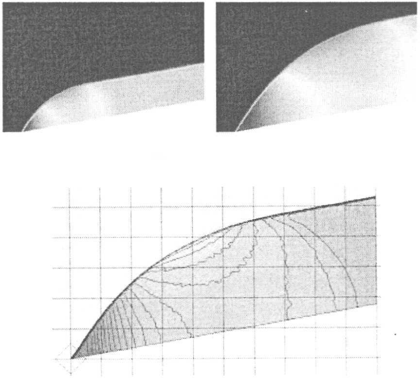
Liu, T.-P.: Multi-dimensional gas flow: some historical perspectives, Bull. Institute of Math., Academia Sinica, **6** (2011), 269–291.

Shock Problem: Prandtl-Meyer Reflection Configuration



Elling-Liu's Numerical Simulations (2009)

Shock Problem: Prandtl-Meyer Reflection Configuration



Elling-Liu's Numerical Simulations (2009)

? There Exists a Global Solution of the Prandtl-Meyer Reflection Configuration when the wedge angle is less than the sonic angle so that the state behind of the attached shock at the tip of wedge is supersonic.

Boundary Value Problem in the Unbounded Domain

Slip boundary condition on the wedge boundary:

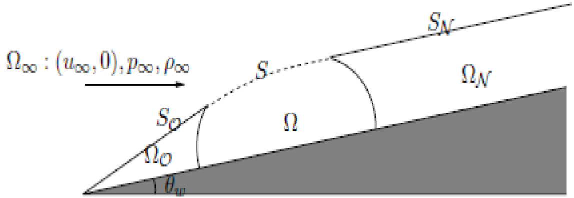
$$D\varphi \cdot \boldsymbol{\nu} = 0 \qquad \text{on} \quad \partial W$$

Asymptotic boundary condition as $r := \sqrt{\xi^2 + \eta^2} \to \infty$:

$$D\varphi - (u_\infty - \xi, -\eta) \to 0 \qquad \eta > \xi \tan\theta_w, \xi > 0.$$

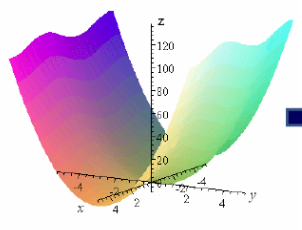
Locations of two shocks $S_{\mathcal{O}}$ and $S_{\mathcal{N}}$ are apriori known.

*Ref: Bae-Chen-Feldman: arXiv 0389822 (2011); Preprint 2012
Also cf. Elling-Liu: CPAM 2009



Outline

Isometric Embedding Problems



g_{ij} ⟷ **Metric**

(First Fundamental Form: $\mathrm{I} = \sum g_{ij} dx^i dx^j$)

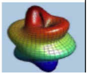
h_{ij} ⟷ **Curvatures**

(Second Fundamental Form: $\mathrm{II} = \sum h_{ij} dx^i dx^j$)

Given a metric g_{ij} and certain curvatures

Inverse Problem: CAN we find a surface in our real world with this metric g_{ij} and corresponding curvatures?

Realization Question?



Question: CAN *we produce even more sophisticated surfaces or thin sheets?*

Fundamental

- Mathematics: Differential Geometry, Topology, ……
- Understanding evolution of sophisticated shapes of surfaces or thin sheets in nature, including
  - --Elasticity, Materials Science, ……
  - --Biology and Algorithmic Origami: Protein Folding, ……
    - *US DARPA's 10th question of the 23 Challenge Questions in the Sciences [US Defense Advanced Research Project Agency]:
    - Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.
- Design, Visual Arts, ……

History: *Schlaefli (1873), Darboux (1894), Hilbert (1901), Weyl (1916), Janet (1926-27), Cartan (1926-27), Lewy (1936), Nash (1954-56), Kuiper (1955), Yau (1980's, 1990's), Gromov (1970, 1986), Günther (1989), Poznyak (1973), Levi (1908), Heinz (1962), Alexandroff (1938, 1942), Pogorelov (late 1940's, 1972), Nirenberg (1953, 1963), Efimov (1963), Bryant-Griffiths-Yan (1983), Lin (1985-86), Hong (1991,1993), ……*

Nash Isometric Embedding Theorem

(C^k embedding theorem, $k \geq 3$)

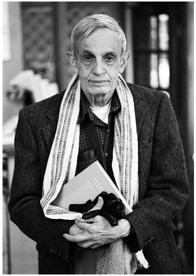
Every n-Dimensional Riemannian manifold (analytic or $C^k, k \geq 3$) can be C^k isometrically imbedded in the Euclidean space $\mathbb{R}^N$:

Compact Case: $N = 3s_n + 4n$

Noncompact Case: $N = (n+1)(3s_n + 4n)$

Gromov (1986): $N = s_n + 2n + 3$

Günther (1989): $N = \max\{s_n + 2n, s_n + n + 5\}$



the Subject of the Hollywood Movie *A Beautiful Mind*

Open Problems Important for Applications

Lowest Target Dimension? Janet-D: $N = s_n = \frac{n(n+1)}{2}$?

Optimal or Assigned Regularity?

$C^{1,1}$ Isometric Embedding? What about $BV(C^1)$?

Current Research Activities, ...

Efimov's Example (1966): No C^2 Isometric Embedding when $n = 2, s_n = 3$.

Gauss-Codazzi System: Compatibility/Constraint

Fundamental Theorem in Differential Geometry:
There exists a surface in $\mathbb{R}^3$ with 1st and 2nd fundamental form coefficients $\{g_{ij}\}$ and $\{h_{ij}\}$, $\{g_{ij}\}$ being positive definite, provided that the coefficients satisfy the Gauss-Codazzi system.

*This theorem holds even when $h_{ij} \in L^p$ (Mardare 2003–05)

Given $\{g_{ij}\}$, $\{h_{ij}\}$ is determined by the Codazzi Eqs. (Compatibility):

$$\begin{cases} \partial_x M - \partial_y L = L\Gamma_{22}^{(2)} - 2M\Gamma_{12}^{(2)} + N\Gamma_{11}^{(2)}, \\ \partial_x N - \partial_y M = -L\Gamma_{22}^{(1)} + 2M\Gamma_{12}^{(1)} - N\Gamma_{11}^{(1)}, \end{cases}$$

satisfying the Gauss Equation (Constraint):

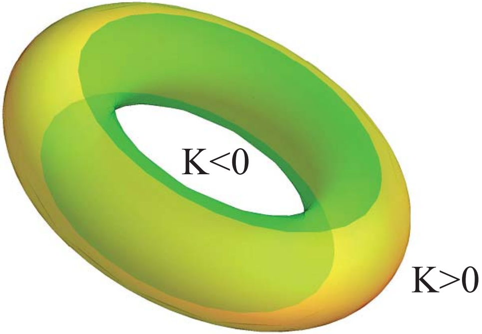
$$LN - M^2 = K,$$

where $L = \frac{h_{11}}{\sqrt{|g|}}$, $M = \frac{h_{12}}{\sqrt{|g|}}$, $N = \frac{h_{22}}{\sqrt{|g|}}$, $|g| = g_{11}g_{22} - g_{12}^2$

$\Gamma_{ij}^{(k)}$—Christoffel symbols, depending on g_{ij} up to their 1st derivatives
$K(x,y)$—Gauss curvature, determined by g_{ij} up to their 2nd derivatives

***Nonlinear PDEs of Mixed Elliptic-Hyperbolic Equations: Sign of K**

Surfaces with Gauss Curvature of Changing Sign



Gauss Curvature K on a Torus:
Toroidal Shell or Doughnut Surface

Fluid Dynamics Formalism for Isometric Embedding

Set $L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p, \quad q^2 = u^2 + v^2.$
Choose p as the Chaplygin type gas: $p = -1/\rho.$

The Codazzi Equations become the Momentum Equations:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -(\rho v^2 + p)\Gamma_{22}^{(2)} - 2\rho uv\Gamma_{12}^{(2)} - (\rho u^2 + p)\Gamma_{11}^{(2)}, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -(\rho v^2 + p)\Gamma_{22}^{(1)} - 2\rho uv\Gamma_{12}^{(1)} - (\rho u^2 + p)\Gamma_{11}^{(1)}, \end{cases}$$

and the Gauss Equation becomes the Bernoulli Relation:

$$p = -\sqrt{q^2 + K}.$$

Define the sound speed: $c^2 = p'(\rho)$. Then $c^2 = 1/\rho^2 = q^2 + K$.

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$c^2 > q^2$ and the "flow" is subsonic when $K > 0$,
$c^2 < q^2$ and the "flow" is supersonic when $K < 0$,
$c^2 = q^2$ and the "flow" is sonic when $K = 0$.

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?? **Existence/Continuity of Isometric Embedding**
$\Leftarrow$ **Weak Convergence Methods: Compensated Compactness**
Chen-Slemrod-Wang: Commun. Math. Phys. 2010

Outline

Gauss-Codazzi-Ricci System for Isometric Embedding of d-D Riemannian Manifolds into $\mathbb{R}^N$: $d \geq 3$

Gauss equations: $h^a_{ji} h^a_{kl} - h^a_{ki} h^a_{jl} = R_{ijkl}$

Codazzi equations:

$$\frac{\partial h^a_{lj}}{\partial x^k} - \frac{\partial h^a_{kj}}{\partial x^l} + \Gamma^m_{lj} h^a_{km} - \Gamma^m_{kj} h^a_{lm} + \kappa^a_{kb} h^b_{lj} - \kappa^a_{lb} h^b_{kj} = 0$$

Ricci equations:

$$\frac{\partial \kappa^a_{lb}}{\partial x^k} - \frac{\partial \kappa^a_{kb}}{\partial x^l} - g^{mn}\left(h^a_{ml} h^b_{kn} - h^a_{mk} h^b_{ln}\right) + \kappa^a_{kc}\kappa^c_{lb} - \kappa^a_{lc}\kappa^c_{kb} = 0$$

*R_{ijkl} is the Riemann curvature tensor, $\kappa^a_{kb} = -\kappa^b_{ka}$ is the coefficients of the connection form (torsion coefficients) on the normal bundle; the indices a, b, c run from 1 to N, and i, j, k, l, m, n run from 1 to $d \geq 3$.

***The Gauss-Codazzi-Ricci system has no type, neither purely hyperbolic nor purely elliptic for general Riemann curvature tensor R_{ijkl}**

*Bryant-Griffiths-Yang (**1983**): Duke Math. J., 102 pages.

*Chen-Slemrod-Wang (**2012**): Positive Symmetry & Entropy

Weak Continuity & Rigidity of the Gauss-Codazzi-Ricci System and the Embedded Surfaces in Geometry

Theorem (Chen-Slemrod-Wang: Proc. Amer. Math. Soc. 2010)

- **Let $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ be a sequence of solutions to the Gauss-Codazzi-Ricci system, which is uniformly bounded in $L^p, p > 2$. Then the weak limit vector field $(h_{ij}^{a}, \kappa_{lb}^{a})$ of the sequence $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ in L^p is still a solution to the Gauss-Codazzi-Ricci system.**
- **There exists a minimizer $(h_{ij}^{a}, \kappa_{lb}^{a})$ for the minimization problem:**

$$\min_S \|(h,\kappa)\|^p_{L^p(\Omega)} := \min_S \int_\Omega \sqrt{|g|}\left(|h_{ij}h_{ij}|^{\frac{p}{2}} + |\kappa_{lb}\kappa_{lb}|^{\frac{p}{2}}\right)dx,$$

where S is the set of weak solutions to the system.

Observations: Div-Curl Structure of the GCR System

$$\text{div}\,(\overbrace{\underbrace{0,\cdots,0,h_{lj}^{a,\varepsilon}}^{k},0,\cdots,-h_{kj}^{a,\varepsilon}}_{l},0,\cdots,0)=R_1,\ \ \text{curl}\,(h_{1j}^{a,\varepsilon},h_{2j}^{a,\varepsilon},\cdots,h_{dj}^{a,\varepsilon})=R_2,$$

$$\text{div}\,(\overbrace{\underbrace{0,\cdots,0,\kappa_{lb}^{a,\varepsilon}}^{k},0,\cdots,-\kappa_{kb}^{a,\varepsilon}}_{l},0,\cdots,0)=R_3,\ \ \text{curl}\,(\kappa_{1b}^{a,\varepsilon},\kappa_{2b}^{a,\varepsilon},\cdots,\kappa_{db}^{a,\varepsilon})=R_4,$$

$$\text{div}\,(\overbrace{\underbrace{0,\cdots,0,h_{li}^{b,\varepsilon}}^{k},0,\cdots,-h_{ki}^{b,\varepsilon}}_{l},0,\cdots,0)=R_5,\quad \text{curl}\,(h_{1i}^{b,\varepsilon},h_{2i}^{b,\varepsilon},\cdots,h_{di}^{b,\varepsilon})=R_6,$$

$$\text{div}\,(\overbrace{\underbrace{0,\cdots,0,\kappa_{lc}^{b,\varepsilon}}^{k},0,\cdots,-\kappa_{kc}^{b,\varepsilon}}_{l},0,\cdots,0)=R_7,\quad \text{curl}\,(\kappa_{1c}^{b,\varepsilon},\kappa_{2c}^{b,\varepsilon},\cdots,\kappa_{dc}^{b,\varepsilon})=R_8.$$

Lemma (Div-Curl Lemma: Murat 1978, Tartar 1979)

Let $\Omega \subset \mathbb{R}^d, d \geq 2,$ *be open bounded. Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Assume that, for any* $\varepsilon > 0$*, two fields*

$$u^\varepsilon \in L^p(\Omega; \mathbb{R}^d), \qquad v^\varepsilon \in L^q(\Omega; \mathbb{R}^d)$$

satisfy the following:

1. $u^\varepsilon \rightharpoonup u$ *weakly in* $L^p(\Omega; \mathbb{R}^d)$ *as* $\varepsilon \to 0$*;*
2. $v^\varepsilon \rightharpoonup v$ *weakly in* $L^q(\Omega; \mathbb{R}^d)$ *as* $\varepsilon \to 0$*;*
3. $\operatorname{div} u^\varepsilon$ *are confined in a compact subset of* $W^{-1,p}_{loc}(\Omega; \mathbb{R})$*;*
4. $\operatorname{curl} v^\varepsilon$ *are confined in a compact subset of* $W^{-1,q}_{loc}(\Omega; \mathbb{R}^{d\times d})$*.*

Then the scalar product of u^ε *and* v^ε *are weakly continuous:*

$$u^\varepsilon \cdot v^\varepsilon \longrightarrow u \cdot v$$

in the sense of distributions.

*Various variations of this lemma for different applications/purposes.

Observations: Div-Curl Structure of the GCR System

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$$\mathrm{div}\,(\overbrace{0,\cdots,0,\kappa^{a,\varepsilon}_{lb}}^{k},\underbrace{,0,\cdots,-\kappa^{a,\varepsilon}_{kb}}_{l},0,\cdots,0) = R_3,\quad \mathrm{curl}\,(\kappa^{a,\varepsilon}_{1b},\kappa^{a,\varepsilon}_{2b},\cdots,\kappa^{a,\varepsilon}_{db}) = R_4,$$

$$\mathrm{div}\,(\overbrace{0,\cdots,0,h^{b,\varepsilon}_{li}}^{k},\underbrace{,0,\cdots,-h^{b,\varepsilon}_{ki}}_{l},0,\cdots,0) = R_5,\quad \mathrm{curl}\,(h^{b,\varepsilon}_{1i},h^{b,\varepsilon}_{2i},\cdots,h^{b,\varepsilon}_{di}) = R_6,$$

$$\mathrm{div}\,(\overbrace{0,\cdots,0,\kappa^{b,\varepsilon}_{lc}}^{k},\underbrace{,0,\cdots,-\kappa^{b,\varepsilon}_{kc}}_{l},0,\cdots,0) = R_7,\quad \mathrm{curl}\,(\kappa^{b,\varepsilon}_{1c},\kappa^{b,\varepsilon}_{2c},\cdots,\kappa^{b,\varepsilon}_{dc}) = R_8.$$

Weak Convergence: Div-Curl $\Longrightarrow$

$$\begin{aligned}
h^{a,\varepsilon}_{lj}h^{b,\varepsilon}_{ki} - h^{a,\varepsilon}_{kj}h^{b,\varepsilon}_{li} &\rightharpoonup h^a_{lj}h^b_{ki} - h^a_{kj}h^b_{li},\\
\kappa^{a,\varepsilon}_{kb}\kappa^{b,\varepsilon}_{lc} - \kappa^{a,\varepsilon}_{lb}\kappa^{b,\varepsilon}_{kc} &\rightharpoonup \kappa^a_{kb}\kappa^b_{lc} - \kappa^a_{lb}\kappa^b_{kc},\\
\kappa^{a,\varepsilon}_{kb}h^{b,\varepsilon}_{li} - \kappa^{a,\varepsilon}_{lb}h^{b,\varepsilon}_{ki} &\rightharpoonup \kappa^a_{kb}h^b_{li} - \kappa^a_{lb}h^b_{ki}
\end{aligned}$$

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where S is the set of weak solutions to the system.

Weak Continuity of Nonlinear Functions/Functionals

- **Rigidity of Embedded Surfaces in Geometry**:
  The weak limit of isometrically embedded surfaces is still an isometrically embedded surface in $\mathbb{R}^d$ for any Riemann curvature tensor R_{ijkl} without restriction
- **Motivation/Connection: Theory of Polyconvexity in Nonlinear Elasticity by Ball (1977):**
  Weak Continuity of Determinants, $\cdots$
- **Stronger Compactness Framework for the Gauss-Codazzi-Ricci System (CSD 2010)**: **Given any sequence of approximate solutions to this system which is uniformly bounded in L^2 and has reasonable bounds on the errors made in the approximation (the errors are confined in a compact subset of H^{-1}_{loc}), then the approximating sequence has a weakly convergent subsequence whose limit is still a solution of the Gauss-Codazzi-Ricci system.**

Nonlinear Partial Differential Equations of Mixed Hyperbolic-Elliptic Type, or even No Type,

naturally arise in many fundamental problems in

Fluid Mechanics
Differential Geometry
Materials Science: Phase Transition, ...
Relativity: Non-Vacuum State, Matter, ...
Optimization, Dynamical Systems
......

The solution to these fundamental problems in the areas greatly requires a **deep understanding of**

Nonlinear Partial Differential Equations of Mixed Hyperbolic-Elliptic Type

- During the last half century, the two different types of nonlinear PDEs have been separately studied.
  Focus: Mathematical tools to understand different properties of solutions; **Great progress has been made**.
- With these achievements, **it is the time to initiate a comparable attack:**
  To analyze systematically nonlinear PDEs of mixed type;
  To explore unified mathematical approaches, ideas, and techniques to deal with such problems.
  In particular, we have presented several fundamental examples of such PDEs, which indicate that **some of the mixed-type problems have been ready to be tractable.**
- **Many important mixed-type problems are wide open and very challenging, which require further new ideas, approaches, techniques, ..., and deserve our special attention and true effort.**