Stokes resolvent estimates in spaces of bounded functions

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Local existence theorem for NS eq.

(NS eq.)
$$\begin{cases} v_t - \Delta v + (v,\nabla)v + \nabla q = 0 & \text{in } (0,T)\times\mathbb{R}^n \\ \operatorname{div} v = 0 & \text{in } (0,T)\times\mathbb{R}^n \\ v = v_0 & \text{on } \{t=0\}\times\mathbb{R}^n \end{cases}$$

- Mild solution:

$$v(t) = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\mathrm{div}(v(s)\otimes v(s))ds$$

- Regularizing effect of $e^{t\Delta}f = \Gamma * f$ as analytic semigroup on L^p, $p\in[1,\infty]$, e.g.

$$||\frac{d}{dt}e^{t\Delta}f||_{L^p} \le \frac{C}{t}||f||_{L^p}$$
$$\left(||e^{t\Delta}\mathbf{P}\mathrm{div}F||_{L^p} \le \frac{C}{t^{1/2}}||F||_{L^p}\right)$$

- $\Gamma = \Gamma(x,t)$: Heat kernel
- $\mathbf{P} = I - \nabla(-\Delta)^{-1}\operatorname{div}$: Helmholtz projection.

Analyticity of the Stokes semigroup

(Stokes eq.)
$$\begin{cases} v_t - \Delta v + \nabla q &= 0 \quad \text{in } (0,T)\times\Omega \\ \operatorname{div} v &= 0 \quad \text{in } (0,T)\times\Omega \\ \text{(B.C.)} \quad v &= 0 \quad \text{on } \partial\Omega \\ \text{(I.C.)} \quad v(x,0) &= v_0 \quad \text{on } \{t=0\} \end{cases}$$

in domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$

- $S(t) : v_0 \mapsto v(\cdot, t), t \geq 0$: **Stokes semigroup**
- Multiply $\mathbf{P}$ to Stokes eq.: $v_t - Av = 0$ with $A = \mathbf{P}\Delta$ (Non-local op.)
- $S(t)$ is an analytic semigroup on $L^p(\Omega), p \in (1,\infty)$ for various Ω, e.g. bdd and exterior domains: Solonnikov '77, Giga '81
- Analyticity results for uniformly regular domains Ω: Farwig-Kozono-Sohr '05, Geissert-Heck-Hieber-Sawada '12

L^∞-type results

Problem: Is $S(t)$ an analytic semigroup on L^∞-type spaces?

- A typical L^∞-type space:

$$C_{0,\sigma}(\Omega) = L^\infty\text{-closure of } \{v \in C_c^\infty(\Omega) \mid \mathrm{div}\ v = 0\}$$

- Analyticity results for $\mathbf{R}^n_+$: Desch-Hieber-Prüss '01, Solonnikov '03, Maremonti-Starita '03
- General elliptic operators in uniform regular Ω (e.g. $A = \Delta$): Masuda '72, Stewart '74
- How to control $\nabla q = (I - \mathbf{P})\Delta v$ on L^∞?

$$||\nabla q||_p \not\lesssim C_p||\Delta v||_p \quad \text{when} \quad p = \infty$$

- $\mathbf{P} : L^\infty(\Omega) \not\to L^\infty(\Omega)$

Admissible domains

Theorem (A-Giga, to appear)

Let Ω be a uniformly C^3-domain. Then $S(t)$ is an analytic semigroup on $C_{0,\sigma}(\Omega)$ provided that Ω is admissible (e.g. bdd.)

- A priori L^∞-estimates:

$$\sup_{0\le t\le T_0} ||N(v,q)||_\infty(t) \le C||v_0||_\infty \quad \text{for } v_0 \in C_{0,\sigma}(\Omega)$$

$$N(v,q)(x,t) = |v(x,t)| + t^{\frac{1}{2}}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|\partial_t v(x,t)| + t|\nabla q(x,t)|$$

- Key idea: **Pressure estimates**

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla q(x,t)| \le C_\Omega ||\mathrm{curl}\ v \times n_\Omega||_{L^\infty(\partial\Omega)}(t)$$

What kinds of domains are admissible?

- Neumann problem for pressure q

$$\Delta q = 0 \text{ in } \Omega,\ \partial q/\partial n_\Omega = \mathrm{div}_{\partial\Omega}(\mathrm{curl}\ v \times n_\Omega) \text{ on } \partial\Omega$$

- Ω is **strictly admissible** if an *a priori* estimate,

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla P(x)| \le C_\Omega \|W\|_{L^\infty(\partial\Omega)}$$

holds for all solutions to

$$\Delta P = 0 \text{ in } \Omega,\ \partial P/\partial n_\Omega = \mathrm{div}_{\partial\Omega} W \text{ on } \partial\Omega$$

- Remarks:
  - Strictly admissible: e.g. bounded, exterior C^3 domains
  - Not strictly admissible: e.g. layer domains

- Conjecture: Ω is strictly admissible $\Leftrightarrow \overline{\lim}_{|x|\to\infty} d_\Omega(x) = \infty$
  (Not *quasicylindrical*)

Angle of analytic semigroups

Definition (Analytic semigroup)

Let X be a Banach space and $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ be a semigroup.
We call $T(t)$ analytic semigroup if $\exists C > 0$ s.t.

$$\left|\left|\frac{dT(t)}{dt}\right|\right|_{\mathcal{L}(\mathrm{X})} \leq \frac{C}{t} \quad \text{for } t \in (0,1].$$

Moreover, if $T(t)$ has an analytic continuation to a sector $\{t \in \mathbb{C} \mid |\arg t| < \theta\}$, we say T(t) is angle θ.

- A priori L^∞-estimates:

$$\sup_{0\leq t\leq T_0} t||v_t||_\infty(t) \leq C||v_0||_\infty$$

- Remarks
  - $S(t)$ is angle $\pi/2$ on L^p_σ
  - $S(t)$ is angle $\varepsilon > 0$ on $C_{0,\sigma}$

Resolvent approach

Question: Is $S(t)$ angle $\pi/2$ on $C_{0,\sigma}$?

(Resolvent eq.) $$\begin{cases} \lambda v - \Delta v + \nabla q &= f \quad \text{in } \Omega \\ \operatorname{div} v &= 0 \quad \text{in } \Omega \\ \text{(B.C.)} \qquad v &= 0 \quad \text{on } \partial\Omega \end{cases}$$

on $\Sigma_{\vartheta,\delta} = \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \vartheta,\ |\lambda| > \delta\}$ with $\vartheta \in (\pi/2, \pi)$ and $\delta > 0$.

Goal: Give a direct estimate

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} \|M_p(v,q)\|_\infty(\lambda) \leq C\|f\|_\infty \quad \text{for } f \in C_{0,\sigma}$$

$$\text{for } M_p(v,q)(x,\lambda) = |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| + |\lambda|^{n/2p}\|\nabla^2 v\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p}\|\nabla q\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})},$$

$\Omega_{x,|\lambda|^{-1/2}} = B(x,|\lambda|^{-1/2}) \cap \Omega$ for $p > n$.

Main results

Theorem (A-Giga-Hieber '12)

Let Ω be a strictly admissible, uniformly C^2-domain. For $\vartheta \in (\pi/2, \pi)$ there exist constants δ and C such that

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} ||M_p(v,q)||_{L^\infty(\Omega)}(\lambda) \leq C||f||_{L^\infty(\Omega)} \quad \text{for } f \in C_{0,\sigma}(\Omega)$$

holds for $p > n$.

Our direct approach clarifies

- $S(t)$ is an analytic semigroup of angle $\pi/2$ on $C_{0,\sigma}$
- Application to the Robin B.C., i.e. $B(v) = 0, v \cdot n_\Omega = 0$ on $\partial\Omega$ where

$$B(v) = \alpha v_{\tan} + (D(v)n_\Omega)_{\tan} \quad \text{with } \alpha \geq 0$$

and $D(v) = (\nabla v + \nabla^T v)/2$.

Sketch of the proof

- Goal:

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} ||M_p(v,q)||_\infty(\lambda) \le C||f||_\infty \quad \text{for } f \in C_{0,\sigma}(\Omega)$$

- Pressure estimate:

$$\sup_{x \in \Omega} d_\Omega(x)|\nabla q(x)| \le C_\Omega ||\nabla v||_{L^\infty(\partial\Omega)}$$

- Masuda-Stewart method:
  - (1) Localization
  - (2) Error estimates for pressure (key step!)
  - (3) Interpolation

Step1 (Localization)

Take

- $x_0 \in \Omega, r > 0$
- $\eta \geq 1$: parameters
- $q_c \in \mathbb{C}$: constant
- θ: cut-off function s.t. $\theta \equiv 1$ in $B_{x_0}(r)$ and $\theta \equiv 0$ in $B_{x_0}((\eta+1)r)^c$

Set

$$u = v\theta \text{ and } p = (q - q_c)\theta.$$

Then (u, p) solves

$$\begin{cases} \lambda u - \Delta u + \nabla p = h & \text{in } \Omega' \\ \text{div } u = g & \text{in } \Omega' \\ v = 0 & \text{on } \partial\Omega' \end{cases}$$

where $\Omega' = B_{x_0}((\eta+1)r) \cap \Omega$ and

$$h = f\theta - 2\nabla v \nabla\theta - v\Delta\theta + (q - q_c)\nabla\theta,$$
$$g = v \cdot \nabla\theta$$

- L^p-estimates:

$$|\lambda| ||u||_{L^p(\Omega')} + |\lambda|^{1/2} ||\nabla u||_{L^p(\Omega')} + ||\nabla^2 u||_{L^p(\Omega')} + ||\nabla p||_{L^p(\Omega')}$$
$$\leq C_p \left(||h||_{L^p(\Omega')} + ||\nabla g||_{L^p(\Omega')} + |\lambda| ||g||_{W_0^{-1,p}(\Omega')} \right)$$

- Estimates for $\theta = \theta_0((x - x_0)/(\eta + 1)r)$:

$$||\theta||_\infty + (\eta + 1)r||\nabla \theta||_\infty + (\eta + 1)^2 r^2 ||\nabla^2 \theta||_\infty \leq K$$

- $h = f\theta - 2\nabla v \nabla \theta - v\Delta\theta + (q - q_c)\nabla\theta$
- Estimates for velocity, e.g.

$$||\nabla v \nabla \theta||_{L^p(\Omega')} \leq C \frac{((\eta + 1)r)^{n/p}}{(\eta + 1)r} ||\nabla v||_\infty$$

- How to estimate $(q - q_c)\nabla\theta$?

$$\left(||(q - q_c)\nabla\theta||_{L^p(\Omega')} \leq C \frac{((\eta + 1)r)^{n/p}}{(\eta + 1)r} ||\nabla v||_\infty \ ? \right)$$

Step2 (Error estimates for pressure)

Lemma (Poincaré-Sobolev-type inequality)

Let D be a bounded domain with C^2-boundary and $p \in [1, \infty)$. There exists a constant C_D s.t.

$$||\varphi - (\varphi)||_{L^p(D)} \leq C_D ||\nabla \varphi||_{L^\infty_d(D)} \quad for\ \nabla\varphi \in L^\infty_d(D)$$

where $(\varphi) = ⨍_D \varphi dx$ and

$$||\nabla \varphi||_{L^\infty_d(D)} = \sup_{x \in D} d_D(x) |\nabla \varphi(x)|.$$

- Ex. $D = B(1)$
  $\varphi(x) = \log(1 - |x|) \in L^p$ even if $|\nabla \varphi| = d_D^{-1} \notin L^p$

- Pressure estimate by velocity:

$$||\nabla q||_{L^\infty_d(\Omega)} \leq C_\Omega ||\nabla v||_{L^\infty(\Omega)}$$

- Localized version for (PS)

$$||\varphi - (\varphi)||_{L^p(\Omega')} \leq Cs^{n/p}||\nabla\varphi||_{L^\infty_d(\Omega)}$$

in $\Omega' = \Omega \cap B(s)$ with $s = (\eta + 1)r$.

- Pressure estimate by velocity:

$$||\nabla q||_{L^\infty_d(\Omega)} \leq C_\Omega||\nabla v||_{L^\infty(\Omega)}$$

- Apply to $\varphi = q$ in Ω' with $q_c = \fint_{\Omega'} q(x)dx$

$$\begin{aligned} ||(q - q_c)\nabla\theta||_{L^p(\Omega')} \leq & C((\eta + 1)r)^{-1}||q - (q)||_{L^p(\Omega')} \\ \leq & C((\eta + 1)r)^{-1}((\eta + 1)r)^{n/p}||\nabla q||_{L^\infty_d(\Omega)} \\ \leq & C((\eta + 1)r)^{-(1-n/p)}||\nabla v||_{L^\infty(\Omega)} \end{aligned}$$

Step 3 (Interpolation)

- Apply the Interpolation inequality:

$$||\varphi||_{L^\infty(\Omega_{x_0,r})} \leq C_I r^{-n/p} \left(||\varphi||_{L^p(\Omega_{x_0,r})} + r||\nabla\varphi||_{L^p(\Omega_{x_0,r})} \right)$$

  for $\varphi = u$ and ∇u.
- Take $r = |\lambda|^{-1/2}$ to get

$$M_p(v,q)(x_0,\lambda) \leq C\left((\eta+1)^{n/p}||f||_{L^\infty(\Omega)} + (\eta+1)^{-(1-n/p)}||M_p(v,q)||_{L^\infty(\Omega)}(\lambda) \right)$$

- $-(1-n/p) < 0 \iff p > n$

- Remark (Robin B.C.)

$B(v) = 0, \; v \cdot n_\Omega = 0$ on $\partial\Omega$ with

$$B(v) = \alpha v_{\tan} + (D(v) n_\Omega)_{\tan} \quad \text{for } \alpha \geq 0,$$
$$D(v) = (\nabla v + \nabla^T v)/2$$

- B.C. for the localized equation:

$$B(u) = k, \; u \cdot n_{\Omega'} = 0 \quad \text{on } \partial\Omega'$$

with

$$k = v_{\tan} \partial\theta / \partial n_{\Omega'}$$

- L^p-estimates with inhomogeneous B.C.:

$$|\lambda| ||u||_{L^p(\Omega')} + |\lambda|^{1/2} ||\nabla u||_{L^p(\Omega')} + ||\nabla^2 u||_{L^p(\Omega')} + ||\nabla p||_{L^p(\Omega')}$$
$$\leq C_p \big(||h||_{L^p(\Omega')} + ||\nabla g||_{L^p(\Omega')} + |\lambda| ||g||_{W_0^{-1,p}(\Omega')}$$
$$+ |\lambda|^{1/2} ||k||_{L^p(\Omega')} + ||\nabla k||_{L^p(\Omega')} \big)$$

Summary

- Blow-up arguments: Stronger estimates

$$\sup_{0\le t\le T_0} ||N(v,q)||_\infty(t) \le C||v_0||_\infty \quad \text{for } v_0 \in C_{0,\sigma}$$

$$N(v,q)(x,t) = |v(x,t)| + t^{\frac{1}{2}}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|\partial_t v(x,t)| + t|\nabla q(x,t)|$$

- Direct approach: Clarifies the angle

$$\sup_{\lambda\in\Sigma_{\vartheta,\delta}} ||M_p(v,q)||_\infty(\lambda) \le C||f||_\infty \quad \text{for } f \in C_{0,\sigma}$$

$$M_p(v,q)(x,\lambda) = |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| + |\lambda|^{n/2p}||\nabla^2 v||_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p}||\nabla q||_{L^p(\Omega_{x,|\lambda|^{-1/2}})}$$