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Multidimensional consistency of discrete and continuous equations

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Introduction

Multidimensional consistency of discrete systems

① Discrete analog of commuting flows

Nijhoff–Walker 2001

② Integrability criterion

Nijhoff 2002, Bobenko–Suris 2002

③ Classification of integrable equations

Adler–Bobenko–Suris 2003 & 2009

Bobenko A., Suris Yu (2008)
Discrete Differential Geometry Integrable Structures
AMS Graduate Studies in Mathematics **98**

Introduction

Symmetries

① Symmetry analysis of the ABS equations

Tongas–Tsoubelis–X 2007

② Symmetries and Yang–Baxter maps

Papageorgiou–Tongas–Veselov 2006

③ Hierarchies of symmetries

X 2009 & X–Papageorgiou 2009

④ Continuous symmetry reductions

Tsoubelis-X 2009

Introduction

Multidimensional consistency of continuous systems

From discrete : a constructive approach

Tsoubelis-X 2009 & Lobb–Nijhoff 2009

Notation

- Independent variables

  Discrete : n_1 , n_2 , ...

  Continuous : α_1 , α_2 , ...

- Dependent variable u

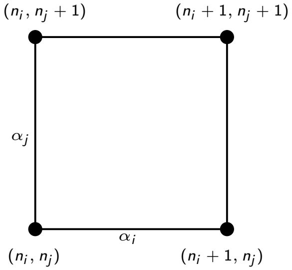
$$u_i = u(\ldots, n_i + 1, \ldots; \alpha_1, \alpha_2, \ldots)$$

$$u_{-j} = u(\ldots, n_j - 1, \ldots; \alpha_1, \alpha_2, \ldots)$$

$$u_{ij} = u(\ldots, n_i + 1, \ldots, n_j + 1, \ldots; \alpha_1, \alpha_2, \ldots)$$

Discrete potential KdV equation (H1)

$$(u - u_{ij})(u_i - u_j) = (\alpha_i - \alpha_j) \qquad (\mathrm{Q}_{ij})$$



An elementary quadrilateral on the lattice

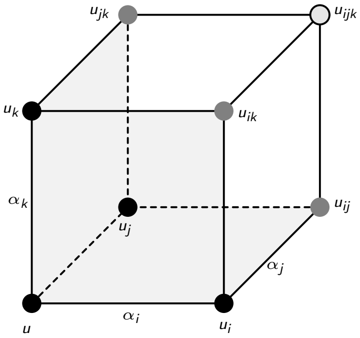
① Autonomous & Affine linear

② Symmetric : $\mathrm{Q}_{ij} \equiv \mathrm{Q}_{ji}$

③ Multidimensionally consistent

Multidimensional consistency

$$
\begin{aligned}
(u - u_{ij})(u_i - u_j) &= \alpha_i - \alpha_j \\
(u - u_{jk})(u_j - u_k) &= \alpha_j - \alpha_k \\
(u - u_{ki})(u_k - u_i) &= \alpha_k - \alpha_i
\end{aligned}
$$



Nijhoff F., Walker A. J. (2001)
The Discrete and Continuous Painlevé VI hierarchy and the Garnier systems
Glasgow Math. J. A **43**

Bäcklund transformation

$$(u - u_{ij})(u_i - u_j) = \alpha_i - \alpha_j$$

$$(u - u_{jk})(u_j - u_k) = \alpha_j - \alpha_k \quad (u - u_{ki})(u_k - u_i) = \alpha_k - \alpha_i$$

- Substitute the shifts of u in the k-direction by a function $\tilde{u}$.
- Replace the variable α_k with λ.
- The resulting equations

$$\left.\begin{array}{l}(u - \tilde{u}_j)(u_j - \tilde{u}) = \alpha_j - \lambda \\ (u - \tilde{u}_i)(\tilde{u} - u_i) = \lambda - \alpha_i\end{array}\right\} \quad (\mathbb{B}_{ij})$$

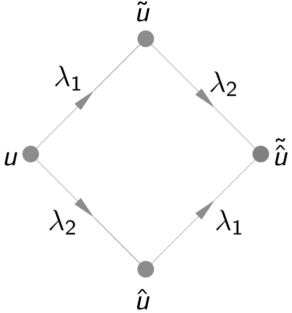
constitute an auto-Bäcklund transformation for Q_{ij}.

Atkinson J. (2008)
Bäcklund transformations for integrable lattice equations
J. Phys. A: Math. Theor. **41**

XP (2009)
Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case
Proceedings of the 4th Workshop "Group Analysis of Differential Equations & Integrable Systems" p 226–242

Bianchi permutability

The composition of two successive Bäcklund transformations according to the Bianchi commuting diagram leads to an algebraic relation similar to the discrete potential KdV.



Bianchi commuting diagram

$$(u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) = \lambda_1 - \lambda_2$$

Lax pair

- A Lax pair can be derived from the consistency property.
- It has the following form

$$\Psi_i = \mathrm{L}^{(i)}\,\Psi\,, \quad \Psi_j = \mathrm{L}^{(j)}\,\Psi$$

where

$$\mathrm{L}^{(i)} := \frac{1}{\sqrt{\lambda - \alpha_i}} \begin{pmatrix} u_i & -1 \\ u u_i - \alpha_i + \lambda & -u \end{pmatrix}$$

Nijhoff F. (2002)
Lax pair for the Adler (lattice Krichever–Novikov) system
Phys. Lett. A **297**

XP (2009)
Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case
Proceedings of the 4th Workshop "Group Analysis of Differential Equations & Integrable Systems" p 226–242

Generalized symmetries

- Q_{ij} admits infinite hierarchies of generalized symmetries.
- They have the following form

$$\frac{\partial u}{\partial \epsilon_{i\,s}} = \mathscr{R}_i^s \left(\frac{1}{u_i - u_{-i}} \right) , \quad \frac{\partial u}{\partial \epsilon_{j\,s}} = \mathscr{R}_j^s \left(\frac{1}{u_j - u_{-j}} \right)$$

where the recursion operators are given by

$$\mathscr{R}_i = \sum_{\ell=-\infty}^{\infty} \frac{\ell}{u_{i+\ell} - u_{-i+\ell}} \, \partial_{u_\ell}$$

XP (2009)
Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case
Proceedings of the 4th Workshop "Group Analysis of Differential Equations & Integrable Systems" p 226–242

Generalized symmetries

- Q_{ij} admits infinite hierarchies of generalized symmetries.
- The equation admits a pair of **extended symmetries** (symmetries acting also on the parameters α)

$$\begin{aligned} \frac{\partial u}{\partial \tau_i} &= \frac{n_i}{u_i - u_{-i}}, \quad \frac{\partial \alpha_i}{\partial \tau_i} = -1 \\ \frac{\partial u}{\partial \tau_j} &= \frac{n_j}{u_j - u_{-j}}, \quad \frac{\partial \alpha_j}{\partial \tau_j} = -1 \end{aligned}$$

which play a particular role : They are **master symmetries** of the first members of each hierarchy.

XP (2009)
Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case
Proceedings of the 4th Workshop "Group Analysis of Differential Equations & Integrable Systems" p 226–242

Continuously invariant solutions

Solutions of Q_{ij} remaining invariant under the action of both of the master symmetries.

- Such solutions will satisfy Q_{ij} and the differential–difference equations

$$\frac{\partial u}{\partial \alpha_i} + \frac{n_i}{u_i - u_{-i}} = 0 \qquad (\mathrm{R}_i)$$

$$\frac{\partial u}{\partial \alpha_j} + \frac{n_j}{u_j - u_{-j}} = 0 \qquad (\mathrm{R}_j)$$

Tsoubelis D, XP (2009)
Continuous symmetric reductions of the Adler–Bobenko–Suris equations
J. Phys. A: Math. Theor. **42**

Differential system

The previous differential–difference system is equivalent to

$$
\begin{aligned}
\frac{\partial u_i}{\partial \alpha_j} &= \frac{u_i - u_j}{\alpha_i - \alpha_j}\left(n_j - (u_i - u_j)\frac{\partial u}{\partial \alpha_j}\right) \\
\frac{\partial u_j}{\partial \alpha_i} &= \frac{u_j - u_i}{\alpha_j - \alpha_i}\left(n_i - (u_j - u_i)\frac{\partial u}{\partial \alpha_i}\right) \\
\frac{\partial^2 u}{\partial \alpha_i \partial \alpha_j} &= 2\,\frac{u_i - u_j}{\alpha_i - \alpha_j}\frac{\partial u}{\partial \alpha_i}\frac{\partial u}{\partial \alpha_j} + \frac{n_i}{\alpha_i - \alpha_j}\frac{\partial u}{\partial \alpha_j} + \frac{n_j}{\alpha_j - \alpha_i}\frac{\partial u}{\partial \alpha_i}
\end{aligned}
\qquad (\mathrm{S}_{ij})
$$

Nijhoff F., Hone A., Joshi N.(2000)
On a Schwarzian PDE associated with the KdV Hierarchy
Phys. Lett. A **267**

Tongas A., Tsoubelis D., XP (2001)
A family of Integrable nonlinear equations of Hyperbolic type
J. Math. Phys. **42**

Properties of the continuous system

1. *Symmetric*
   Interchanging the indices, the system remains invariant :
   $\mathrm{S}_{ij} \equiv \mathrm{S}_{ji}$.

2. *Involution*
   S_{ij} represents an involution of a fourth order equation

   $$\mathscr{R}(u; n_i, n_j) = 0\,, \quad \mathscr{R}(u_i; n_i{+}1, n_j) = 0\,, \quad \mathscr{R}(u_j; n_i, n_j{+}1) = 0$$

3. *Multidimensionally consistent*
   Systems S_{ij}, S_{jk} and S_{ki} are compatible.

4. *Integrable*
   S_{ij} admits an auto-Bäcklund transformation and a Lax pair.

Multidimensional consistency

- It follows directly from the consistency of its discrete analog, i.e. the system (Q_{ij}), (R_i), (R_j).

- The two different ways to evaluate $\partial_{\alpha_j}\partial_{\alpha_k}u_i$ lead to the same result. The same holds for the three different ways yielding $\partial_{\alpha_i}\partial_{\alpha_j}\partial_{\alpha_k}u$.

Thus, we can consider all the S_{ij}'s as an infinite dimensional system of the following form

$$\begin{aligned} \frac{\partial u_i}{\partial \alpha_j} &= \frac{u_i - u_j}{\alpha_i - \alpha_j}\left(n_j - (u_i - u_j)\frac{\partial u}{\partial \alpha_j}\right), \quad i \neq j \\ \frac{\partial^2 u}{\partial \alpha_i \partial \alpha_j} &= 2\,\frac{u_i - u_j}{\alpha_i - \alpha_j}\frac{\partial u}{\partial \alpha_i}\frac{\partial u}{\partial \alpha_j} + \frac{n_i}{\alpha_i - \alpha_j}\frac{\partial u}{\partial \alpha_j} + \frac{n_j}{\alpha_j - \alpha_i}\frac{\partial u}{\partial \alpha_i} \end{aligned}$$

Bäcklund transformation

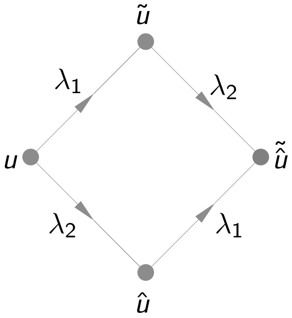
Employing the consistency of S_{ij}, one can derive the following auto-Bäcklund transformation for it

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial \alpha_i} &= \frac{\tilde{u} - u_i}{\lambda - \alpha_i}\left(n_i - (\tilde{u} - u_i)\frac{\partial u}{\partial \alpha_i}\right) \\
\frac{\partial \tilde{u}}{\partial \alpha_j} &= \frac{\tilde{u} - u_j}{\lambda - \alpha_j}\left(n_j - (\tilde{u} - u_j)\frac{\partial u}{\partial \alpha_j}\right) \qquad (\mathbb{B}_{ij}) \\
(u - \tilde{u}_j)(u_j - \tilde{u}) &= \alpha_j - \lambda \\
(u - \tilde{u}_i)(\tilde{u} - u_i) &= \lambda - \alpha_i
\end{aligned}
$$

Tsoubelis D, XP (2009)
Continuous symmetric reductions of the Adler–Bobenko–Suris equations
J. Phys. A: Math. Theor. **42**

Bianchi permutability

The Bianchi permutability diagram and the Bäcklund transformation lead again to the discrete KdV equation!



Bianchi commuting diagram

$$(u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) = \lambda_1 - \lambda_2$$

$$(u_i - \hat{\tilde{u}}_i)(\tilde{u}_i - \hat{u}_i) = \lambda_1 - \lambda_2$$

$$(u_j - \hat{\tilde{u}}_j)(\tilde{u}_j - \hat{u}_j) = \lambda_1 - \lambda_2$$

Lax pair

From the auto-Bäcklund transformation one can derive the following Lax pair for S_{ij}

$$\frac{\partial \Psi}{\partial \alpha_i} = \mathrm{M}^{(i)}\,\Psi\,, \quad \frac{\partial \Psi}{\partial \alpha_j} = \mathrm{M}^{(j)}\,\Psi$$

where

$$\mathrm{M}^{(i)} := \frac{1}{\alpha_i - \lambda} \begin{pmatrix} \frac{1}{2}\left(n_i + 2u_i \frac{\partial u}{\partial \alpha_i}\right) & -\frac{\partial u}{\partial \alpha_i} \\ u_i\left(n_i + u_i \frac{\partial u}{\partial \alpha_i}\right) & -\frac{1}{2}\left(n_i + 2u_i \frac{\partial u}{\partial \alpha_i}\right) \end{pmatrix}$$

Tsoubelis D, XP (2009)
Continuous symmetric reductions of the Adler–Bobenko–Suris equations
J. Phys. A: Math. Theor. **42**

Darboux transformations

The Lax pair for the discrete KdV

$$\Psi_i = \mathrm{L}^{(i)}\,\Psi := \frac{1}{\sqrt{\lambda-\alpha_i}}\begin{pmatrix} u_i & -1 \\ uu_i-\alpha_i+\lambda & -u \end{pmatrix}\Psi$$

and the one for the continuous system

$$\frac{\partial\Psi}{\partial\alpha_i} = \mathrm{M}^{(i)}\,\Psi := \frac{1}{\alpha_i-\lambda}\begin{pmatrix} \frac{1}{2}\left(n_i+2u_i\frac{\partial u}{\partial\alpha_i}\right) & -\frac{\partial u}{\partial\alpha_i} \\ u_i\left(n_i+u_i\frac{\partial u}{\partial\alpha_i}\right) & -\frac{1}{2}\left(n_i+2u_i\frac{\partial u}{\partial\alpha_i}\right) \end{pmatrix}\Psi$$

are compatible.

$$\mathrm{D}_{\alpha_i}\mathrm{L}^{(i)} + \mathrm{L}^{(i)}\mathrm{M}^{(i)} = \mathrm{M}_i^{(i)}\mathrm{L}^{(i)}$$

$$\mathrm{D}_{\alpha_j}\mathrm{L}^{(i)} + \mathrm{L}^{(i)}\mathrm{M}^{(j)} = \mathrm{M}_i^{(j)}\mathrm{L}^{(i)}$$

Summary

Discrete KdV was used as an illustrative example.

Actually, all of the ABS equations have the following properties :

1. Multidimensional consistent
2. Infinite hierarchies of symmetries
3. Reduce to differential equations.

The last property leads to systems of PDEs which are :

1. Multidimensional consistent
2. Integrable