Yang-Baxter maps and integrability

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Complement to the lectures at UK-Japan Winter School, Manchester 2010
History


Yang-Baxter equation in quantum theory and statistical mechanics

Set-theoretical solutions of quantum Yang-Baxter equation:

E.K. Sklyanin

Classical limits of SU(2)-invariant solutions of the Yang-Baxter equation.


V.G. Drinfeld

On some unsolved problems in quantum group theory.


Dynamical point of view:

A.P. Veselov

Yang-Baxter maps and integrable dynamics.


Yang-Baxter equation in quantum theory and statistical mechanics

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Set-theoretical solutions of quantum Yang-Baxter equation:


Dynamical point of view:

Quantum Yang-Baxter equation

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \]

where
\[ R : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{V} \]

is a linear operator.

\[ \frac{1}{2} \left( \sum_{i,j=1}^{n} R_{ij} \right) = \frac{1}{2} \left( \sum_{i,j=1}^{n} R_{ij} \right) \]

Figure: Yang-Baxter relation

Important consequence: Transfer-matrices
\[ T(\lambda) = \text{tr} 0 R_{01} \ldots R_{0n} \]

commute:
\[ T(\lambda) T(\mu) = T(\mu) T(\lambda) \]
Quantum Yang-Baxter equation

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Figure: Yang-Baxter relation

Important consequence: **Transfer-matrices** \( T(\lambda) = tr_0 R_{0n} \ldots R_{01} \) commute:

\[ T(\lambda) T(\mu) = T(\mu) T(\lambda). \]
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$$R_{21} R = Id.$$
Yang-Baxter maps (= Set-theoretical solutions of YBE)

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**Figure:** Reversibility
One can consider also the parameter-dependent Yang-Baxter maps $R(\lambda, \mu)$ with $\lambda, \mu$ from some parameter set $\Lambda$, satisfying

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2)$$

and reversibility condition

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Although this case can be considered as a particular case of the previous one by introducing $\tilde{X} = X \times \Lambda$ and $\tilde{R}(x, \lambda; y, \mu) = R(\lambda, \mu)(x, y)$ it is often convenient to keep the parameter separately.
Example 1: Interaction of matrix solitons

Matrix KdV equation

\[ \frac{\partial U}{\partial t} + 3UU_x + 3U_x U + U_{xxx} = 0 \]

has the soliton solution of the form

\[ U = 2\lambda^2 P \text{sech}^2(\lambda x - 4\lambda^3 t), \]

where "polarisation" \( P \) must be a projector: \( P^2 = P \).
Example 1: Interaction of matrix solitons

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The change of polarisations \( P \) after the soliton interaction is non-trivial:

\[ \tilde{L}_1 = (I + \frac{2\lambda_2}{\lambda_1 - \lambda_2} P_2)L_1, \]

\[ \tilde{L}_2 = (I + \frac{2\lambda_1}{\lambda_2 - \lambda_1} P_1)L_2, \]

where \( L \) is the image of \( P \) (Goncharenko, AV (2003)).

Darboux transformation

\[
L = -\frac{d^2}{dx^2} + u(x) = A^* A \rightarrow L_1 = AA^*.
\]

\[
A = \frac{d}{dx} - f(x), \quad A = -\frac{d}{dx} - f(x).
\]
Example 2: KdV and Adler map

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\[ (f_i + f_{i+1})' = f_i^2 - f_{i+1}^2 + \alpha_i, \quad i = 1, \ldots, 2m + 1. \]
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V. Adler (1993): symmetry of dressing chain

\[ \tilde{f}_1 = f_2 - \frac{\beta_1 - \beta_2}{f_1 + f_2} \]

\[ \tilde{f}_2 = f_1 - \frac{\beta_2 - \beta_1}{f_1 + f_2} \]
Geometric realisation: Recuttings of polygon
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1
2
3
4
5
Geometric realisation: Recuttings of polygon
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Define the \textbf{transfer maps}

\[ T_i^{(n)} : X^n \to X^n, \ i = 1, \ldots, n \]

by

\[ T_i^{(n)} = R_{i+n} R_{i+n-1} \cdots R_{i+1}, \]

where the indices are considered modulo \( n \). In particular

\[ T_1^{(n)} = R_{1n} R_{1n-1} \cdots R_{12}. \]
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where the indices are considered modulo \( n \). In particular \( T_1^{(n)} = R_{1n} R_{1n-1} \cdots R_{12} \).

For any reversible Yang-Baxter map \( R \) the transfer maps \( T_i^{(n)} \) **commute with each other**:

\[ T_i^{(n)} T_j^{(n)} = T_j^{(n)} T_i^{(n)} \]

and satisfy the property

\[ T_1^{(n)} T_2^{(n)} \cdots T_n^{(n)} = \text{Id}. \]

Conversely, if \( T_i^{(n)} \) satisfy these properties then \( R \) is a reversible YB map.
Commutativity of the transfer maps

Figure: Commutativity of the transfer maps
Recutting of polygons: dynamics
Some other initial data
Some other initial data
Matrix $A(x, \lambda, \zeta)$ with spectral parameter $\zeta \in \mathbb{C}$ is called **Lax matrix** of the map $R$ if it satisfies the relation

$$A(x, \lambda; \zeta)A(y, \mu; \zeta) = A(\tilde{y}, \mu; \zeta)A(\tilde{x}, \lambda; \zeta),$$

whenever $(\tilde{x}, \tilde{y}) = R(\lambda, \mu)(x, y)$. 

**Define** monodromy matrix $M = A(x_n, \lambda_n, \zeta)A(x_{n-1}, \lambda_{n-1}, \zeta) \ldots A(x_1, \lambda_1, \zeta)$. The transfer maps $T^n_i$ preserve the spectrum of $M$ for all $\zeta$. The coefficients of the characteristic polynomial $\chi = \det(A(x, \lambda, \zeta) - \mu I)$ are the integrals of the transfer-dynamics.
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Suris, AV (2003):
Suppose that the Yang-Baxter map $R(\lambda, \mu)$ has the following special form:

$$\tilde{x} = B(y, \mu, \lambda)[x], \quad \tilde{y} = A(x, \lambda, \mu)[y]$$

for some action of $GL(N)$ on $X$. Then both $A(x, \lambda, \zeta)$ and $B^T(x, \lambda, \zeta)$ are Lax matrices for $R$. 
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Indeed, LHS gives $z_{12} = A(y_1, \mu, \nu)A(x_2, \lambda, \nu)[z]$, while the RHS gives $z_{12} = A(x, \lambda, \nu)A(y, \mu, \nu)[z]$. 
For Adler map

\[ \tilde{x} = y - \frac{\lambda - \mu}{x + y} \]

\[ \tilde{y} = x - \frac{\mu - \lambda}{x + y} \]

we can write

\[ \tilde{y} = x - \frac{\mu - \lambda}{x + y} = \frac{x^2 + xy - (\mu - \lambda)}{x + y} = A(x, \lambda, \mu)[y], \]

so we come to the Lax matrix

\[ A = \begin{pmatrix} x & x^2 + \lambda - \zeta \\ 1 & x \end{pmatrix}, \]

(which was actually known from the theory of the dressing chain).
Bianchi (1880s):

**Superposition of Bäcklund transformations:**

\[
\begin{align*}
&v \quad \rightarrow \quad v_1 \\
&\downarrow \quad \quad \quad \quad \downarrow \\
&v_2 \quad \rightarrow \quad v_{12}
\end{align*}
\]
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Bianchi’s important observation was the results of these commuting transformations satisfy an algebraic relation.
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Superposition of Bäcklund transformations:

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\end{align*} \]

Bianchi's important observation was the results of these commuting transformations satisfy an **algebraic relation**.

In KdV case the Darboux transformations satisfy

\[(\nu_{12} - \nu)(\nu_1 - \nu_2) = \beta_1 - \beta_2,\]

which is the discrete KdV equation.
Discrete integrability: 3D consistency condition

Yang-Baxter versus 3D consistency condition

Figure: “Cubic” representation of the Yang–Baxter relation
Papageorgiou, Tongas, AV (2006): symmetry approach

Discrete KdV equation

$$(v_{12} - v)(v_1 - v_2) = \beta_1 - \beta_2$$

is invariant under the translation $v \rightarrow v + \text{const}$. 
Papageorgiou, Tongas, AV (2006): symmetry approach

Discrete KdV equation

\[(v_{12} - v)(v_1 - v_2) = \beta_1 - \beta_2\]

is invariant under the translation \(v \rightarrow v + \text{const.}\).

The invariants

\[x_1 = v_1 - v, \quad x_2 = v_{1,2} - v_1, \quad y_1 = v_{1,2} - v_2, \quad y_2 = v_2 - v,\]

satisfy the relation

\[x_1 + x_2 = y_1 + y_2\]

and the equation itself:

\[(x_1 + x_2)(x_1 - y_2) = \beta_1 - \beta_2.\]
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satisfy the relation

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and the equation itself:

$$(x_1 + x_2)(x_1 - y_2) = \beta_1 - \beta_2.$$ 

This leads to the following YB map

$$y_1 = x_2 + \frac{\beta_1 - \beta_2}{x_1 + x_2}, \quad y_2 = x_1 - \frac{\alpha_1 - \beta_2}{x_1 + x_2},$$

which is nothing else but the Adler map.
Weinstein and Xu (1992), Reshetikhin, AV (2005)

Suppose that $X$ can be embedded as a symplectic leaf in a Poisson Lie group $G$: $\phi_\lambda : X \to G$ and define the correspondence $R(\lambda, \mu) : X \times X \to X \times X$ by the relation

$$\phi_\lambda(x) \phi_\mu(y) = \phi_\mu(\tilde{y}) \phi_\lambda(\tilde{x}).$$
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Suppose that $X$ can be embedded as a symplectic leaf in a Poisson Lie group $G$: $\phi_\lambda : X \to G$ and define the correspondence $R(\lambda, \mu) : X \times X \to X \times X$ by the relation

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Define the symplectic structure $\Omega^{(N)}$ on $X^{(N)}$ as

$$\Omega^{(N)} = \omega_{\lambda_1} \oplus \omega_{\lambda_2} \oplus \ldots \oplus \omega_{\lambda_N}.$$ 

Then $R(\lambda, \mu)$ is a reversible Yang-Baxter Poisson correspondence and transfer dynamics is Poisson with respect to $\Omega^{(N)}$. 
Other relations: "box-ball" systems, geometric crystals

Hatayama, Hikami, Inoue, Kuniba, Noumi, Okado, Takagi, Tokihiro, Yamada (2000-): Takahashi-Satsuma "box-ball" systems and Kashiwara’s crystal theory


Yang-Baxter map:

\[ R : X \times X \rightarrow X \times X, \quad X = \mathbb{C}^n \]

\[ \tilde{x}_j = x_j \frac{P_j}{P_{j-1}}, \quad \tilde{y}_j = y_j \frac{P_{j-1}}{P_j}, \quad j = 1, \ldots, n, \]

where

\[ P_j = \sum_{a=1}^{n} \left( \prod_{k=1}^{a-1} x_{j+k} \prod_{k=a+1}^{n} y_{j+k} \right). \]

with the subscripts \( j + k \) taken modulo \( n \).
Adler, Bobenko, Suris (2004):

Quadrirational case, $X = \mathbb{C}P^1$

\begin{align*}
  &u = \alpha yP, \quad v = \beta xP, \quad P = \frac{(1 - \beta)x + \beta - \alpha + (\alpha - 1)y}{\beta(1 - \alpha)x + (\alpha - \beta)yx + \alpha(\beta - 1)y}, \quad (1) \\
  &u = \frac{y}{\alpha} P, \quad v = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y + \beta - \alpha}{x - y}, \quad (2) \\
  &u = \frac{y}{\alpha} P, \quad v = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y}{x - y}, \quad (3) \\
  &u = yP, \quad v = xP, \quad P = 1 + \frac{\beta - \alpha}{x - y}, \quad (4) \\
  &u = y + P, \quad v = x + P, \quad P = \frac{\alpha - \beta}{x - y}, \quad (5)
\end{align*}
Figure: A quadrirational map on a pair of conics
Yang-Baxter property = Geometric theorem

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Papageorgiou, Suris, Tongas, V (2009):

\[ u = yQ^{-1}, \quad v = xQ, \quad Q = \frac{(1 - \beta)xy + (\beta - \alpha)y + \beta(\alpha - 1)}{(1 - \alpha)xy + (\alpha - \beta)x + \alpha(\beta - 1)}, \]  
\[ (6) \]

\[ u = yQ^{-1}, \quad v = xQ, \quad Q = \frac{\alpha + (\beta - \alpha)y - \beta xy}{\beta + (\alpha - \beta)x - \alpha xy}, \]  
\[ (7) \]

\[ u = \frac{y}{\alpha}Q, \quad v = \frac{x}{\beta}Q, \quad Q = \frac{\alpha x + \beta y}{x + y}, \]  
\[ (8) \]

\[ u = yQ^{-1}, \quad v = xQ, \quad Q = \frac{\alpha xy + 1}{\beta xy + 1}, \]  
\[ (9) \]

\[ u = y - P, \quad v = x + P, \quad P = \frac{\alpha - \beta}{x + y}. \]  
\[ (10) \]

The last map is the Adler map.
Some open questions

- Classification

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▶ Soliton interaction ⇒ Integrable hierarchy

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- Soliton interaction $\Rightarrow$ Integrable hierarchy

- Alternative transfer-dynamics
  - Papageorgiou, AV: transfer KdV correspondences
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- Discrete hierarchies and tropicalization
  Inoue, Takenawa (2008): tropical algebraic geometry