Hypersurface Geometry

–with some applications–

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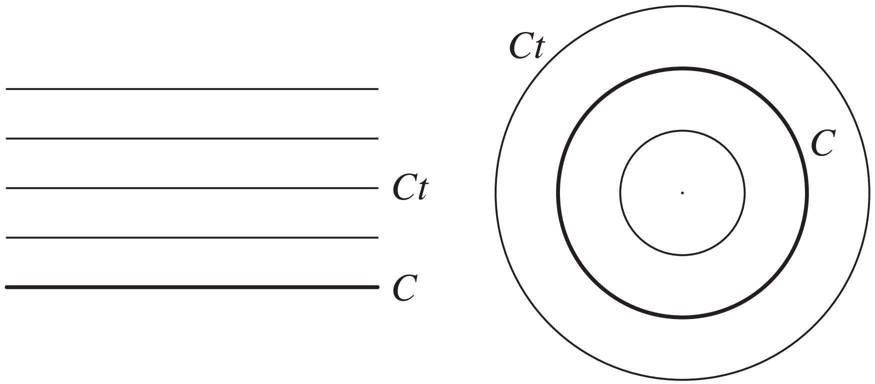
1 Introduction

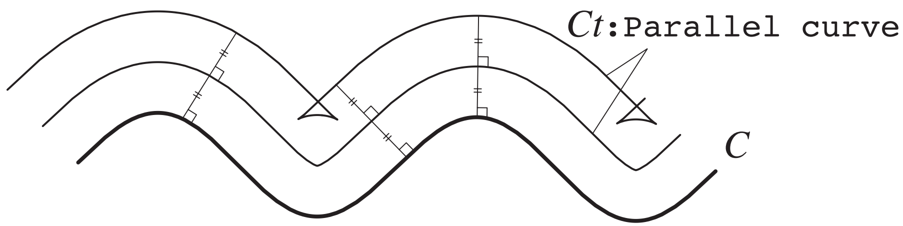
$\overline{M} = \mathbb{R}^n, H^n, S^n$

M: an embedded hypersurface in $\overline{M}$

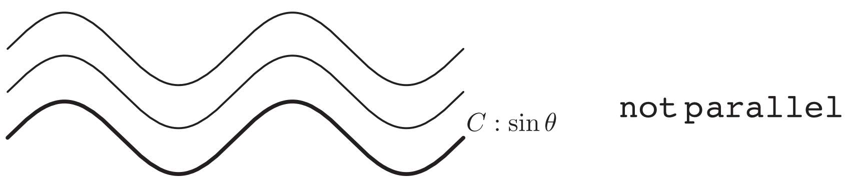
We consider M as a wave front developping with time.



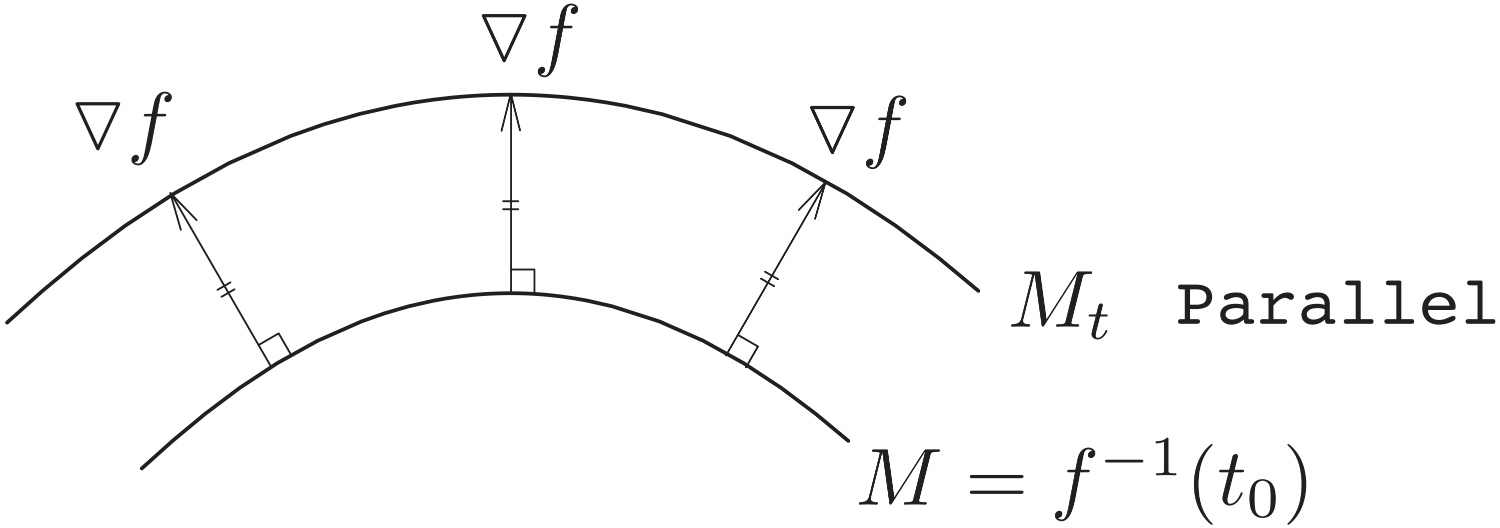
Ct
C
Ct
C



Ct: Parallel curve
C

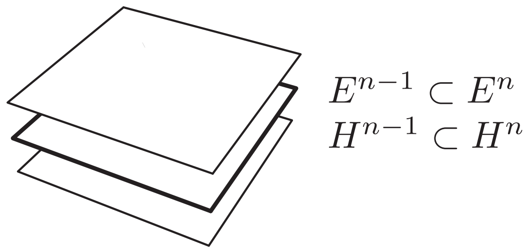


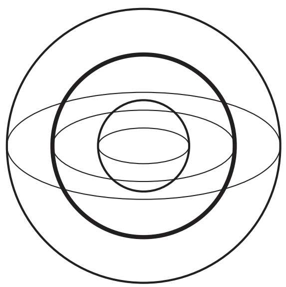
C : sin θ
not parallel

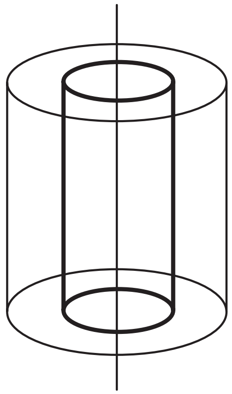

∇f
∇f
∇f
M_t Parallel
$M = f^{-1}(t_0)$

Singularities occur in general.

When all parallel hypersurfaces are regular ?







Historical origin:

Geometric Optics (Laura, Somigliana, '18)

$\Phi_t(p) = \{x \in \mathbb{R}^3 \mid \text{the light from } p \text{ arrives at } x \text{ in time } t\}$

M: a surface in $\mathbb{R}^3$.

$M_t = \{x \in \mathbb{R}^3 \mid \text{the light from } p \in M \text{ arrives at } x \text{ in time } t\}$

Huygens principle: M_t = the envelope of $\Phi_t(p)$

If the velocity is constant, M_t is parallel to $M = M_0$.

$\Rightarrow \{M_t\}$ a family of parallel (local) hypersurfaces.

Q. When all M_t are regular?

Definition. $f : \overline{M} \to \mathbb{R}$ is an isoparametric function

$\Leftrightarrow$ (i) $|\nabla f|^2 = p(f)$ (ii) $\triangle f = q(f)$

where p, q are functions defined on the range of f in $\mathbb{R}$.

A level set $M_t = f^{-1}(t)$ satisfies:

(i) $\Rightarrow$ $|\nabla f|$ is constant along M_t $\Rightarrow$ M_t are mutually parallel.

(ii) $\Rightarrow$ $\triangle f$ is constant along M_t $\Rightarrow$ each M_t has constant mean curvature (CMC).

Definition.

A level set M_t of an isoparametric function is called

an isoparametric hypersurface if t is a regular value,

a focal submanifold (regular) if t is a critical value.

Remark. The regularity of each M_t follows from (i), if $p(f)$ is of class C^2 (**Q. M. Wang, '87**).

Theorem 1. (É. Cartan '37) Let $\{M_t\}$ be a family of parallel hypersurfaces. Then

$$\{M_t : \text{isoparametric hypersurface}\}$$
$$\Leftrightarrow \text{All } M_t \text{ are CMC}$$
$$\Leftrightarrow \text{Some } M_t \text{ has constant principal curvatures}$$

Remark. This is remarkable because a local notion induces a global notion.

Examples: {homogeneous h'surfaces in $\overline{M}$ = orbits of certain subgroup of $\mathrm{ISO}(\overline{M})$} $\subset^*$ {isoparametric h'surfaces}

$\overline{M}$	M^{n-1}		
$\mathbb{R}^n$	$\mathbb{R}^{n-1}$ or S^{n-1}	$\mathbb{R}^k \times S^{n-k-1}$	–
H^n	H_{eq}, H_0 or S^{n-1}	$H_{eq}^k \times S^{n-k-1}$	–
S^n	S^{n-1}	$S^k \times S^{n-k-1}$	more

H_{eq} : an equidistant h's, H_0 : a horosphere.

In the cases $\mathbb{R}^n$ and H^n, the equality holds in $*$. (Cartan).

In the case S^n, all homogeneous h'surfaces are classified by Hsiang-Lawson ('71), however, not all isoparametric h'surfaces are classified yet.

Let $\overline{M} = S^n$ **from now on**.

Let g be the number of distinct principal curvatures of an isoparametric hypersurface.

Cartan hypersurfaces: $g = 3$

Theorem 2. (Cartan '38) Isoparametric hypersurfaces with $g = 3$ are given by tubes over the standard embedding of the projective planes $\mathbb{F}P^2$ in S^4, S^7, S^{13} and S^{25}, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathcal{C}$ (Cayley numbers).

They are called the Cartan hypersurfaces, and denoted by $C_{\mathbb{F}}^{3d}$, $d = 1, 2, 4, 8$. They are all homogeneous.

Remember $C_{\mathbb{C}} \cong SU(3)/T$, where T is the maximal torus of $SU(3)$.

Theorem 3. (Münzner, '81) An isoparametric function f on S^n is given by the restriction of a homogeneous polynomial $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ of degree g, satisfying

$$\begin{cases} |DF|^2 = g^2 r^{2g} & (1) \\ \triangle F = c r^{g-2}, \quad r = |x| & (2) \end{cases}$$

for a constant $c = (m_2 - m_1)g^2/2$, and

$$f = F|_{S^n} : S^n \to [-1, 1]$$

F is called a <u>Cartan-Münzner Polynomial</u>.

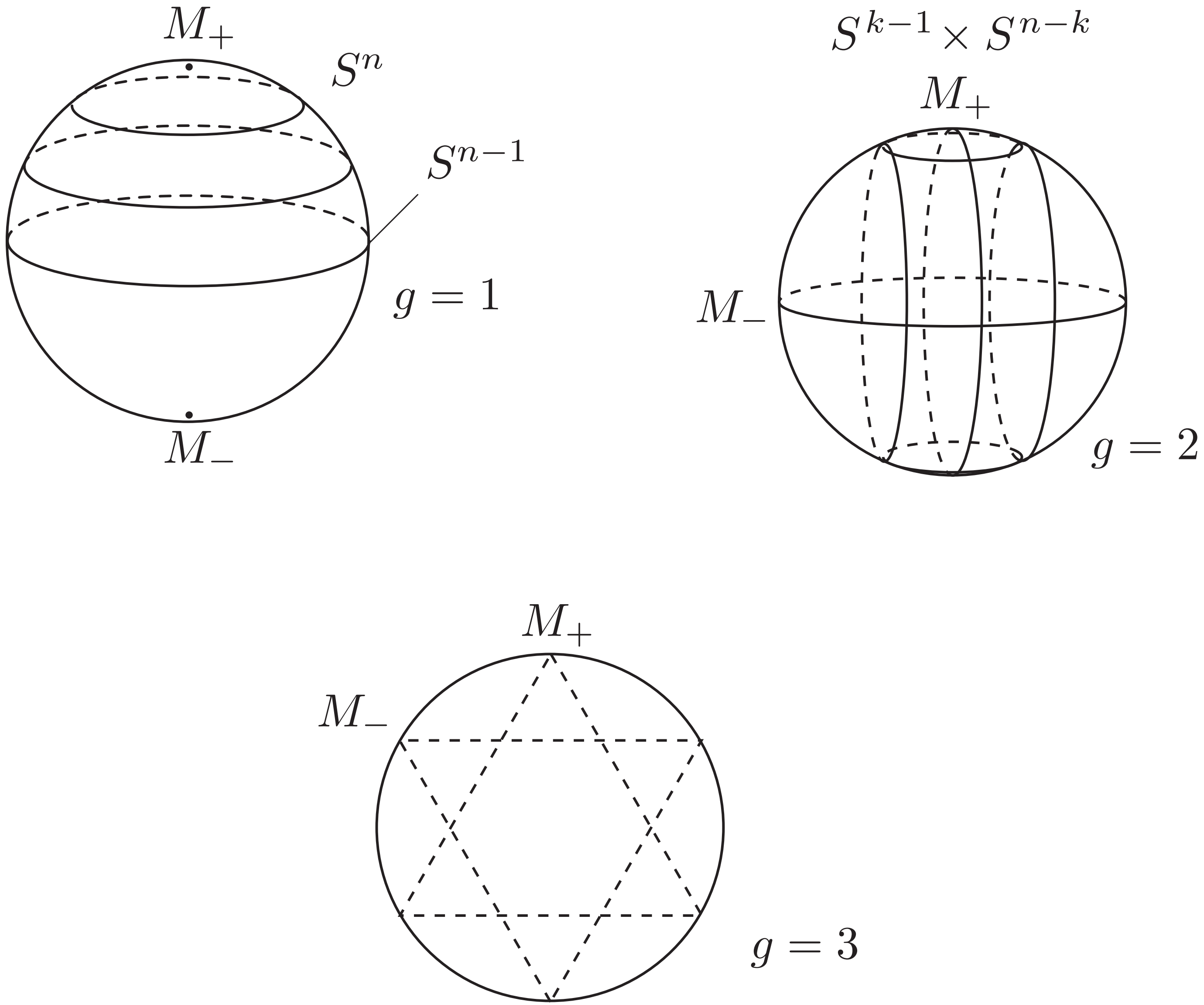
e.g. $g = 1 : F(x) = x_{n+1}$

$g = 2 : F(x) = \sum_{i=1}^{k+1} x_i^2 - \sum_{j=2}^{n-k+1} x_{k+j}^2$

$M_t = f^{-1}(t)$: isoparametric hypersurface for $t \in (-1, 1)$
$M_\pm = f^{-1}(\pm 1)$: focal submanifolds

• M_t sweeps out $S^n = \cup_{t\in[-1,1]} M_t$, i.e., S^n has a singular foliation.

• Moreover, S^n is decomposed into two disk bundles $B_+ \cup B_-$, where $B_\pm \to M_\pm$ and $B_+ \cap B_- = M_t$.



M_+
S^n
S^{n-1}
$g = 1$
M_-
$S^{k-1} \times S^{n-k}$
M_+
M_-
$g = 2$
M_+
M_-
$g = 3$

Theorem 4. (Münzner, '81)

(1) $g \in \{1, 2, 3, 4, 6\}$,

(2) For principal curvature $\lambda_1 > \cdots > \lambda_g$, let m_i be the multiplicity of λ_i. Then $m_i = m_{i+2}$ follows.

$g \leq 3 \Rightarrow$ all homogeneous (Cartan)

Q. Are all isoparametric hypersurfaces in S^n homogeneous?

2 Non-homogeneous examples: $g = 4$

Theorem 5. **(Ozeki-Takeuchi, '76)** There exist infinitely many non-homogeneous isoparametric hypersurfaces with $g = 4$ in S^n.

Method: Construction of Cartan-Münzner polynomials of degree 4 by using the representation of certain Clifford algebras.

Theorem 6. (Ferus-Karcher-Münzner, '81) O-T method can be generalized into any Clifford algebras.

We call these hypersurfaces of OT-FKM type.

Remark. Some of OT-FKM type are homogeneous, and others are non-homogeneous.

Theorem 7. (Cecil-Chi-Jensen, '07, Immervoll, '08) Isoparametric hypersurfaces with $g = 4$ are either homogeneous or of OT-FKM type, except for the cases $(m_1, m_2) = (3, 4), (4, 5), (6, 9), (7, 8)$.

Open Problem. Classify the remaining cases.

3 Isoparametric h'surfaces : $g = 6$

Theorem 8. (Abresch, '83) When $g = 6$, $m_i = m \in \{1, 2\}$.

For each case there is a homogeneous example:

$m = 1$: isotropy orbits N_t^6 of $G_2/SO(4)$ in S^7.

$m = 2$: isotropy orbits M_t^{12} of $G_2 \times G_2/G_2$ in S^{13}.

Geometric properties of these orbits:

Theorem 9. (M. '93) The homogeneous hy'surface N^6 with $(g, m) = (6, 1)$ is given by $\pi^{-1}(C_{\mathbb{R}})$ where $C_{\mathbb{R}}$ is the Cartan h'surface in S^4, and $\pi : S^7 \to S^4$ is the Hopf map. Thus $N^6 \cong C_{\mathbb{R}} \times S^3$.

Theorem 10. (M. '08) The homogeneous h'surface M^{12} with $(g, m) = (6, 2)$ has a Kähler fibration $\pi : M \to S^6$ with fiber the Cartan h'surface $C_{\mathbb{C}} = SU(3)/T^2$.

Classification in the case $g = 6$:

Theorem 11. (Dorfmeister-Neher, '85, M. '09) Isoparametric h's with $(g, m) = (6, 1)$ are homogeneous, i.e., the $SO(4)$ orbits.

Theorem 12. (M. '09) The isoparametric h's with $(g, m) = (6, 2)$ are homgeneous, i.e., the G_2 orbits.

Key Lemma. (M, '93, '98) Isoparametric h's with $g = 6$ are homogeneous $\Leftrightarrow$ the shape operators of a focal submfd have the kernel indep. of the normal directions.

Classification of isoparametric h'surfaces in S^n

g	1	2	3	4*	6
M	S^{n-1}	$S^k \times S^{n-k-1}$	$C_{\mathbb{F}}$	homogeneous or of OT-FKM type	N^6, M^{12}

* some exceptions.

4 Application 1: Special metrics

Theorem 13. (N.Koiso, '81) Every real analytic Riemannian manifold M with constant scalar curvature can be embedded into certain Einstein manifold $\overline{M}$ as a totally geodesic hypersurface.

Basically, $\overline{M}$ is given by $M \times \mathbb{R}$ with an Einstein metric obtained by solving an ODE. Here, M is a h's of $\overline{M}$.

• 1-parameter family of h's is a nice tool to find certain metrics.

Construction of special metrics

• Special metrics mean metrics with special holonomy, Ricci flat (Kähler) metrics, etc.

• Such metrics are often constructed on a vector bundle over a Riemannian manifold with nice properties (Einstein etc).

Examples:

◦ Bryant-Salamon's G_2-metrics and Spin(7)-metrics.

◦ Lü-Page-Pope metrics

Idea: $\pi : Y \to M$: a vector bundle

$\Rightarrow Y = \cup_{r \geq 0} X_r$ where X_r is the sphere bundle $X_r \to M$ consisting of fiber vectors of constant length r.

- X_r is a hypersurface of Y for $r > 0$.

If there is a nice metric g_r on each X_r, we may obtain a nice metric allover Y (solving certain ODE w.r.t r).

Special holonomy: (by Berger, ‘55)

Hol(g)	$U(n)$	$Sp(n)$	G_2	$Spin(7)$
structure	Calabi-Yau	Hyperkähler	G_2	$Spin(7)$
M	M^{2n}	M^{4n}	M^7	M^8

(the first two: complex geometry, well investigated)

the last two : firstly, metrics are constructed by Bryant (‘87), and complete metrics by Bryant-Salamon (‘89), both are in an explicit way.

$\Rightarrow$ important in physics, e.g. treated in M-theory by Atiyah-Witten.

Bryant-Salamon's G_2-metric

$Y^7 = \Lambda^2_-(M) \to M$: the ASD bundle of $M = S^4$ or $\mathbb{C}P^2$.

Theorem 14. (BS, '89) For a constant $\lambda > 0$, a metric on Y^7

$$g_\lambda = (\lambda + r^2)^{1/2} g_b + \frac{1}{(\lambda + r^2)^{1/2}} g_f$$

is a complete Ricci flat metric with $\mathrm{Hol}(g) = G_2$, where g_b and g_f are metrics of the base and the fibers, respectively.

A complete G_2 metric is also constructed on the spin bundle $Y^7 = \mathcal{S}$ over S^3.

Let

$$X_r^6 = \{\text{fiber vectors of length } r\} \subset Y^7,$$

then

$$\begin{aligned} X_r \cong \quad & \mathbb{C}P^3 && \text{for } Y^7 = \Lambda^2_-(S^4) \\ & \underline{SU(3)/T^2} && \text{for } Y^7 = \Lambda^2_-(\mathbb{C}P^2) \\ & \underline{S^3 \times S^3} && \text{for } Y^7 = \mathcal{S} \end{aligned}$$

Remark. BS metrics are homogeneous on X_r, and hence a cohomogeneity one metric on Y^7 (S. Salamon's talk in the Winter School 2002.)

Relation with isoparametric h'surfaces

$S^3 \times S^3$ and $SU(3)/T^2$ appear as isoparametric h's in $\underline{S^7}$.

However, the metric used by Bryant-Salamon is completely **different** from that of the isoprametric hypersurfaces.

In the case $SU(3)/T^2$, the former is non-Kähler Einstein metric, while the latter is Kähler non-Einstein.

We have a topological correspondence between Y^7 and a part of S^7.

The topology of $\Lambda^2_-(\mathbb{C}P^2)$

Recall:

$$Y = \Lambda^2_-(\mathbb{C}P^2) = \cup_{r\geq 0} X_r, \qquad X_r \cong C_{\mathbb{C}},\ r > 0.$$

On the other hand, the Cartan h'surfaces $C_{\mathbb{C}} \cong M_t$ and two focal submanifolds $M_{\pm} \cong \mathbb{C}P^2$ give a singular foliation $S^7 = \cup_{t\in[-1,1]} M_t$ by a 1- parameter family, via the theory of isoparametric h'surfaces.

Thus we obtain

$$\Lambda^2_-(\mathbb{C}P^2) \cong \cup_{r\geq 0} X_r \cong \cup_{t\in[-1,1)} M_t \cong S^7 \setminus \mathbb{C}P^2.$$

because

$$X_0 \cong \mathbb{C}P^2 = M_- \text{ and } S_\infty \cong \mathbb{C}P^2 = M_+,$$

by identifying X_r with M_t where

$$t = \frac{r-1}{r+1}, \quad r \geq 0$$

In the case of the spin bundle $\mathcal{S}$ over S^3, $X_r = S^3 \times S^3$, which can be identified with the isoparametric family $\{M_t\}$ in S^7, and we obtain

Theorem 15. **[M, '05]**

$$\boxed{\begin{array}{ccc} \mathcal{S} & \cong & S^7 \setminus S^3 \\ \Lambda^2_-(\mathbb{C}P^2) & \cong & S^7 \setminus \mathbb{C}P^2 \end{array}}$$

Recall open Calabi-Yau problem

$\overline{M}$: compact Kähler mfd with Ricci> 0,

D : a suitable divisor

Bando-Kobayashi [BK], Tian-Yau [TY] obtain a complete Ricci flat Kähler Einstein metric on $\overline{M} \setminus D$.

Real version:

Construct a complete Ricci-flat, non-flat metric on a manifold $M = \overline{M} \setminus D$, where $\overline{M}$ is a compact Riemannian manifold with positive Ricci curvature, and D is some submanifold of M.

In particular,

For each isoparametric family $\{M_t\}$ in S^n, does there exist a complete Ricci flat, non-flat metric on $S^n \setminus M_\pm$?

Theorem 16. **[Lü-Page-Pope, 2004]** There exists a complete Ricci flat metric on $S^m \times \mathbb{R}^{n+2}$ for any $n, m \geq 1$ (generalization of Taub-NUT metric).

Since $S^m \times \mathbb{R}^{n+2} = \cup_{r \geq 0} S^m \times S^{n+1}(r)$, and $S^m \times S^{n+1}(r)$ is identified with an isoparametric h'surface in S^{m+n+1}, where $S^m = M_-$ and $S^{n+1} = M_+$, we obtain

Corollary 17. On $S^{n+m+1} \setminus S^{n+1}$, for any $n, m \geq 1$, there exists a complete Ricci flat metric.

5 Application 2: Calibrated geometry

(M, g) : Riemannian manifold

$\varphi \in \Omega^p$: a closed p-form is a calibration $\Leftrightarrow$ for any p-plane T in TM,

$$\varphi(T) \leq 1$$

N : a p-dimensional submanifold of M is calibrated

$\Leftrightarrow \varphi(T_x N) = 1$ at any $x \in N$

($\Rightarrow$ N is volume minimizing in the same homology class)

Example

(1) Complex submanifolds N^{2p} of a Kähler manifold M,

$\varphi = \omega^p/(p!)$, $\quad \omega$: Kähler form of M

(2) Special Lagrangian submanifolds N^p of a Calabi-Yau manifold M ($\Leftrightarrow$ Ricci-flat Kähler),

$\varphi = \Re(e^{i\theta}\Omega)$, $\quad \Omega$: the holomorphic $(n, 0)$ volume form on M

Special Lagrangian submanifolds:

$\mathbb{C}^{n+1}$ has a calibration: $\varphi = \Re(e^{i\theta} dz_0 \wedge \cdots \wedge dz_n)$

Def. A submanifold N of a Riemannian mfd is **austere**

$\Leftrightarrow$ any shape operators of N have eigenvalues in pairs $\{\pm\lambda_j\}$, and the multiplicities of $\pm\lambda_j$ coincide.

N : austere in $S^n \Rightarrow$ the cone over N : austere in $\mathbb{R}^{n+1}$

Example. (i) minimal surface = austere surface.

(ii) a complex submanifold of a Kähler manifold

Theorem 18. [Ishikawa-Kimura-M. 2002]

(i) Minimal isoparametric h'surfaces with principal curvatures having the same multiplicity are austere, namely,

$$M_1 = S^{n-1}, \quad M_2 = S^{(n-1)/2} \times S^{(n-1)/2} \quad (n: \text{ odd})$$
$$M_3 = C_{\mathbb{F}}, \, (m = 1, 2, 4, 8), \quad M_4^{4m}, \quad M_6^{2m}, \, (m = 1, 2)$$

where M_g denotes an isoparametric h'surface with g principal curvatures.

(ii) The focal submanifolds of **any** isoparametric hypersurfaces are austere.

Remark. (i) : all homogeneous. (ii) includes both homogeneous and non-homogeneous ones.

Theorem 19. **[Harvey-Lawson, 1982]** The conormal bundle of the cone of an <u>austere</u> submanifolds in S^n is a special Lagrangian submanifold of $\mathbb{C}^{n+1} = T^*\mathbb{R}^{n+1}$.

Theorem 20.[Kariggianis and Min-Oo, 2004] The conormal bundle of <u>austere</u> submanifolds in S^n is a special Lagrangian submanifold of T^*S^n with the Stenzel metric.

Stenzel metric : a generalization of Eguchi-Hanson metric on T^*S^2 (Ricci flat Kähler).

Corollary 21. **[IKM]**

(i) The conormal bundle of the cone over submanifolds given in Theorem 18 are special Lagrangian submanifolds of $\mathbb{C}^{n+1}$.

(ii) The conormal bundle of the submanifolds given in Theorem 18 are special Lagrangian submanifolds of T^*S^n with the Stenzel metric.

6 Brezis' question

Q. (Brezis, 1999) Let $u : \mathbb{R}^N \to \mathbb{R}^N$ be a solution of the Ginzburg-Landau system:

$$\triangle u = u(|u|^2 - 1) \quad N \geq 3,$$

with $|u(x)| \to 1$ as $|x| \to \infty$. Assume $\deg(u, \infty) = \pm 1$. Does u have the form

$$u(x) = \frac{x}{|x|} h(|x|),$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function, such that $h(0) = 0$ and $h(\infty) = 1$?

Counterexamples to this question were constructed by Farina (2004) and Ge-Xie (2009), by using Cartan-Münzner polynomials.

In fact, consider the Cartan-Münzner polynomial

$$F : \mathbb{R}^N \to \mathbb{R} \quad \text{and put} \quad \Phi = \frac{\nabla F}{g} : \mathbb{R}^N \to \mathbb{R}^N$$

When $m_1 = m_2$, F satisfies

$$\text{(i) } |DF|^2 = g^2 |x|^{2g-2}, \quad \text{(ii) } \triangle F = 0$$

Then putting

$$u(x) = \Phi\left(\frac{x}{|x|}\right) h(|x|)$$

and solving h so that $u(x)$ be a solution to the Ginzburg-Landau system, we obtain conterexamples to Brezis' question if $(g, m) = (6, 1)$: Farina, 2004, and

$(g, m) = (4, 1)$: Ge-Xie, 2009

Remark. No other Cartan-Münzner polynomials give counterexamples (Ge-Xie, because of the degree condition).

7 Other directions

I. Hamiltonian stability of Lagrangian submfds:

Theorem 22. **[B.Palmer]** The image of the Gauss map $G : M \to Q^{n-1}$ of an isoparametric hypersurface M in S^n, defined by $G(p) = p \wedge \xi_p$ where ξ_p is the unit normal to M at p, is a Lagrangian submanifold of the complex hyperquadric $Q^{n-1} \cong \mathrm{Gr}_2^+(n+1, \mathbb{R})$ (1997).

• Hamiltonian stability of $G(M)$ is studied by **M. Hui and Y. Ohnita** (2009) in the homogeneous cases.

II. Moment maps

Theorem 23. [S. Fujii, '09] For homogeneous hypersurfaces with $g = 4$ obtained by the isotropy action of a Hermitian symmetric space of classical type, the corresponding Cartan-Münzner polynomial is given by a square norm of the moment map of this action.

III. Relation with integrable systems:

Theorem 24. **[Ferapontov, 95]** Homogeneous hypersurfaces in S^n correspond to completely integrable n-wave systems.

Related works: Dubrovin, Novikov, Tsarëv, etc.

Q. How we can say about the non-homogeneous OT-FKM type?

Concluding remarks

Hypersurface geometry is important in itself, and also in relation with various fields, as a tool constructing special metrics, special Lagrangian submanifolds, counterexamples of Brezis' question, etc.

In particular, isoparametric h'surfaces are interesting, as they include both homogeneous and non-homogeneous cases.

One of the most interesting problems is to investigate non-homogeneous OT-FKM type hypersurfaces from a view point of group actions.

Thank you for your attention.

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