Snapback repellers and border collision bifurcations

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Introduction

Aim:

- What and why border collision bifurcations?
- How to prove systems are chaotic
- Border collisions: normal forms and chaos
- border collisions and chaos: snapback repellers

Bringing together two fundamental ideas in
continuous but noninvertible discrete dynamical systems

Border collisions: what?

Suppose that phase space (think of the plane) is divided into open connected regions separated by codimension one hyperplanes (boundary) $\partial\Sigma$ and such that

- in each region the dynamics is defined by a smooth invertible map depending smoothly on parameters;
- the maps are continuous across $\partial\Sigma$ but derivatives are not;

then a border collision bifurcation occurs at a parameter value if $\partial\Sigma$ intersects the boundary of an invariant set (e.g. fixed point, periodic orbit,...) of the system.

Examples in hybrid systems, models of friction.... see Chris Budd's lectures.

Border collision: interesting example

The fully chaotic skew tent map is:

$$f(x) = \begin{cases} sx & \text{if } 0 < x \leq s^{-1} \\ \frac{s}{s-1}(1-x) & \text{if } s^{-1} \leq x < 1 \end{cases}$$

with $s > 1$. Now define a system by

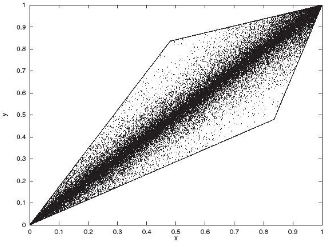
$$\begin{aligned} x_{n+1} &= (1-\varepsilon)f(x_n) + \varepsilon f(y_n) \\ y_{n+1} &= \varepsilon f(x_n) + (1-\varepsilon)f(y_n) \end{aligned}$$

with $(x,y) \in [0,1] \times [0,1]$. Studied by Pikovsky and Grassberger (1991) and their conjectures later proved by Glendinning (2001) – blowout bifurcations (cf. Milnor attractors and Prof Kaneko's talk).

Unit square divided into four regions by the lines $x = s^{-1}$ and $y = s^{-1}$ on which the dynamics is linear; and the diagonal line (synchronized state) $x = y$ is invariant and on this line the dynamics is given by the skew tent map itself.

Example of dynamics: $s = 1.8$

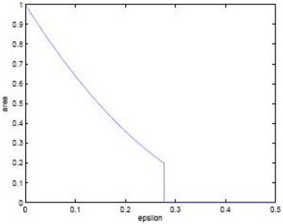
For small ε ($\varepsilon = 0.24$ is shown) typical trajectories fill a lozenge:



For larger ε most trajectories tend to the synchronized state although the lozenge persists (containing a dense set of periodic orbits), and at still larger ε ($\varepsilon \geq 1/2s$) the lozenge disappears and the synchronized state (the diagonal) attracts all solutions.

Creation of the lozenge

Figure shows the volume of the lozenge as a function of the parameter ε with $s = 1.8$.



It jumps!

How? A plethora of border collision bifurcations (including some novel ones we are still in the process of analyzing) immediately after the initial loss of transverse stability of the synchronized state create the required orbits. (Ivan's thesis!)

Mechanisms to create chaos (invertible case)

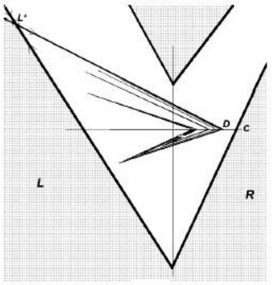
Two-dimensional invertible dynamics: the basic mechanism is the creation of a horseshoe, and the simplest way to do this is via a homoclinic orbit where

a homoclinic orbit is an orbit which approaches the same fixed point (or periodic orbit) of the map in both forwards and backwards time.

In other words it is in the intersection of the stable and unstable manifolds of the fixed point (or periodic orbit) leading to a homoclinic tangle.

Mechanisms to create chaos (non-invertible case): I

Can of course have the same essential mechanism as the invertible case: a fixed point with one dimensional stable and unstable manifolds which intersect.



(from Banerjee, Ranjan and Grebogi, 2000)

But there is another way....

Snapback repellers

Suppose a fixed point is unstable (has a two-dimensional local unstable manifold in $\mathbb{R}^2$). If the system is invertible, then no orbit can tend to this point in forwards time. But if it is non-invertible this becomes possible.

1D Example: $x_{n+1} = T(x_n) = 4x_n(1 - x_n)$ $x = 0$ is fixed and unstable, but $T(\frac{1}{2}) = 1$ and $T(1) = 0$, and there exists (choosing preimages of the left branch) points $y_n \to 0$ such that $T^n(y_n) = \frac{1}{2})$.

It is no coincidence that this occurs when the map is fully chaotic!

Marotto, 1978, 2005

Marotto defined a snapback repeller of a non-invertible map F on $\mathbb{R}^2$ to be a fixed point $\mathbf{p}$ such that

1. the eigenvalues $s_\pm$ of the Jacobian at $\mathbf{p}$ satisfy $|s_+| \geq |s_-| > 1$;
2. there is a point $\mathbf{x}_0 \neq \mathbf{p}$ such that $F(\mathbf{x}_0) = \mathbf{p}$; and
3. there exists a sequence $\mathbf{x}_i$ in which tends to $\mathbf{p}$ as $i \to \infty$ such that $F(\mathbf{x}_{i+1}) = \mathbf{x}_i$, $i = 1, 2, 3, \ldots$ and $F(\mathbf{x}_1) = \mathbf{x}_0$.

Theorem: If F has a snapback repeller then there exists a chaotic invariant set.

Several recent papers make this clearer, more precise and more general (heteroclinic loops).

Sketch Proof

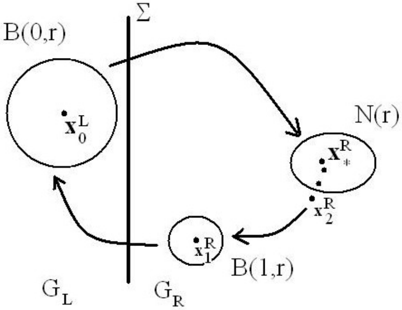


Figure: The geometry of a simple snap-back repeller.

Normal form (Nusse and Yorke, 1992)

Suppose that a fixed point strikes $\partial\Sigma$ at parameter $\mu = 0$. Aim to work with leading order terms (constants and linear): choose coordinates so that $\partial\Sigma$ is the $y-$axis, $x = 0$, then assuming no degeneracies the map takes the form (with $\mathbf{x} = (x, y)$)

$$\mathbf{x}_{n+1} = \begin{cases} A_L \mathbf{x}_n + \mathbf{m}_L & \text{if} \quad x_n \leq 0 \\ A_R \mathbf{x}_n + \mathbf{m}_R & \text{if} \quad x_n \geq 0 \end{cases}$$

locally.

Impose continuity on $x = 0$

$$\mathbf{x}_{n+1} = \begin{cases} A_L \mathbf{x}_n + \mathbf{m} & \text{if} \quad x_n \leq 0 \\ A_R \mathbf{x}_n + \mathbf{m} & \text{if} \quad x_n \geq 0 \end{cases}$$

(i.e. $\mathbf{m}_L = \mathbf{m}_R = \mathbf{m}$) and

$$\text{if } A_L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \; \textit{then } A_R = \begin{pmatrix} a & B \\ c & D \end{pmatrix}$$

Linear transform keeping $x = 0$ invariant, shift y, $\mu = 0$ has border collision

$$\mathbf{x}_{n+1} = \begin{cases} A_L \mathbf{x}_n + \mathbf{m} & \text{if} \quad x_n \leq 0 \\ A_R \mathbf{x}_n + \mathbf{m} & \text{if} \quad x_n \geq 0 \end{cases}$$

$\mathbf{m} = (\mu, 0)^T$ and

$$A_L = \begin{pmatrix} T_L & 1 \\ -D_L & 0 \end{pmatrix} \text{ and } A_R = \begin{pmatrix} T_R & 1 \\ -D_R & 0 \end{pmatrix}$$

This is the normal form for the standard border collision bifurcation. (Note that by rescaling can set $\mu \in \{-1, 0, 1\}$.)

Simple Remarks

Fixed Points

$$x_*^\alpha = \frac{\mu}{1 - T_\alpha + D_\alpha}, \quad y_*^\alpha = -D_\alpha x_*^\alpha, \quad \alpha = L, R$$

and $\mathbf{x}_*^R$ exists provided $x_*^R > 0$, with a similar inequality for the existence of $\mathbf{x}_*^L$.

Image of left/right half plane

If $x_n > 0$ then $y_{n+1} = -D_R x_n$. So if $D_R > 0$ (resp. $D_R < 0$) then the image of the right half plane is the lower (resp. upper) half plane.
If $x_n < 0$ then $y_{n+1} = -D_L x_n$. So if $D_L > 0$ (resp. $D_L < 0$) then the image of the left half plane is the upper (resp. lower) half plane.

Local non-invertibility if $D_L D_R < 0$.

Area contraction

If $|D_\alpha| < 1$, $\alpha = R, L$, then both maps contract areas and so we expect to see stable objects. Sequences of papers in the late 1990s and early 2000s by the Maryland group (particularly Banerjee, Yorke and Grebogi), and Bristol/Bath established a basic pattern of results including

- 'no bifurcation': border crossing;
- 'saddlenode' style bifurcation;
- fixed point to stable periodic orbit;
- fixed point to many periodic orbits;
- Lozi-type chaotic attractor ('Robust Chaos')

Area expansion

The possibility that (e.g.) $D_R > 1$ might be interesting had not been considered – possibly because it was thought either uninteresting or could be obtained from the modulus less than one case by reversing time. The example at the start shows it might be interesting, and it certainly cannot be obtained by time reversal.

Snapback repellers

Fix

$$D_R > 1, \quad D_L < 0$$

and

$$T_R > 2, \quad T_R^2 > 4D_R, \quad 1 - T_R + D_R > 0$$

which ensures that there is an unstable node in $x > 0$ for $\mu > 0$. Since $D_L < 0$ this fixed point has a preimage in $x < 0$ and it is then a case of ensuring this in turn has a preimage in $x > 0$ and then that the backward orbit of this latter point remains in $x > 0$, where it tends to the fixed point. We obtain some tiresome inequalities which are sufficient for the existence of a snapback repeller and hence chaos!

The tiresome inequalities

Suppose that

$$D_R > 1, \quad D_L < 0, \quad T_R > 2, \quad T_R^2 > 4D_R, \quad 1 - T_R + D_R > 0$$

as before. If, in addition,

$$T_R D_L - T_L D_R - D_L D_R > 0$$

(which ensures that the preimage in $x < 0$ of the fixed point in $x > 0$ has $y < 0$) and

$$s_+ D_R(D_R - D_L) + (s_+ T_R - D_R)(T_R D_L - T_L D_R) \geq 0$$

where s_+ is the larger eigenvalue of A_R (which ensures that the preimages converge back to the fixed point in $x > 0$) **then there is a snapback repeller in the normal form** if $\mu > 0$.

Application to the Pikovsky-Grassberger example

One of the simplest border collisions in this example is for orbits of period three. If $s = 1.8$ and $\varepsilon = 0.18$, then there are two period 3 orbits, one with a point close to $x = s^{-1} \approx 0.555$ at $(0.553, 0.737)$ and the other with a point at $(0.559, 0.736)$. As ε increases these tend to the boundary, one from the left and the other from the right, and there is a border collision at $\varepsilon \approx 0.1845$. For the third iterate of the map these are repelling fixed points, and (again for the third iterate) $T_L \approx -9.83$, $D_L \approx 21.77$, $T_R \approx -1.44$ and $D_R \approx -27.21$.

In this case we checked analytically that the fixed point (of the third iterate) in $x < s^{-1}$ is a snapback repeller.

The proof of the pudding...

We therefore expect to see orbits with each iterate in $x > s^{-1}$ separated by several iterates in $x < s^{-1}$.

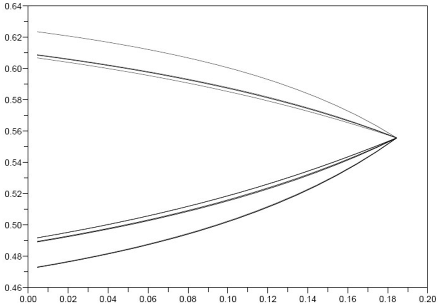


Figure: Bifurcating orbits in the (ϵ, x) plane for the third iterate of the map for the border collision with $s = 1.8$. Except for the period three orbit in $x < a^{-1}$, each orbit has one point in $x > s^{-1}$ and n in $x < s^{-1}$; orbits with $n = 0, 1, 2, 3, 4$

Conclusion

- interesting bifurcations in non-invertible/hybrid systems (border collision);
- snapback repellers a way of proving existence of complicated trajectories;
- snapback repellers occur in (area expanding) border collisions.