An Introduction to Dynamical Systems and Fractals: 2

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Overview

In yesterday's talk we covered the first two parts of the plan

- Some properties of the Cantor set
- Ways to characterise fractals
- Hyperbolic iterated function systems (IFSs)
- Semi-infinite strings of symbols seen as a metric space
- Topological equivalence with the Cantor set
- Non-hyperbolic IFSs
- Digital forcing/controlling of an inverted pendulum

Now let's move on to discuss iterated function systems.

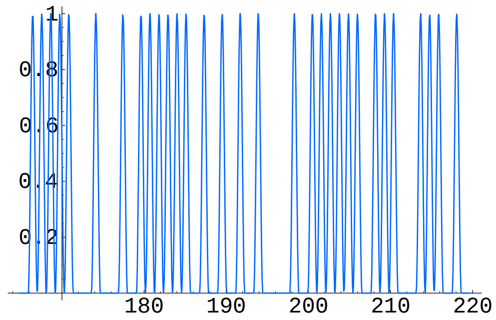
A Simple Digital Channel

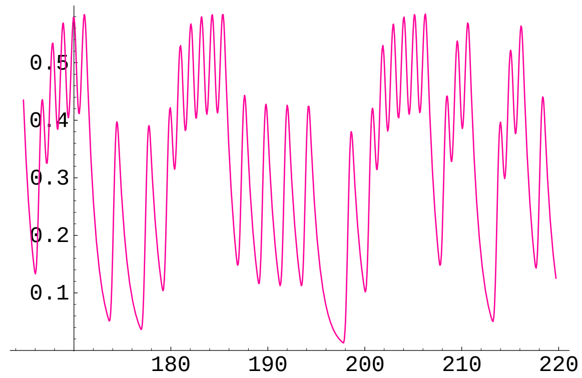
- First order system driven by a clocked random pulse sequence:

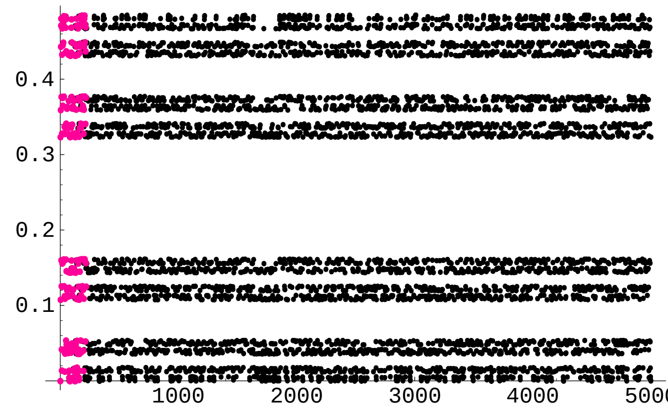
$$\frac{dy}{dt} = -\gamma y + \xi(t)$$

$$\xi(t) = \sum_p a_p g(t - p\tau)$$

- $\{a_p\} \in \{0,1\}^{\mathbb{N}}$ are the input symbols
- g is supported on $(0, \tau)$.
- Symbols input at constant rate, τ^{-1}.
- In the examples, g is a raised cosine.
- In the numerical example $\gamma\tau = \log 3$
- Plotting the samples $\{y_n \stackrel{\text{def}}{=} y(n\tau)\}$ gives a Cantor set.







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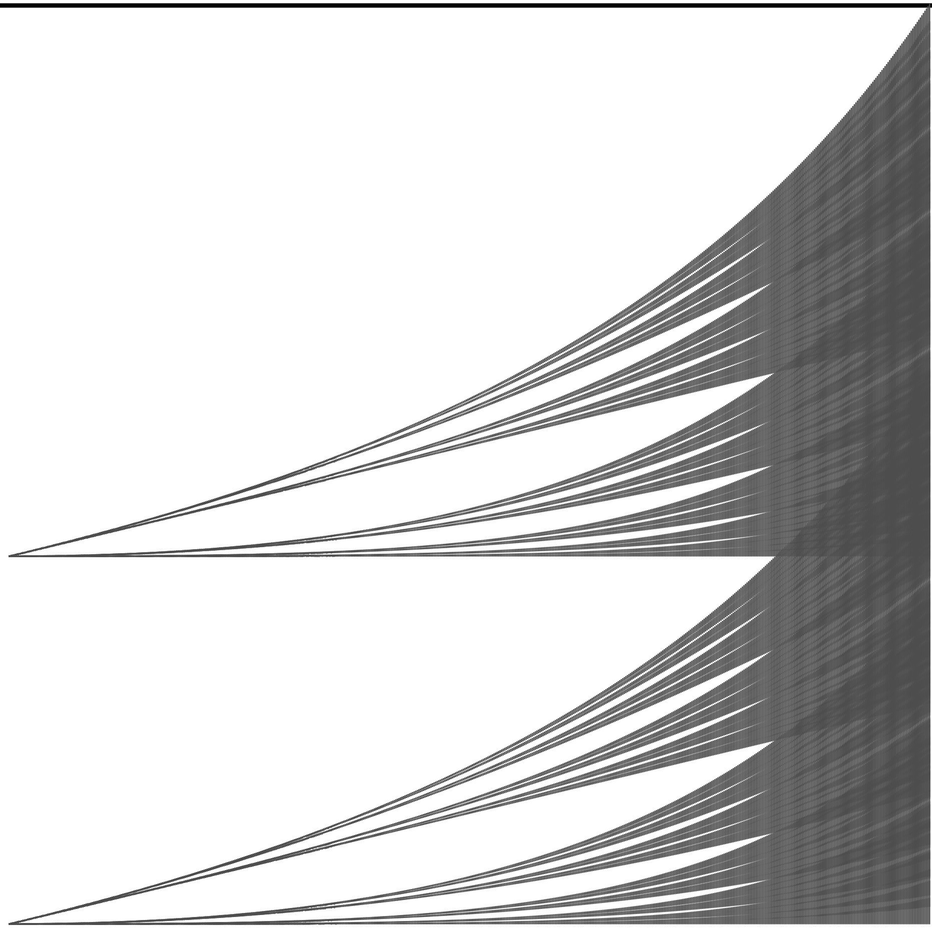
- The sampled output is given by random composition
- For a random sequence $\{a_0, \ldots, a_p, \ldots\} \in \{0,1\}^{\mathbb{N}}$ the sequence of output values is $\{y_1, y_2, y_3, \ldots\}$ where:

$$
\begin{aligned}
y_1 &= f_{a_0}(y_0) \\
y_2 &= f_{a_1}(y_1)
\end{aligned}
$$

and so on

- Generally, $y_n = f_{a_{n-1}} \circ \ldots \circ f_{a_1} \circ f_{a_0}(y_0)$

Parameter dependence



- The channel example: rescale $y \mapsto y/b$ to get:

$$\{y \mapsto \lambda y, y \mapsto \lambda y + 1 : \lambda = e^{-\gamma\tau} \in [0, 1)\}$$

- Plot: random iterates (plotted vertically) for each λ

The gasket examples

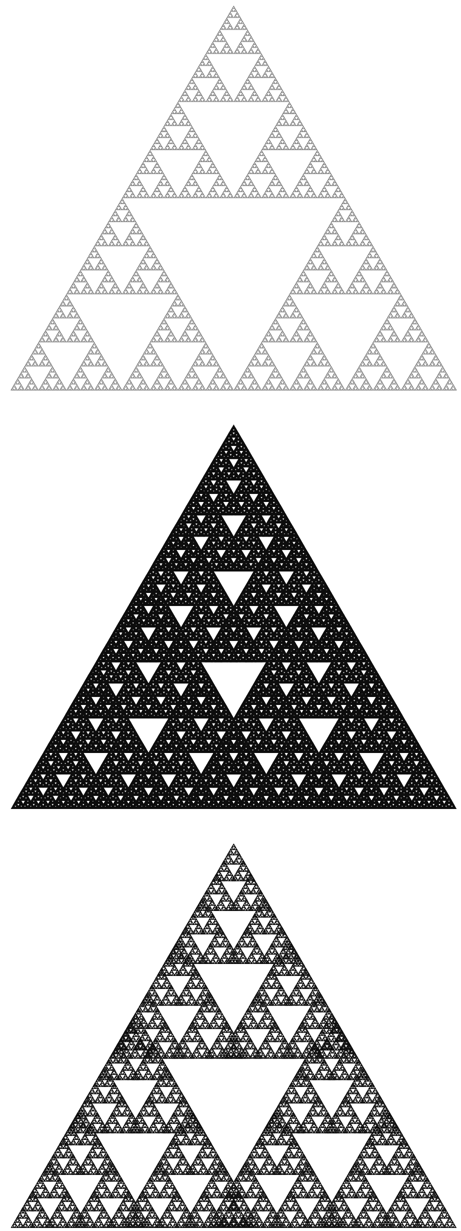
- The gasket examples can be thought of as IFSs
- Let's call the set of vertices of the triangle $P = \{p_1, p_2, p_3\}$. Then we have three maps that can be applied:

$$x \mapsto f_i(x) = s(x - p_i) + p_i$$

- The random walk is given by random composition of these.
- For a random choices of the vertices $\{i_0, \ldots, i_n, \ldots\} \in P^{\mathbb{N}}$ the sequence of point visited in the plane is $\{x_1, x_2, x_3, \ldots\}$ where:

$$\begin{aligned} x_1 &= f_{i_0}(x_0) \\ x_2 &= f_{i_1}(x_1) \end{aligned} \qquad \text{and so on}$$

- So that $x_2 = f_{i_1} \circ f_{i_0}(x_0)$ etc.



The symbol space

- We have been speaking of spaces $\Sigma = \mathcal{A}^{\mathbb{N}}$ which consists of (all) semi-infinite strings of symbols from $\mathcal{A}$, a finite alphabet.
- This can be made into a metric space, say, by introducing

$$d(\mathsf{a}, \mathsf{a}') = 2^{-\rho(\mathsf{a}, \mathsf{a}')}$$

where $\rho(\mathsf{a}, \mathsf{a}')$ is the maximum length of substrings starting at the left, over which the strings a and a' agree.

- It will be natural to think of the left shift dynamics $\sigma : \Sigma \to \Sigma$ on this space
- We will also be interested in the inverse of this the right shift

$$\sigma^{-1}\mathsf{a} = \{\sigma_a^{-1}\mathsf{a} = (a, \mathsf{a}) | a \in \mathcal{A}\}$$

- For the application, think of a string a as the history of inputs to the system.

Hyperbolic Iterated Function Systems

- Let $\mathcal{C}$ be the set of nonempty compact subsets of a complete metric space, $(\mathbf{X}, \mathbf{d})$
- Define the map $F : \mathcal{C} \to \mathcal{C}$

$$F(U) = \bigcup_{a=1}^{|\mathcal{A}|} f_a(U)$$

$\mathcal{A}$ is a finite alphabet and the maps $\{f_a : a \in \mathcal{A}\}$ act on $\mathbf{X}$.

- If each of the $\{f_a : a \in \mathcal{A}\}$ is a contraction mapping:

$$\mathbf{d}(f_a(x) - f_a(x')) \leq c_a \mathbf{d}(x - x') \quad \text{with } c_a < 1 \quad \text{for every } x, x' \in \mathbf{X}$$

- Then there is a set K which is the unique fixed point of F

$$K = \bigcup_{a=1}^{|\mathcal{A}|} f_a(K)$$

- For any $U \in \mathcal{C}$, $F^k(U) \to K$ in the Hausdorff metric.

Addressing K

- In the case of two symbols $\mathcal{A} = \{0, 1\}$, K satifies

$$K = f_0(K) \cup f_1(K)$$

- The maps are contractions so $f_0(K)$ and $f_1(K)$ are two smaller copies of K
- We can rewrite the Ks on the right hand side

$$K = f_0(f_0(K)) \cup f_0(f_1(K)) \cup f_1(f_0(K)) \cup f_1(f_1(K))$$

- Now K is seen as the union of 4 even smaller copies of itself
- The process is sometimes called backward iteration
- Consider the taking the following limit of this process

$$\bigcap_{n \geq 0} f_{a_0} \circ f_{a_1} \circ \ldots \circ f_{a_n}(K)$$

- This converges uniformly to a single point in K which depends only on the infinite sequence $(a_0, a_1, \ldots) \in \Sigma$.
- This defines a continuous surjection $\pi : \Sigma \to K$.

IFSs with overlaps

- In the case of two symbols $\mathcal{A} = \{0, 1\}$, K satifies

$$K = f_0(K) \cup f_1(K)$$

- All points in $f_0(K)$ have "0" as the first symbol in their addresses, and all points in $f_1(K)$ have "1" as their first symbol
- If $f_0(K) \cap f_1(K) \neq \emptyset$ then there are points which have at least two addresses, one beginning with "0" and one beginning with "1".
- Digital channel example

$$\begin{aligned} y &\mapsto f_0(y) = \lambda y \\ y &\mapsto f_1(y) = \lambda y + b \end{aligned}$$

- If $f_0([0, b/(1-\lambda)]) \cap f_1([0, b/(1-\lambda)]) \neq \emptyset$ then $K = [0, b/(1-\lambda)]$ and so we have overlap
- Condition for overlap is then $b \leq \lambda b/(1-\lambda) \iff \lambda \geq 1/2$

The channel example and Cantor sets

- Digital channel example

$$\begin{aligned} y &\mapsto f_0(y) = \lambda y \\ y &\mapsto f_1(y) = \lambda y + b \end{aligned}$$

- If $\lambda < 1/2$ we have the strong separation condition $f_0(K) \cap f_1(K) = \emptyset$
- This implies the open set condition.
- It is also true that $\pi : \Sigma \to K$ is homeomorphism
- The middle thirds Cantor set is therefore topologically conjugate to (Σ, d)
- So we can find the Hausdorff dimension of K using the result for self-similar sets satisfying the open set condition:

$$\dim_H(K) = \dim_B(K) = \frac{\log 2}{\log \lambda^{-1}} < 1$$

- So for all $\lambda < 1/2$ the attractor K is totally disconnected
- All these attractors are therefore conjugate to the Cantor set.

Forward iteration

- The iteration process represented by

$$x_n = f_{a_{n-1}} \circ \ldots \circ \; f_{a_1} \circ f_{a_0}(x_0)$$

is known as forward iteration

- Think of $\mathsf{a} = (a_{-1}, a_{-2}, a_{-3} \ldots) \in \Sigma$ as the history of symbols input to the channel .
- We are currently at $x = \pi(\mathsf{a}) = \bigcap_{n \geq 1} f_{a_{-1}} \circ f_{a_{-2}} \circ \ldots \circ f_{a_{-n}}(K)$
- If the next input is $a_0 \in \mathcal{A}$, so we move to $\mathsf{a}' = (a_0, \mathsf{a}) \in \sigma^{-1}\mathsf{a}$
- And in **x** we move to $x' = f_{a_0}(x)$ by forward iteration
- But $x' = \pi((a_0, \mathsf{a})) = \bigcap_{n \geq 0} f_{a_0} \circ f_{a_{-1}} \circ \ldots \circ f_{a_{-n}}(K)$
- We have a (semi-)conjugacy between the IFS dynamics on K and the right shift on Σ

$$f_{a_0} \circ \pi = \pi \circ \sigma_{a_0}^{-1}$$

where $\sigma_{a_0}^{-1}(\mathsf{a}) = (a_0, \mathsf{a})$ is a possible value of the right shift.

Conclusion

We have completed the first 5 parts of the plan:

- Some properties of the Cantor set
- Ways to characterise fractals
- Hyperbolic iterated function systems (IFSs)
- Semi-infinite strings of symbols seen as a metric space
- Topological equivalence with the Cantor set
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