# An Introduction to Dynamical Systems and Fractals: 1

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**and the University of Manchester.**

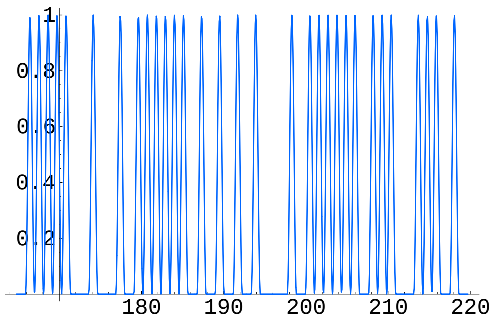
# A Simple Digital Channel



- First order system driven by a clocked random pulse sequence:

$$\frac{dy}{dt} = -\gamma y + \xi(t)$$

$$\xi(t) = \sum_p a_p g(t - p\tau)$$

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- $g$ is supported on $(0, \tau)$.
- Symbols input at constant rate, $\tau^{-1}$.
- In the examples, $g$ is a raised cosine.
- In the numerical example $\gamma\tau = \log 3$

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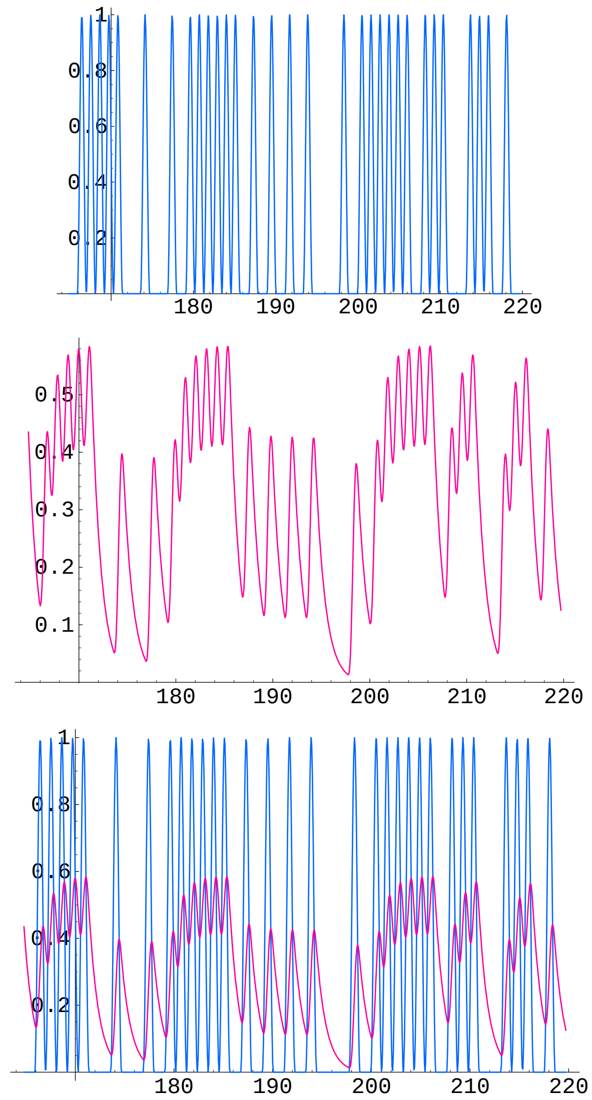

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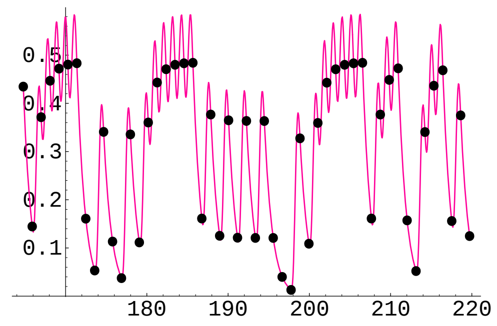

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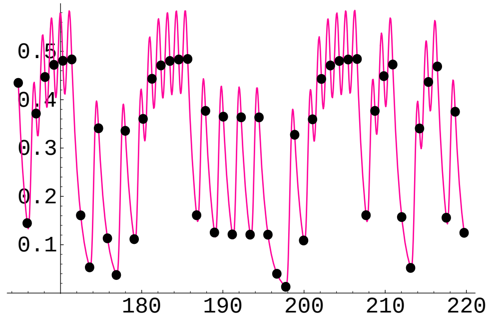


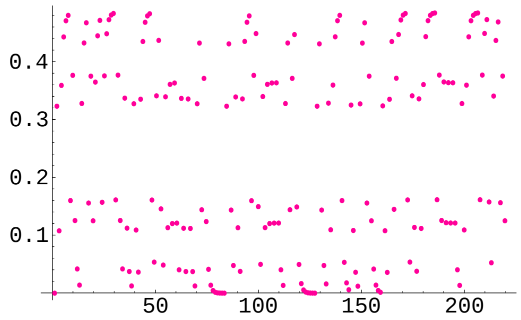

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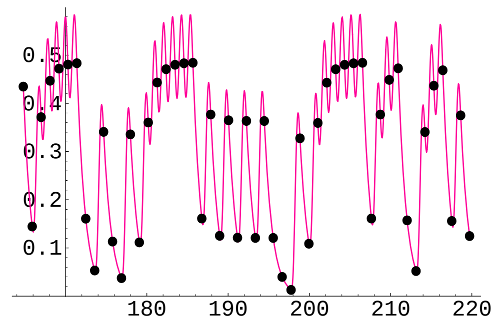
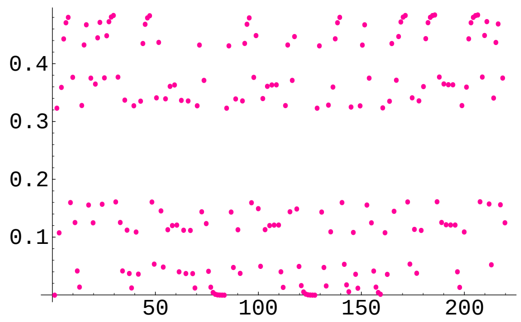
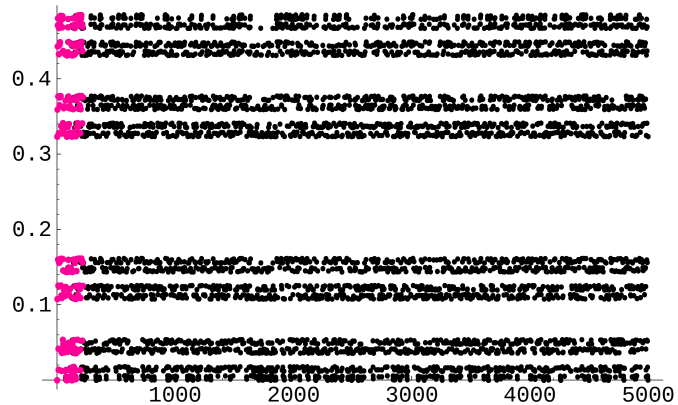



- This becomes clearer with more data
- Apparently, we have sampled a Cantor set.



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- Digital forcing/controlling of an inverted pendulum

# Cantor's (rather small) set

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- This implies that $l(C) \leq (2/3)^j$ for any $j \in \mathbb{N}$

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- For example it contains $1/4 = .\overline{02}_3$

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- A switch $2 \leftrightarrow 0$ in the $k$th digit of $x$ gives $y = x \pm 2 \cdot 3^{-k}$. By construction, $y \in C$ and is in the $\epsilon$-neighbourhood of $x$.

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- Thus $C$ is *nowhere dense* in $[0, 1]$.

# Summary



- In the sense of measure, the Cantor set is very small since it has zero length.
- Its cardinality is, however, that of the unit interval.
- It is an example of a perfect set—a topological property
- And yet, topologically, it is sparse in the sense that it is a nowhere dense subset of the unit interval.

- Let's consider now what makes the Cantor set a basic example in fractal geometry.

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- This property is known as *self-similarity*
  $\longrightarrow$ $C$ is the union of two similar copies of itself.

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- A *similarity* is a transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ with the following property:

$$\|S(x) - S(y)\| = c\|x - y\| \quad x, y \in \mathbb{R}^n$$

for some fixed positive $c$

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- A *similarity* is a transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ with the following property:

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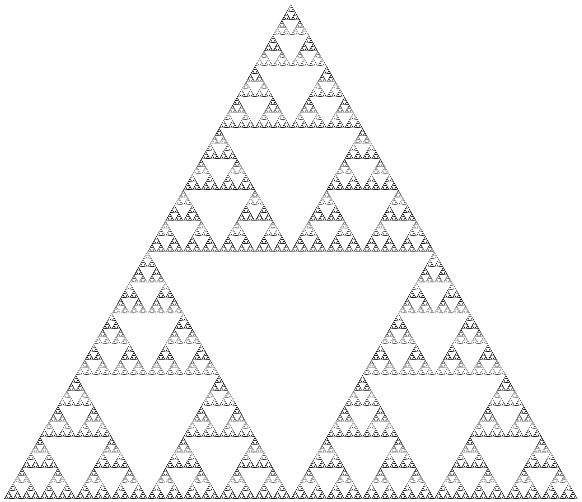
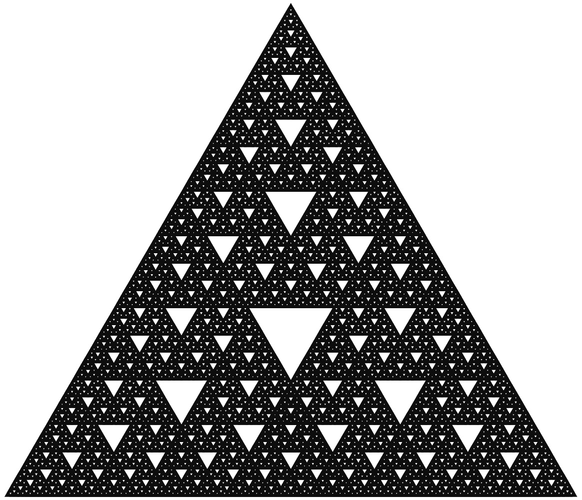
- Many examples—like the middle thirds Cantor set—are fractal sets, having structure on arbitrary small scales





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- The value of $\mathcal{H}^s(K)$ when $s = \dim_H(K)$ may be 0 or $\infty$ or something in between.

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- The upper and lower box-counting dimensions of $K$ are:

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- The small overlap idea is captured by the open set condition. There should exist a bounded, non-empty, open set $V$ such that

$$\bigcup_{i=1}^{N} S_i(V) \subset V$$

with the union disjoint.

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- We showed that $C = S_0(C) \cup S_1(C)$ with

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- This illustrates a general result that a set with Hausdorff dimension less than unity is totally disconnected.

# Conclusion



We have probably overdosed on the middle thirds Cantor set.

but in doing so we have completed the first two parts of the plan:

- Some properties of the Cantor set
- Ways to characterise fractals
- Hyperbolic iterated function systems (IFSs)
- Semi-infinite strings of symbols seen as a metric space
- Topological equivalence with the Cantor set
- Non-hyperbolic IFSs
- Digital forcing/controlling of an inverted pendulum