Polar Actions on Symmetric Spaces

Jürgen Berndt
King’s College London

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Polar representations

- $H$ compact connected Lie group acting on $V$ real vector space with $H$-invariant inner product
- $\pi : H \rightarrow O(V)$ representation
- $v \in V$, $\Sigma_v \subset V$ cross-section of action at $v$
- $\Sigma_v$ minimal $\iff$ $\dim H \cdot v$ maximal

**Definition.** $\pi : H \rightarrow O(V)$ **polar** if all orbits intersect a minimal cross-section orthogonally

**Examples.**
- standard representation $\pi : SO_2 \rightarrow O(\mathbb{R}^2)$ is polar
- $M = G/K$ Riemannian symmetric space, $o \in M$ with $K \cdot o = o$, isotropy representation $\pi : K \rightarrow O(T_o M)$ is polar

**Dadok 1985:** Polar representations on $\mathbb{R}^n$ are orbit equivalent to isotropy representations of Riemannian symmetric spaces
Polar actions

$M$ connected Riemannian manifold, $H \subset \Gamma(M)$ connected subgroup

**Definition.** The action of $H$ on $M$ is **polar** if there exists a connected closed submanifold $\Sigma$ of $M$ such that

- $\forall p \in M : \Sigma \cap H \cdot p \neq \emptyset$
- $\forall p \in \Sigma : T_p \Sigma \subset \nu_p(H \cdot p)$

Such a submanifold $\Sigma$ is called a **section** of the action.

**Fact.** Sections are *totally geodesic* submanifolds

**Definition.** A polar action is **hyperpolar** if it admits a flat section.

**Problem.** Classification of polar actions on Riemannian symmetric spaces

$S^n$ and $\mathbb{RH}^n$: apply Dadok’s result
Compact symmetric spaces

Podestà, Thorbergsson 1999: Classification of polar actions on projective spaces

Kollross 2002: Classification of hyperpolar actions on irreducible Riemannian symmetric spaces of compact type and rank $\geq 2$

Every polar action on an irreducible Riemannian symmetric spaces of compact type and rank $\geq 2$ is hyperpolar

- Podestà-Thorbergsson 2002: $SO_{n+2}/SO_nSO_2$, $n \geq 3$
- Biliotti-Gori 2005: $SU_{n+k}/S(U_nU_k)$, $n \geq k \geq 2$
- Biliotti 2006: Hermitian symmetric spaces
- Kollross 2007: Simple isometry group
- Kollross 2009: $G_2$, $F_4$, $E_6$, $E_7$, $E_8$
- Lytchak 2011: Cohomogeneity is $\geq 3$
- Kollross-Lytchak 2011: Cohomogeneity is 2
Compact vs noncompact

Some observations:

- **Cohomogeneity one actions**: Every Riemannian symmetric space of noncompact type admits cohomogeneity one actions (not true for compact type)

- **Polar and hyperpolar actions**: Every Riemannian symmetric space of noncompact type admits polar actions which are not hyperpolar (not true for compact type and higher rank)

- Concept of *duality* between symmetric spaces of compact type and of noncompact type is useful only for special situations, e.g. actions by algebraic reductive subgroups (*Kollross 2011*)

- In the compact case one can restrict to actions of compact groups (well understood!), whereas in the noncompact case one needs to consider noncompact groups (not well understood!)
Current state of affairs

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Joint work with

- José Carlos Díaz-Ramos (Santiago de Compostela)
- Hiroshi Tamaru (Hiroshima)
Polar foliations of complex hyperbolic spaces

- $\mathbb{CH}^n = SU_{n,1}/S(U_n U_1) = G/K$
- $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ restricted root space decomposition
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ Iwasawa decomposition, $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$
- $\mathbb{CH}^n = AN$ solvable Lie group with left-invariant metric
- $V = \{0\}$ or $V = a$; $\mathfrak{w} \subset \mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ real subspace
- $\mathfrak{s}_{V,w} = (a \ominus V) \oplus (n \ominus w)$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$
- $S_{V,w}$ corresponding subgroup of $AN$

Berndt-DiazRamos 2012:

- The orbits of $S_{V,w}$ form a homogeneous polar foliation of $\mathbb{CH}^n$
- Every homogeneous polar foliation of $\mathbb{CH}^n$ is holomorphically congruent to one of these foliations
Proof relies on following result (Gorodski 2004 for compact case):

Let $M = G/K$ be a Riemannian symmetric space of noncompact type and $H$ be a connected closed subgroup of $G$ whose orbits form a regular foliation $\mathcal{F}$ of $M$. Consider the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and define

$$\mathfrak{h}_p^\perp = \{ \xi \in \mathfrak{p} : \langle \xi, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h} \}.$$ 

Then the action of $H$ on $M$ is polar if and only if

- $\mathfrak{h}_p^\perp$ is a Lie triple system in $\mathfrak{p}$, and
- $\mathfrak{h}$ is orthogonal to the subalgebra $[\mathfrak{h}_p^\perp, \mathfrak{h}_p^\perp] \oplus \mathfrak{h}_p^\perp$ of $\mathfrak{g}$.

In this case, let $H_p^\perp$ be the connected subgroup of $G$ with Lie algebra $[\mathfrak{h}_p^\perp, \mathfrak{h}_p^\perp] \oplus \mathfrak{h}_p^\perp$. Then the orbit $\Sigma = H_p^\perp \cdot o$ is a section of the $H$-action on $M$. 
The case of codimension one

- horosphere foliation
- foliation with exactly one minimal leaf $S = \text{ruled real hypersurface associated to a horocycle in a totally geodesic } \mathbb{R}H^2 \subset \mathbb{C}H^n$
Polar actions on $\mathbb{C}H^2$

- $N$ horosphere in $\mathbb{C}H^2$; $n = g_\alpha \oplus g_{2\alpha}$; $N$ is a 3-dim Heisenberg group
- $S$ ruled real hypersurface in $\mathbb{C}H^2$ generated by a horocycle in $\mathbb{R}H^2 \subset \mathbb{C}H^2$; $s = a \oplus g^{R}_\alpha \oplus g_{2\alpha}$
- $N \cap S$ is a Euclidean plane $\mathbb{E}^2$ embedded in $N$ as a minimal surface and in $\mathbb{C}H^2$ as a real surface with nonzero constant mean curvature; $n \cap s = g^{R}_\alpha \oplus g_{2\alpha}$

**Berndt-DiazRamos 2012:** Every polar action on $\mathbb{C}H^2$ is orbit equivalent to the action of the invariance group of one of the following geometric objects in $\mathbb{C}H^2$:

- Cohom 1: $\{ o \}$, $\mathbb{C}H^1$, $\mathbb{R}H^2$, $N$, $S$
- Cohom 2: $\{ o \} \subset \mathbb{C}H^1$ (full flag), $\mathbb{R}H^1$, horocycle in $\mathbb{C}H^1$, $\mathbb{E}^2$
Outline of proof

- Possible cohomogeneity is 1 or 2
- Cohomogeneity 1: known by earlier work
- Assume cohomogeneity 2
- 0-dimensional orbit: group is compact and action has a fixed point, only possibility is $S(U_1U_1U_1)$
- 1-dimensional orbit, no fixed point: Lie-theoretical arguments, technical
- regular foliation: known by earlier work
The general setting

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  - $G$ noncompact semisimple real Lie group
  - $K$ maximal compact subgroup of $G$
  - $o \in M$ with $K \cdot o = o$
- $H$ connected closed subgroup of $G$ acting on $M$ polarly
Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\alpha$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left( \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

- $\Lambda$ set of simple roots for $\Sigma$
- $\Phi$ subset of $\Lambda$, $\Sigma_\Phi = \Sigma \cap \text{span}\{\Phi\}$
- $l_\Phi = \mathfrak{g}_0 \oplus \left( \bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right)$, $n_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\alpha$
- $l_\Phi$ reductive subalgebra, $n_\Phi$ nilpotent subalgebra
- $q_\Phi = l_\Phi \oplus n_\Phi$ parabolic subalgebra (Chevalley decomposition)
- Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to $q_\Phi$ for some subset $\Phi \subset \Lambda$
Parabolic subalgebras (II)

- \( l_\Phi = m_\Phi \oplus a_\Phi \) with \( a_\Phi \) split component of \( l_\Phi \)
  - \( m_\Phi \) reductive subalgebra, \( a_\Phi \) abelian subalgebra
- \( q_\Phi = m_\Phi \oplus a_\Phi \oplus n_\Phi \) (Langlands decomposition)
- \( M_\Phi \cdot o = B_\Phi \) semisimple symmetric space with rank equal to \( |\Phi| \), totally geodesic in \( M \), boundary component of \( M \) with respect to maximal Satake compactification
- \( A_\Phi \cdot o = \mathbb{E}^{r-|\Phi|} \) Euclidean space, totally geodesic in \( M \)
- \( L_\Phi \cdot o = F_\Phi = B_\Phi \times \mathbb{E}^{r-|\Phi|} \) totally geodesic in \( M \)
- \( M = B_\Phi \times \mathbb{E}^{r-|\Phi|} \times N_\Phi \) (horospherical decomposition)
- The action of \( N_\Phi \) on \( M \) is polar
- The action of \( N_\Phi \) on \( M \) is hyperpolar \( \iff \Phi = \emptyset \)
Examples of hyperpolar foliations

- $V$ linear subspace of $\mathbb{E}^m$
  $\implies \mathcal{F}_V^m = \{ p + V \mid p \in \mathbb{E}^m \}$ homogeneous hyperpolar foliation of $\mathbb{E}^m$

- $F \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \}$, $M = G/K = FH^n$
  $s = a \oplus (g_\alpha \ominus \ell) \oplus g_{2\alpha}$, $\ell$ line in $g_\alpha$
  $\implies \mathcal{F}_F^n$ homogeneous codimension one foliation of $\mathbb{F}H^n$ with unique minimal leaf

\[ \mathcal{F}_{F_1}^{n_1} \times \cdots \times \mathcal{F}_{F_k}^{n_k} \times \mathcal{F}_V^m \text{ homogeneous hyperpolar foliation of } \mathbb{F}_1H^{n_1} \times \cdots \times \mathbb{F}_kH^{n_k} \times \mathbb{E}^m \]
Examples of hyperpolar foliations (II)

- $M = G/K$ symmetric space of noncompact type
- $\Phi$ orthogonal set of simple roots, $k = |\Phi|$  
- $q_\Phi = m_\Phi \oplus a_\Phi \oplus n_\Phi$ Langlands decomposition of parabolic subalgebra $q_\Phi$ of $g$
- $F_\Phi \cong \underbrace{F_1 H^{n_1} \times \cdots \times F_k H^{n_k}}_{M_\Phi \cdot o} \times \underbrace{E^{r-k}}_{A_\Phi \cdot o}$
- $F_{F_1}^{n_1} \times \cdots \times F_{F_k}^{n_k} \times F_V^{r-k}$ homogeneous hyperpolar foliation of $F_\Phi$
- $F_{\Phi, V} = F_{F_1}^{n_1} \times \cdots \times F_{F_k}^{n_k} \times F_V^{r-k} \times N_\Phi$ homogeneous hyperpolar foliation of $M = F_\Phi \times N_\Phi$
- $F_{\emptyset, \{0\}}$ horocycle foliation of $M$
Classification of homogeneous hyperpolar foliations

Berndt-DiazRamos-Tamaru 2010: Let $M$ be a symmetric space of noncompact type. Every homogeneous hyperpolar foliation on $M$ is isometrically congruent to $\mathcal{F}_{\Phi, V}$ for some orthogonal set $\Phi$ of simple roots and some linear subspace $V \subset \mathbb{R}^{r-|\Phi|}$. 
The symmetric space $SL_{r+1}(\mathbb{R})/SO_{r+1}$

- Dynkin diagram

\[
\begin{array}{c}
\circ & \circ & \circ \\
\alpha_1 & \alpha_2 & \alpha_{r-1} & \alpha_r
\end{array}
\]

- $\Phi \subset \Lambda = \{\alpha_1, \ldots, \alpha_r\}$ orthogonal, $k = |\Phi|$

- horospherical decomposition:

$SL_{r+1}(\mathbb{R})/SO_{r+1} \cong \mathbb{R}H^2 \times \ldots \times \mathbb{R}H^2 \times \mathbb{E}^{r-k} \times N_\Phi$

$k$ factors

- $N_\Phi$ corresponds to the set of all upper block diagonal matrices with certain $2 \times 2$ and $1 \times 1$ diagonal blocks, diagonal entries are 1
The symmetric space $SL_{r+1}(\mathbb{R})/SO_{r+1}$

- horospherical decomposition:
  $SL_{r+1}(\mathbb{R})/SO_{r+1} \cong \mathbb{R}H^2 \times \ldots \times \mathbb{R}H^2 \times \mathbb{E}^{r-k} \times N_\Phi$

- On each $\mathbb{R}H^2$ select the foliation

- On $\mathbb{E}^{r-k}$ select a foliation by parallel affine subspaces

- On $N_\Phi$ select the foliation with one leaf $N_\Phi$

- The product foliation is hyperpolar, and every homogeneous hyperpolar foliation of $SL_{r+1}(\mathbb{R})/SO_{r+1}$ arises in this way