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ON HOMOGENEOUS HYPERSURFACES IN RIEMANNIAN SYMMETRIC SPACES

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1. Introduction

The aim of this note is to present a survey about the classification problem of homogeneous hypersurfaces in Riemannian symmetric spaces. Let (M,g) be a Riemannian manifold and $I(M,g)$ its isometry group. A *homogeneous submanifold* of (M,g) is a connected submanifold N of M which is an orbit of some closed subgroup G of $I(M,g)$. If the codimension of N is one, then N is called a *homogeneous hypersurface*. Suppose N is a homogeneous hypersurface of M. Then there exists some closed subgroup G of $I(M,g)$ having N as an orbit. Since the codimension of N is one, the regular orbits of the action of G on M have codimension one, that is, the action of G on M is of cohomogeneity one. Thus the classification of homogeneous hypersurfaces is equivalent to the classification of cohomogeneity one actions up to orbit equivalence. Therefore we start with some general remarks about cohomogeneity one actions. I would like to thank Andreas Kollross for explaining me details of his results in an early stage.

2. The orbit structure of cohomogeneity one actions

Let (M,g) be a complete Riemannian manifold and G a closed subgroup of $I(M,g)$ acting on M with cohomogeneity one. We equip the orbit space M/G with the quotient topology relative to the canonical projection $M \to M/G$. Then M/G is a one-dimensional Hausdorff

space homeomorphic to the real line $\mathbb{R}$, the circle S^1, the half-open interval $[0,\infty)$, or the closed interval $[0,1]$. This was proved by Mostert [20] in the case G is compact and by Bérard-Bergery [2] in the general case. The following basic examples illustrate these four cases. Consider a one-parameter group of translations in $\mathbb{R}^2$. Then the orbits are parallel lines in $\mathbb{R}^2$, and the space of orbits is homeomorphic to $\mathbb{R}$. Rotating a torus around its vertical axis through the center leads to an orbit space homeomorphic to S^1. And rotating a sphere around some axis through its center yields an orbit space homeomorphic to $[0,1]$. Eventually, rotating a plane around some fixed point leads to an orbit space homeomorphic to $[0,\infty)$.

If M/G is homeomorphic to $\mathbb{R}$ or S^1, each orbit of the action of G is regular and the orbits form a codimension one Riemannian foliation on M. In the case M/G is homeomorphic to $[0,\infty)$ or $[0,1]$ there exist one or two singular orbits, respectively. If a singular orbit has codimension greater than one, then each regular orbit is geometrically a tube around this singular one. And if the codimension of a singular orbit is one, then each regular orbit is an equidistant hypersurface to it, and the natural projection from a regular orbit to the singular one is a covering map.

If M is simply connected and compact, then for topological reasons M/G must be homeomorphic to $[0,1]$ and each singular orbit must have codimension greater than one. Thus each regular orbit is a tube around any of the two singular orbits, and each singular orbit is a focal set of any regular orbit. If M is simply connected and non-compact, then M/G must be homeomorphic to $\mathbb{R}$ or $[0,\infty)$. In the latter case the singular orbit must have codimension greater than one, and each regular orbit is a tube around the singular one. This discussion gives us a rough idea of what homogeneous hypersurfaces look like.

3. Homogeneous hypersurfaces in real space forms

Any isometry is an affine map with respect to the Riemannian connection. As a consequence from the Weingarten formula we thus see that any homogeneous hypersurface has constant principal curvatures. In a space of constant curvature a hypersurface has constant principal curvatures if and only if it is isoparametric. An isoparametric hypersurface is a level hypersurface of an isoparametric function. This is a

smooth function $f : M \to \mathbb{R}$ so that $\|df\|^2 = a(f)$ and $\Delta f = b(f)$ for some smooth function a and some continuous function b on $\mathbb{R}$. Isoparametric functions and hypersurfaces in $\mathbb{R}^n$ were classified by Levi-Civita [17] in the case $n = 3$ and by Segre [23] in the general case. It is easy to see that each of these isoparametric hypersurfaces is homogeneous. This establishes the classification of homogeneous hypersurfaces in Euclidean spaces.

Theorem 1. *A hypersurface in $\mathbb{R}^n$, $n \geq 3$, is homogeneous if and only if it is*

(1) a hypersphere in $\mathbb{R}^n$, or
(2) an affine hyperplane in $\mathbb{R}^n$, or
(3) a tube around a k-dimensional affine subspace of $\mathbb{R}^n$ for some $1 \leq k \leq n-2$.

The isometry group of $\mathbb{R}^n$ is the semi-direct product $I(\mathbb{R}^n) = O(n) \times \mathbb{R}^n$, where $\mathbb{R}^n$ acts on itself by translations. As closed subgroups of $I(\mathbb{R}^n)$ giving the homogeneous hypersurfaces of type (1), (2), (3) one may choose $SO(n)$, $\mathbb{R}^{n-1}$, $SO(n-k) \times \mathbb{R}^k$, respectively.

In a series of papers [7], [8], [9], [10], Cartan made an attempt to classify the isoparametric hypersurfaces in the real hyperbolic space $\mathbb{R}H^n$ and the sphere S^n. He succeeded in the case of the hyperbolic space, and obtained various results in the case of the sphere. As concerns the hyperbolic space, the crucial step in the classification is a formula which is derived from the equations of Gauss and Codazzi and describes a relation among the principal curvatures. This formula implies that the number of distinct principal curvatures is at most two. If there is just one principal curvature, then the hypersurface is umbilical and hence a horosphere, a geodesic hypersphere, a totally geodesic real hyperbolic hyperplane or an equidistant hypersurface to it. If there are two distinct principal curvatures, one can use theory of focal sets to deduce that the hypersurface is a tube around some totally geodesic real hyperbolic subspace. All these spaces are in fact homogeneous, which implies

Theorem 2. *A hypersurface in $\mathbb{R}H^n$, $n \geq 3$, is homogeneous if and only if it is*

(1) a geodesic hypersphere in $\mathbb{R}H^n$, or

(2) a horosphere in $\mathbb{R}H^n$, or
(3) a totally geodesic $\mathbb{R}H^{n-1}$ or an equidistant hypersurface to it, or
(4) a tube around a k-dimensional totally geodesic $\mathbb{R}H^k$ in $\mathbb{R}H^n$ for some $1 \leq k \leq n-2$.

As subgroups of the identity component $SO^0(1,n)$ of the isometry group of $\mathbb{R}H^n$ one may choose (1) the isotropy group $SO(n)$; (2) the nilpotent part in some Iwasawa decomposition of $SO^0(1,n)$, which is isomorphic to the abelian Lie group $\mathbb{R}^{n-1}$; (3) $SO^0(1,n-1)$; (4) $SO^0(1,k) \times SO(n-k)$.

For spheres Cartan's formula does not provide sufficient information to determine the possible number of distinct principal curvatures. Only later it was proved by Münzner [21], using sphere bundles and methods from algebraic topology, that the number g of distinct principal curvatures of an isoparametric hypersurface in S^n equals $1, 2, 3, 4$ or 6. Already Cartan classified the isoparametric hypersurfaces with at most three distinct principal curvatures. They all turn out to be homogeneous. Surprisingly, for $g = 4$ there are non-homogeneous isoparametric hypersurfaces. The first such examples were discovered by Ozeki and Takeuchi [22], later Ferus, Karcher and Münzner [13] constructed further series of examples by using representations of Clifford algebras. It was shown by Abresch [1] that the case $g = 6$ occurs only in S^7 and S^{13}. Dorfmeister and Neher [12] proved that in S^7 an isoparametric hypersurface must be homogeneous. This is still an open problem in the case of S^{13}. Now, as concerns the classification of homogeneous hypersurfaces, the following result by Hsiang and Lawson [15] settles also the remaining cases $g = 4, 6$:

Theorem 3. *A hypersurface in S^n is homogeneous if and only if it is a principal orbit of the isotropy representation of some Riemannian symmetric space of rank two.*

We can therefore read off the classification of homogeneous hypersurfaces in spheres from the classification of compact, simply connected, Riemannian symmetric spaces. In detail, we get the following homogeneous hypersurfaces N in $S^n = SO(n+1)/SO(n)$:

$g = 1$: Then N is a geodesic hypersphere in S^n. A suitable subgroup of $SO(n+1)$ is the isotropy group $SO(n)$, and the corresponding Riemannian symmetric space of rank two is

$$(SO(2) \times SO(n+1))/SO(n) = S^1 \times S^n \ .$$

$g = 2$: Then N is a Riemannian product of two spheres, namely

$$S^k(r_1) \times S^{n-k-1}(r_2) \ , \ r_1^2 + r_2^2 = 1 \ , \ 0 < r_1, r_2 < 1 \ , \ 0 < k < n-1 \ .$$

A suitable subgroup of $SO(n+1)$ is $SO(k+1) \times SO(n-k)$, and the corresponding Riemannian symmetric space of rank two is

$$(SO(k+2) \times SO(n-k+1))/(SO(k+1) \times SO(n-k)) = S^{k+1} \times S^{n-k} \ .$$

$g = 3$: Then N is congruent to a tube around the Veronese embedding of $\mathbb{R}P^2$ into S^4, or of $\mathbb{C}P^2$ into S^7, or of $\mathbb{H}P^2$ into S^{13}, or of $\mathbb{O}P^2$ into S^{25}. The corresponding Riemannian symmetric spaces of rank two are

$$SU(3)/SO(3) \ , \ SU(3) \ , \ SU(6)/Sp(3) \ , \ E_6/F_4 \ ,$$

respectively. These homogeneous hypersurfaces might also be described as the principal orbits of the natural action of $SO(3)$, $SU(3)$, $Sp(3)$, F_4 on the unit sphere in the linear subspace of all traceless matrices in the Jordan algebra of all 3×3-Hermitian matrices with coefficients in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, respectively. The singular orbits of these actions give the Veronese embeddings of the corresponding projective spaces.

$g = 4$: Then N is a principal orbit of the isotropy representation of

$$Sp(2) \ , \ SO(10)/U(5) \ , \ E_6/Spin(10) \cdot T \ ,$$

or of a two-plane Grassmannian

$$\begin{aligned} G_2^+(\mathbb{R}^{k+2}) &= SO(k+2)/SO(k) \times SO(2) \ (k \geq 3) \ , \\ G_2(\mathbb{C}^{k+2}) &= SU(k+2)/S(U(k) \times U(2)) \ (k \geq 3) \ , \\ G_2(\mathbb{H}^{k+2}) &= Sp(k+2)/Sp(k) \times Sp(2) \ (k \geq 2) \ . \end{aligned}$$

The homogeneous hypersurfaces related to $G_2^+(\mathbb{R}^{k+2})$ are the principal orbits of the action of $SO(k) \times SO(2)$ on the unit sphere S^{2k-1} in $Mat(k \times 2, \mathbb{R}) \approx \mathbb{R}^{2k}$ defined by

$$(A, B) \cdot X := AXB^{-1} ,$$

with $A \in SO(k)$, $B \in SO(2)$ and $X \in Mat(k \times 2, \mathbb{R})$. The homogeneous hypersurfaces related to the complex and quaternionic Grassmannians are obtained from the analogous actions of $S(U(k) \times U(2))$ and $Sp(k) \times Sp(2)$ on the unit sphere in $Mat(k \times 2, \mathbb{C}) \approx \mathbb{C}^{2k} \approx \mathbb{R}^{4k}$ and $Mat(k \times 2, \mathbb{H}) \approx \mathbb{H}^{2k} \approx \mathbb{R}^{8k}$, respectively.

The homogeneous hypersurfaces related to $Sp(2) \approx Spin(5)$ are the principal orbits of the adjoint representation of $Sp(2)$ in the unit sphere S^9 of its Lie algebra $\mathfrak{sp}(2) \approx \mathbb{R}^{10}$.

The unitary group $U(5)$ acts on $\mathbb{C}^5$ and hence on $\Lambda^2\mathbb{C}^5 \approx \mathbb{C}^{10} \approx \mathbb{R}^{20}$ in a natural way. The principal orbits of this action in the unit sphere S^{19} correspond to the principal orbits of the action of the isotropy representation of $SO(10)/U(5)$.

Denote by Δ^+ and Δ^- the two real half-spin representations of $Spin(10)$ on $\mathbb{R}^{32} \approx \mathbb{C}^{16}$, and by ξ the canonical representation of $T \approx U(1)$ on $\mathbb{C}^{16}$ by multiplication with unit complex numbers. Then the isotropy representation of $E_6/Spin(10) \cdot T$ is equivalent to $\Delta^+ \otimes \xi^3 + \Delta^- \otimes \xi^{-3}$, and its principal orbits in the unit sphere $S^{31} \subset \mathbb{R}^{32}$ are homogeneous hypersurfaces.

$g = 6$: Then N is a principal orbit of the isotropy representation of $G_2/SO(4)$ or of the compact exceptional Lie group G_2. The isomorphisms $Spin(4) \approx Sp(1) \times Sp(1)$ and $\mathbb{R}^8 \approx \mathbb{H}^2$ give rise to an action of $Spin(4)$ on $\mathbb{R}^8$ by means of

$$(\lambda, \mu) \cdot (z, v) := (\lambda z, v\mu^{-1}) ,$$

where $(\lambda, \mu) \in Sp(1) \times Sp(1)$ and $(z, v) \in \mathbb{H} \oplus \mathbb{H}$. The principal orbits of this action in the unit sphere S^7 are homogeneous hypersurfaces with six distinct principal curvatures. Miyaoka [19] proved that the orbits of this action are precisely the inverse images under the Hopf

map $S^7 \to S^4$ of the orbits of the action of $SO(3)$ on S^4 as described in the case $g = 3$.

The principal orbits in the unit sphere S^{13} of the Lie algebra $\mathfrak{g}_2 \approx \mathbb{R}^{14}$ of the adjoint representation of the Lie group G_2 are homogeneous hypersurfaces with six distinct principal curvatures, all of whose multiplicities are two.

5. Homogeneous hypersurfaces in projective spaces

For real projective spaces the classification of homogeneous hypersurfaces can be deduced from the one on spheres by using the two-fold covering $S^n \to \mathbb{R}P^n$. An interesting fact is that in complex projective spaces the theories of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures are different. In fact, Wang [25] showed that certain non-homogeneous isoparametric hypersurfaces in spheres project to isoparametric hypersurfaces in complex projective spaces with non-constant principal curvatures. Here, projection is with respect to the Hopf map $S^{2n+1} \to \mathbb{C}P^n$. It is still an open problem whether any hypersurface with constant principal curvatures is isoparametric in $\mathbb{C}P^n$. The classification of homogeneous hypersurfaces in $\mathbb{C}P^n$ was achieved by Takagi [24]. Every homogeneous hypersurface in $\mathbb{C}P^n$ is the projection of a homogeneous hypersurface in S^{2n+1}. But not every homogeneous hypersurface in S^{2n+1} is invariant under the S^1-action and hence does not project to a homogeneous hypersurface in $\mathbb{C}P^n$. In fact, Takagi proved that those which do project are precisely those which arise from isotropy representations of *Hermitian* symmetric spaces of rank two. In detail, this gives the following classification:

Theorem 4. *A hypersurface in $\mathbb{C}P^n$, $n \geq 2$, is homogeneous if and only if it is congruent to*

(1) a tube around a k-dimensional totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for some $0 \leq k \leq n-1$, or

(2) a tube around the complex quadric $\{[z] \in \mathbb{C}P^n \mid z_0^2 + \ldots + z_n^2 = 0\}$ in $\mathbb{C}P^n$, or

(3) a tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$, or

(4) a tube around the Plücker embedding of the complex Grassmann manifold $G_2(\mathbb{C}^5)$ into $\mathbb{C}P^9$, or

(5) a tube around the half spin embedding of the Hermitian symmetric space $SO(10)/U(5)$ *into* $\mathbb{C}P^{15}$.

The corresponding Hermitian symmetric spaces of rank two are (1) $\mathbb{C}P^{k+1} \times \mathbb{C}P^{n-k}$, (2) $G_2^+(\mathbb{R}^{n+3})$, (3) $G_2(\mathbb{C}^{n+3})$, (4) $SO(10)/U(5)$, (5) $E_6/Spin(10) \cdot T$.

For the quaternionic projective space $\mathbb{H}P^n$ an analogous argument was carried out by D'Atri [11] in order to obtain the classification of homogeneous hypersurfaces:

Theorem 5. *A hypersurface in* $\mathbb{H}P^n$, $n \geq 2$, *is homogeneous if and only if it is a tube around some totally geodesic* $\mathbb{H}P^k$ *in* $\mathbb{H}P^n$ *for some* $k \in \{0, \ldots, n-1\}$, *or around some totally geodesic* $\mathbb{C}P^n$ *in* $\mathbb{H}P^n$.

The tubes around $\mathbb{H}P^k$ are the principal orbits of the action of $Sp(k+1) \times Sp(n-k) \subset Sp(n+1)$ on $\mathbb{H}P^n$. The two singular orbits of this action are totally geodesic $\mathbb{H}P^k$ and $\mathbb{H}P^{n-k-1}$. The tubes around $\mathbb{C}P^n$ are the principal orbits of the action of $U(n+1) \subset Sp(n+1)$ on $\mathbb{H}P^n$.

The methods used by Takagi and D'Atri do not work in the case of the Cayley projective plane $\mathbb{O}P^2$. In this situation, the classification follows from a more general result by Kollross [16] which will be discussed in more detail in the next section.

Theorem 6. *A hypersurface in* $\mathbb{O}P^2$ *is homogeneous if and only if it is a geodesic hypersphere or a tube around some totally geodesic* $\mathbb{H}P^2$ *in* $\mathbb{O}P^2$.

The geodesic hyperspheres are obviously the orbits of the isotropy group $Spin(9) \subset F_4$. The second singular orbit of this action is a totally geodesic $\mathbb{O}P^1 = S^8$. The tubes around $\mathbb{H}P^2$ are the principal orbits of the action of $Sp(3) \times SU(2) \subset F_4$. Here, the second singular orbit is $Sp(3)/Sp(2) = S^{11}$.

6. Homogeneous hypersurfaces in Riemannian symmetric spaces of compact type

The classification of homogeneous hypersurfaces in irreducible, simply connected, Riemannian symmetric spaces of compact type is a part of the more general classification of hyperpolar actions (up to

orbit equivalence) on these spaces due to Kollross [16]. Hyperpolar actions on symmetric spaces are sometimes viewed as generalizations of s-representations, that is, of isotropy representations of semisimple Riemannian symmetric spaces. An isometric action of a closed Lie group on a semisimple Riemannian symmetric space M is said to be hyperpolar if there exists a closed, totally geodesic, flat submanifold of M meeting each orbit of the action and intersecting always perpendicularly. It is obvious that the cohomogeneity of a hyperpolar action must be less or equal than the rank of the symmetric space. In particular, the hyperpolar actions on Riemannian symmetric spaces of rank one are precisely the isometric actions of cohomogeneity one, whose classification up to orbit equivalence we described in the previous section. A large class of hyperpolar actions was discovered by Hermann [14]. Suppose (G, K) and (G, H) are two Riemannian symmetric pairs of compact type. Then the action of H on the Riemannian symmetric space G/K is hyperpolar. Also, the action of $H \times K$ on G given by $(h, k) \cdot g := hgk^{-1}$ is hyperpolar. Note that in particular the action of the isotropy group of a semisimple Riemannian symmetric space is hyperpolar.

We describe the idea for the classification by Kollross in the special case when the action is of cohomogeneity one and the symmetric space $M = G/K$ is of rank ≥ 2 and not of group type. Suppose H is a maximal closed subgroup of G. If H is not transitive on M, then its cohomogeneity is at least one. Since the cohomogeneity of the action of any closed subgroup of H is at least the cohomogeneity of the action of H, and we are interested only in classification up the orbit equivalence, it suffices to consider only maximal closed subgroups of G. But it may happen that H acts transitively on G/K. This happens precisely in four cases, where we write down $G/K = H/(H \cap K)$:

$$SO(2n)/U(n) = SO(2n-1)/U(n-1) \ (n \geq 4) \ ,$$
$$SU(2n)/Sp(n) = SU(2n-1)/Sp(n-1) \ (n \geq 3) \ ,$$
$$G_2^+(\mathbb{R}^7) = SO(7)/SO(2) \times SO(5) = G_2/U(2) \ ,$$
$$G_3^+(\mathbb{R}^8) = SO(8)/SO(3) \times SO(5) = Spin(7)/SO(4) \ .$$

In these cases one has to go one step further and consider maximal closed subgroups of H which then never happen to act also transitively. Thus it is sufficient to consider maximal closed subgroups of G, with the few exceptions just mentioned. In order that a closed subgroup H acts with cohomogeneity one it obviously must satisfy $\dim H \geq \dim M - 1$. This rules already out a lot of possibilities. For the remaining maximal closed subgroups one has to calculate case by case the cohomogeneity. One way to do this is to calculate the codimension of the slice representation at $[K]$, this is the action of the isotropy group $H \cap K$ on the normal space of the orbit through $[K]$. This procedure eventually leads to the classification of all cohomogeneity one actions up to orbit equivalence, and hence to the classification of homogeneous hypersurfaces, on $M = G/K$. It turns out that with five exceptions all homogeneous hypersurfaces arise via the construction of Hermann. The exceptions come from the following actions:

1. The action of $G_2 \subset SO(7)$ on $SO(7)/U(3) = SO(8)/U(4) = G_2^+(\mathbb{R}^8)$.
2. The action of $G_2 \subset SO(7)$ on $SO(7)/SO(3) \times SO(4) = G_3^+(\mathbb{R}^7)$.
3. The action of $Spin(9) \subset SO(16)$ on $SO(16)/SO(2) \times SO(14) = G_2^+(\mathbb{R}^{16})$.
4. The action of $Sp(n)Sp(1) \subset SO(4n)$ on $SO(4n)/SO(2) \times SO(4n-2) = G_2^+(\mathbb{R}^{4n})$.
5. The action of $SU(3) \subset G_2$ on $G_2/SO(4)$.

7. Homogeneous hypersurfaces in Riemannian symmetric spaces of non-compact type

We have already discussed above the classification of homogeneous hypersurfaces in real hyperbolic space $\mathbb{R}H^n$. The method of Cartan does not work for the hyperbolic spaces $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$. The reason is that the equations of Gauss and Codazzi become too complicated. Nevertheless, we can apply the method of Cartan to the special class of curvature-adapted hypersurfaces. For these hypersurfaces the equations of Gauss and Codazzi simplify considerably. A hypersurface N of a Riemannian manifold M is called curvature-adapted if its shape operator and its normal Jacobi operator commute with each other. Recall that the normal Jacobi operator of N is the self-adjoint (local)

tensor field on N defined by $R(.,\xi)\xi$, where R is the Riemannian curvature tensor of M and ξ is a (local) unit normal vector field of N. If M is a space of constant curvature, then the normal Jacobi operator is a multiple of the identity at each point, whence any hypersurface is curvature-adapted. But for more general ambient spaces this condition is quite restrictive. For instance, in a non-flat complex space form, say $\mathbb{C}P^n$ or $\mathbb{C}H^n$, a hypersurface N is curvature-adapted if and only if the structure vector field on N is a principal curvature vector everywhere. Recall that the structure vector field of N is the vector field obtained by rotating a local unit normal vector field to a tangent vector field using the ambient Kähler structure. In [3] the author derived the classification of all curvature-adapted hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures. Any of these is homogeneous, which leads to

Theorem 7. *A hypersurface in $\mathbb{C}H^n$, $n \geq 2$, is curvature-adapted and homogeneous if and only if it is*

(1) a geodesic hypersphere in $\mathbb{C}H^n$, or
(2) a tube around some totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $1 \leq k \leq n-1$, or
(3) a horosphere in $\mathbb{C}H^n$, or
(4) a tube around some totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$.

The geodesic hyperspheres are obviously the principal orbits of the isotropy group $S(U(1) \times U(n))$ of $SU(1,n)$ at some point. The tubes around a totally geodesic $\mathbb{C}H^k$ are the principal orbits of the action of $S(U(1,k) \times U(n-k)) \subset SU(1,n)$. The horospheres arise as the orbits (there are only principal orbits in this case) of the nilpotent part in any Iwasawa decomposition of $SU(1,n)$. Note that this nilpotent part is isomorphic to the $(2n-1)$-dimensional Heisenberg group. Eventually, the tubes around $\mathbb{R}H^n$ are the principal orbits of the action of $SO(1,n) \subset SU(1,n)$.

A hypersurface N in a quaternionic space form M, say $\mathbb{H}P^n$ or $\mathbb{H}H^n$, is curvature-adapted if and only if the three-dimensional distribution, which is obtained by rotating the normal bundle of N into the tangent bundle of N by the almost Hermitian structures in the quaternionic Kähler structure of M, is invariant under the shape operator of N. The curvature-adapted hypersurfaces in $\mathbb{H}P^n$ and $\mathbb{H}H^n$ with constant

principal curvatures were classified in [4]. In the case of the quaternionic hyperbolic space this leads to

Theorem 8. *A hypersurface in $\mathbb{H}H^n$, $n \geq 2$, is curvature-adapted and homogeneous if and only if it is*

(1) a geodesic hypersphere in $\mathbb{H}H^n$, or
(2) a tube around some totally geodesic $\mathbb{H}H^k$ in $\mathbb{H}H^n$ for some $1 \leq k \leq n-1$, or
(3) a horosphere in $\mathbb{H}H^n$, or
(4) a tube around some totally geodesic $\mathbb{C}H^n$ in $\mathbb{H}H^n$.

The geodesic hyperspheres are the principal orbits of the isotropy group $Sp(1) \times Sp(n)$ of $Sp(1,n)$ at some point. The tubes around a totally geodesic $\mathbb{H}H^k$ are the principal orbits of the action of $Sp(1,k) \times Sp(n-k) \subset Sp(1,n)$. The horospheres arise as the orbits of the nilpotent part in any Iwasawa decomposition of $Sp(1,n)$. And the tubes around $\mathbb{C}H^n$ are the principal orbits of the action of $SU(1,n) \subset Sp(1,n)$.

Of course, the question arises naturally whether any homogeneous hypersurface in $\mathbb{C}H^n$ or $\mathbb{H}H^n$ is curvature-adapted. As the classifications by Takagi and D'Atri show, the answer for the corresponding question in $\mathbb{C}P^n$ and $\mathbb{H}P^n$ is yes. But recently Lohnherr and Reckziegel [18] found an example of a homogeneous ruled hypersurface in $\mathbb{C}H^n$ which is not curvature-adapted. Consider a horocycle in a totally geodesic and totally real $\mathbb{R}H^2 \subset \mathbb{C}H^n$. At each point of the horocycle we attach a totally geodesic $\mathbb{C}H^{n-1}$ orthogonal to the complex hyperbolic line determined by the tangent vector of the horocycle at that point. By varying with the points on the horocycle we get a homogeneous ruled hypersurface in $\mathbb{C}H^n$. In [5] the author constructed this hypersurface by an algebraic method. Using this method more examples of homogeneous hypersurfaces in $\mathbb{C}H^n$ were found. Also, this method generalizes to the other Riemannian symmetric spaces of non-compact type, as well as to some other homogeneous Hadamard manifolds, to produce examples of homogeneous hypersurfaces. We shall now describe this construction in more detail.

Suppose $M = G/K$ is a Riemannian symmetric space of non-compact type with G equal to the identity component of the full isometry group

of M. The Iwasawa decomposition $G = KAN$ of G implies that M can be realized as a solvable Lie group $S = AN$ equipped with some left-invariant Riemannian metric. Obviously, every closed subgroup of S of codimension one determines a Riemannian foliation of $M = S$ by homogeneous hypersurfaces. To determine the closed subgroups with codimension one it is easier to work on the Lie algebra level. Denote by $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ the Lie algebra of $S = AN$.

We first discuss the familiar case of real hyperbolic space $\mathbb{R}H^n$. Here we have $G = SO^0(1,n)$, $K = SO(n)$, $A = \mathbb{R}$ and $N = \mathbb{R}^{n-1}$, N being equipped with the standard abelian Lie group structure. Then A is one-dimensional and hence N is a closed subgroup of codimension one in S. The orbits of N are the horospheres in $\mathbb{R}H^n$ centered at the point at infinity determined by A. Let V be some unit vector in $\mathfrak{n}$. Then the orthogonal complement $V^\perp$ of $\mathbb{R}V$ in $\mathfrak{s}$ is a Lie subalgebra of codimension one. The corresponding closed subgroup of S has a totally geodesic $\mathbb{R}H^{n-1}$ as an orbit, the other orbits are the equidistant hypersurfaces to it. These two Lie subalgebras describe, up to congruence, all homogeneous hypersurfaces in $\mathbb{R}H^n$ which can be obtained via this construction. Of course, we do not get anything new. But imitating this construction for the hyperbolic spaces over $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ we get new examples of homogeneous hypersurfaces in these spaces. We describe this now in a more general context, details may be found in [5].

Let $m \in \mathbb{N}$, q the standard negative definite quadratic form on $\mathfrak{z} := \mathbb{R}^m$, and $J : Cl(\mathfrak{z}, q) \to \mathrm{End}(\mathfrak{v})$, $Z \mapsto J_Z$ a real representation of the Clifford algebra $Cl(\mathfrak{z}, q)$ on some n-dimensional real vector space $\mathfrak{v}$. If $m \not\equiv 3 (\mathrm{mod}\ 4)$, then there exists a unique irreducible Clifford module $\mathfrak{d}$ over $Cl(\mathfrak{z}, q)$ and $\mathfrak{v}$ is the k-fold direct sum of $\mathfrak{d}$ for some $k \in \mathbb{N}$. If $m \equiv 3 (\mathrm{mod}\ 4)$, then there are two inequivalent irreducible Clifford modules $\mathfrak{d}_1$ and $\mathfrak{d}_2$ over $Cl(\mathfrak{z}, q)$ and $\mathfrak{v} = (\oplus^{k_1} \mathfrak{d}_1) \oplus (\oplus^{k_2} \mathfrak{d}_2)$ for some $k_1, k_2 \in \mathbb{N}$. On the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ of the vector spaces $\mathfrak{v}$ and $\mathfrak{z}$ we define an inner product $< .,. >$ as follows. On $\mathfrak{z}$ the inner product is just minus the polarization of the quadratic form q. The vector spaces $\mathfrak{v}$ and $\mathfrak{z}$ are supposed to be perpendicular with respect to the inner product. Finally we require that for any unit vector $Z \in \mathfrak{z}$ the map J_Z is an orthogonal map on $\mathfrak{v}$ with respect to the induced inner product

on $\mathfrak{v}$. It can be shown that such an inner product exists and is unique. We then define a skew-symmetric bilinear map $[.,.] : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ by

$$<[U+X, V+Y], W+Z> = <J_Z U, V>$$

for all $U, V, W \in \mathfrak{v}$ and $X, Y, Z \in \mathfrak{z}$. This turns $\mathfrak{n}$ into a two-step nilpotent Lie algebra with m-dimensional center $\mathfrak{z}$, a so-called generalized Heisenberg algebra. Choosing $m = 1$ yields the classical Heisenberg algebras. The connected and simply connected Lie group N with Lie algebra $\mathfrak{n}$ and equipped with the left-invariant Riemannian metric induced from the inner product on $\mathfrak{n}$ is called a generalized Heisenberg group.

Next, we extend $\mathfrak{n}$ to a solvable Lie algebra. Let $\mathfrak{a}$ be a one-dimensional real vector space and $A \in \mathfrak{a}$ a non-zero vector. By defining

$$[A, V] = \frac{1}{2} V \quad , \quad [A, Z] = Z$$

for all $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$ the Lie algebra structure on $\mathfrak{n}$ extends to a Lie algebra structure on $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$. Since the derived subalgebra of $\mathfrak{s}$ is the nilpotent Lie algebra $\mathfrak{n}$, $\mathfrak{s}$ is solvable. We extend the inner product on $\mathfrak{n}$ to an inner product on $\mathfrak{s}$ by requiring that $\mathfrak{a}$ and $\mathfrak{n}$ are perpendicular to each other and A is of unit length. The connected, simply connected, solvable Lie group S with Lie algebra $\mathfrak{s}$ and equipped with the left-invariant Riemannian metric induced from the inner product on $\mathfrak{s}$ is called a Damek-Ricci space. A Damek-Ricci space S is symmetric if and only if

(i) $m = 1$; then S is isometric to the complex hyperbolic space $\mathbb{C}H^{n/2+1}$, or

(ii) $m = 3$ and $\mathfrak{v}$ is an isotypic module; then S is isometric to the quaternionic hyperbolic space $\mathbb{H}H^{n/4+1}$, or

(iii) $m = 7$ and $\mathfrak{v}$ is irreducible; then S is isometric to the Cayley hyperbolic plane $\mathbb{O}H^2$.

In all these cases the symmetric metric is the one for which the sectional curvature lies between -1 and $-1/4$. Moreover, the Lie group S in this construction coincides with the one arising from the Iwasawa decomposition as described above. The non-symmetric Damek-Ricci

spaces are harmonic manifolds with non-positive sectional curvature taking values between -1 and 0. For more information about generalized Heisenberg groups and Damek-Ricci spaces we refer the reader to the book [6] by the author, Tricerri and Vanhecke.

Let $\mathfrak{m}$ be a linear subspace of $\mathfrak{s}$ with codimension one. Then $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{s}$ if and only if there exist some unit vector $V_0 \in \mathfrak{v}$ and some $\theta \in \mathbb{R}$ such that

$$\mathfrak{m}^{\perp} = \mathbb{R}(\cos\theta\, V_0 + \sin\theta\, A) \ .$$

For any unit vector $V_0 \in \mathfrak{v}$ and any $\theta \in \mathbb{R}$ we denote the orthogonal complement of $\mathbb{R}(\cos\theta\, V_0 + \sin\theta\, A)$ in $\mathfrak{s}$ by $\mathfrak{s}(\theta, V_0)$. Clearly, we may assume $\theta \in]-\pi/2, \pi/2]$. The Lie algebra $\mathfrak{s}(\pi/2, V_0)$ is the generalized Heisenberg algebra $\mathfrak{n}$ and hence two-step nilpotent. For $\theta \neq \pi/2$ the Lie algebra $\mathfrak{s}(\theta, V_0)$ is solvable but not nilpotent. Under the action of the group of orthogonal automorphisms of $\mathfrak{s}$, the pairwise non-congruent members in the family of Lie subalgebras $\mathfrak{s}(\theta, V_0)$ are indexed by $\theta \in [0, \pi/2]$. Let $\theta \in [0, \pi/2[$. Then any two Lie subalgebras $\mathfrak{s}(\theta, V_0)$ of $\mathfrak{s}$ are congruent by an orthogonal automorphism of $\mathfrak{s}$ if and only if

(i) $\mathfrak{s}$ is the Lie algebra of a symmetric Damek-Ricci space S, that is, of complex hyperbolic space, of quaternionic hyperbolic space, or of Cayley hyperbolic plane; or
(ii) the dimension of $\mathfrak{z}$ is two; or
(iii) the dimension of $\mathfrak{z}$ is five or six and $\mathfrak{v}$ is an irreducible module.

We denote by $S(\theta, V_0) := \mathrm{Exp}(\mathfrak{s}(\theta, V_0))$ the connected, simply connected Lie subgroup of S with Lie algebra $\mathfrak{s}(\theta, V_0)$. Since $\mathrm{Exp} : \mathfrak{s} \to S$ is a diffeomorphism, $S(\theta, V_0)$ is a closed subgroup of S. Since S acts on itself isometrically by left translations, the subgroup $S(\theta, V_0)$ acts on S in an obvious manner isometrically with cohomogeneity one. The orbits of this action form a Riemannian foliation $\mathfrak{F}(\theta, V_0)$ on S consisting of homogeneous hypersurfaces of S. Each orbit of the action of $S(0, V_0)$ on S is a left translate of $S(\theta, V_0)$ for some suitable $\theta \in]-\pi/2, \pi/2[$. In particular, for each $\theta \in]-\pi/2, \pi/2[$ the Riemannian foliations $\mathfrak{F}(0, V_0)$ and $\mathfrak{F}(\theta, V_0)$ on S coincide up to some left translation in S. The Riemannian foliation $\mathfrak{F}(\pi/2, V_0)$ on S is the horosphere foliation containing

the generalized Heisenberg group N in S as a horosphere. Each orbit of the action of $S(\pi/2, V_0)$ on S is a suitable left translate of N.

The homogeneous hypersurface $S(\theta, V_0)$, $\theta \in [0, \pi/2]$, of S has three (for $\theta \neq \pi/2$) or two (for $\theta = \pi/2$) distinct principal curvatures

$$\frac{1}{2}\sin\theta\ ,\ \frac{3}{4}\sin\theta - \frac{1}{4}\sqrt{1+3\cos^2\theta}\ ,\ \frac{3}{4}\sin\theta + \frac{1}{4}\sqrt{1+3\cos^2\theta}$$

with multiplicities $n-m, m, m$ or n, m, respectively. The mean curvature of $S(\theta, V_0)$ is $[(n+2m)/(2n+2m)]\sin\theta$. In particular, $S(0, V_0)$ is a minimal homogeneous hypersurface of S. Moreover, the homogeneous hypersurface $S(\theta, V_0)$ of S is curvature-adapted if and only if $\theta = \pi/2$. Suppose S is $\mathbb{C}H^{k+1}$, $\mathbb{H}H^{k+1}$ or $\mathbb{O}H^2$. Then the homogeneous hypersurface $S(0, V_0)$ is a ruled hypersurface of S whose generators are totally geodesic $\mathbb{C}H^k$, $\mathbb{H}H^k$ and $\mathbb{O}H^1$ in S, respectively. In case of $\mathbb{C}H^{k+1}$ the base curve of $S(0, V_0)$ is a horocycle in some suitable totally real, totally geodesic $\mathbb{R}H^2$ in $\mathbb{H}H^{k+1}$. Hence $S(0, V_0)$ coincides with the homogeneous ruled hypersurface constructed in [18]. The homogeneous hypersurfaces $S(\theta, V_0)$, $\theta \in]0, \pi/2]$, are not ruled in the above sense. A complete classification of the homogeneous hypersurfaces in $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$, as well as in the other Riemannian symmetric spaces of non-compact type and rank ≥ 2, is still unknown.

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