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On Curvature and Submanifolds of Complex Two-Plane Grassmannians

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1. Introduction

The aim of this note is to present some aspects of the Riemannian geometry of complex two-plane Grassmannians. One interesting fact about these Grassmannians is that they are equipped with both a Kähler structure and a quaternionic Kähler structure. There are various spaces which are equipped with these two structures, but the Grassmann manifold $G_2(\mathbb{C}^{m+2})$ of all two-planes in $\mathbb{C}^{m+2}$ is distinguished in some sense. To make this more precise, let M be a connected quaternionic Kähler manifold of real dimension at least eight. It is well-known that M is an Einstein manifold [2]. Moreover, vanishing Ricci curvature is equivalent to M being a locally hyperkähler manifold [2]. Hyperkähler manifolds provide examples of manifolds which are equipped with both a Kähler and a quaternionic Kähler structure. The simplest such space is the quaternionic vector space $\mathbb{H}^m$. But the point with hyperkähler manifolds is that the Kähler structure is incorporated in the quaternionic Kähler structure. If the Ricci curvature is non-zero, then this cannot happen, the Kähler structure and the quaternionic Kähler structure are independent of each other. This follows by some holonomy argument. In fact, M being Kähler means that the holonomy group of M is contained in some unitary group $U(2m)$, where $2m$ is the complex dimension of M. And M being quaternionic Kähler means that its holonomy group is contained in $Sp(m)Sp(1)$. Non-zero Ricci curvature implies that the holonomy group contains the $Sp(1)$-factor [5] (Lemma 14.46). Using the list of Lie groups acting transitively on spheres [11] one can then deduce

that the restricted holonomy group of M cannot act transitively on the unit sphere in some tangent space when M is both Kähler and quaternionic Kähler with non-zero Ricci curvature. As any quaternionic Kähler manifold with non-zero Ricci curvature is locally irreducible [2], it follows from the classification of holonomy groups of Riemannian manifolds due to Berger [1] that M is a locally symmetric space. Therefore, assuming M is complete, its Riemannian universal covering space $\tilde{M}$ is a Riemannian symmetric space. It follows from the classification of simply connected, irreducible, Riemannian symmetric spaces that $\tilde{M}$ is isometric to $G_2(\mathbb{C}^{m+2})$ or its non-compact dual $G_2^*(\mathbb{C}^{m+2})$. The first one has positive Ricci curvature, the latter one negative Ricci curvature. A result by S. Salamon [12] says that a compact quaternionic Kähler manifold with positive Ricci curvature is simply connected. Equivalently we may apply a result of S. Kobayashi [9] which says that every compact Kähler manifold with positive definite Ricci curvature is simply connected to conclude:

Each $4m$-dimensional ($m \geq 2$) compact Kähler and quaternionic Kähler manifold with positive Ricci curvature is isometric to $G_2(\mathbb{C}^{m+2})$.

Note that for negative Ricci curvature there is no similar result. Each non-compact Riemannian symmetric space admits several compact quotients [6], and for $G_2^*(\mathbb{C}^{m+2})$ such quotients provide examples of compact, Kähler and quaternionic Kähler manifolds with negative Ricci curvature.

A basic example of a compact Kähler resp. quaternionic Kähler manifold is the complex projective space $\mathbb{C}P^m$ resp. the quaternionic projective space $\mathbb{H}P^m$. As we will see, the Riemannian geometry of $G_2(\mathbb{C}^{m+2})$ is partially a mixture of these two geometries, and partially it exhibits new geometric phenomena. In the next section we describe a new model for $G_2(\mathbb{C}^{m+2})$. This model is used in Section 3 to derive an expression for the Riemannian curvature tensor of $G_2(\mathbb{C}^{m+2})$ in terms of the Riemannian metric, the Kähler structure, and the quaternionic Kähler structure. We also discuss some general aspects of curvature in $G_2(\mathbb{C}^{m+2})$. In Section 4 we present the classification of all maximal totally geodesic submanifolds of $G_2(\mathbb{C}^{m+2})$ and outline its proof. In the last section we discuss the homogeneous real hypersurfaces of $G_2(\mathbb{C}^{m+2})$ and a characterization of them in terms of geometrical properties of the shape operator. Proofs and more details can be found in [3] and [4].

2. A model for $G_2(\mathbb{C}^{m+2})$

Briefly, the model arises in the following manner. Consider the $(m+1)$-dimensional complex projective space $\mathbb{C}P^{m+1}$ embedded in the canonical way as

a totally geodesic submanifold in the $(m+1)$-dimensional quaternionic projective space $\mathbb{H}P^{m+1}$. The focal set Q^{m+1} of $\mathbb{C}P^{m+1}$ in $\mathbb{H}P^{m+1}$ is a submanifold with codimension three. At each point of Q^{m+1} the null space of the shape operator is independent of the direction of the normal vectors and determines a one-dimensional Riemannian foliation $\mathcal{F}$ on Q^{m+1} by closed geodesics in $\mathbb{H}P^{m+1}$. The orbit space $B^{m+1} := Q^{m+1}/\mathcal{F}$, equipped with the Riemannian structure for which the canonical projection $Q^{m+1} \to B^{m+1}$ becomes a Riemannian submersion, is isometric to the Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$. Thus, a two-dimensional complex linear subspace of $\mathbb{C}^{m+2}$, which represents a point in $G_2(\mathbb{C}^{m+2})$ in the standard model, is here replaced by a closed geodesic in the focal set of $\mathbb{C}P^{m+1}$ in $\mathbb{H}P^{m+1}$. All geometrical information about $G_2(\mathbb{C}^{m+2})$ is encoded in the intrinsic and extrinsic structure of the focal set Q^{m+1} of $\mathbb{C}P^{m+1}$ in $\mathbb{H}P^{m+1}$. The main tools to get information about $G_2(\mathbb{C}^{m+2})$ via this construction are the theories of focal sets and Riemannian submersions.

We now discuss some more details of this construction. The standard isometric action of $SU(m+2)$ on $\mathbb{H}P^{m+1}$ is as follows. First, $SU(m+2)$ acts isometrically on $S^{4m+7} \subset \mathbb{H}^{m+2} = \mathbb{C}^{m+2} \oplus \mathbb{C}^{m+2}j$ by means of

$$SU(m+2) \times S^{4m+7} \to S^{4m+7} \ , \ (A, z+vj) \mapsto (Az) + (Av)j \ ,$$

where $z, v \in \mathbb{C}^{m+2}$ with $|z|^2 + |v|^2 = 1$. This action leaves the fibres of the Hopf map $S^{4m+7} \to \mathbb{H}P^{m+1}$ invariant and hence descends to an action on $\mathbb{H}P^{m+1}$ by isometries. This action has cohomogeneity one, that is, the codimension of the principal orbits is one. Thus there are two singular orbits, one of which is a totally geodesic $\mathbb{C}P^{m+1} \subset \mathbb{H}P^{m+1}$. For geometrical reasons, the second singular orbit must be the focal set Q^{m+1} of $\mathbb{C}P^{m+1}$ in $\mathbb{H}P^{m+1}$. The isotropy of $SU(m+2)$ at some point in Q^{m+1} turns out to be isomorphic to $SU(m) \times SU(2)$. It follows that Q^{m+1} is isometric to the Riemannian homogeneous space $SU(m+2)/SU(m) \times SU(2)$ equipped with some $SU(m+2)$-invariant Riemannian metric. Moreover, the codimension of this homogeneous space in $\mathbb{H}P^{m+1}$ can easily be seen to be three. Note that Q^2 is isometric to a geodesic hypersphere of a certain radius in $\mathbb{C}P^3$.

The extrinsic geometry of Q^{m+1} in $\mathbb{H}P^{m+1}$ is described by its shape operator. Starting from the trivial extrinsic geometry of $\mathbb{C}P^{m+1}$ in $\mathbb{H}P^{m+1}$ one may calculate the shape operator of its focal set Q^{m+1} by means of standard methods from Jacobi field theory. It turns out that with respect to any unit normal vector of Q^{m+1} there are three distinct principal curvatures $-1, 0, +1$. At each point p the zero eigenspace is one-dimensional and independent of the choice

of the unit normal vector at that point. The direct sum of the zero eigenspace and the three-dimensional normal space at p forms a one-dimensional quaternionic subspace of the tangent space of $\mathbb{H}P^{m+1}$ at p. The zero eigenspaces of the shape operator form a one-dimensional autoparallel distribution on the focal set Q^{m+1}. The corresponding totally geodesic foliation is generated by a unit Killing vector field U and hence Riemannian. Its integral curves are closed geodesics in Q^{m+1} as well as in $\mathbb{H}P^{m+1}$. They are the orbits of the $U(1)$-action on Q^{m+1} induced by left multiplication with e^{it} on $\mathbb{H}^{m+2}$. The $+1$- and -1-eigenspaces have the same dimension and are mapped into each other by a suitable almost Hermitian structure in the quaternionic Kähler structure of $\mathbb{H}P^{m+1}$. In particular, it follows that Q^{m+1} is a minimal submanifold of $\mathbb{H}P^{m+1}$. We mention that Q^{m+1} is a Sasakian globally φ-symmetric space.

Since the leaves of $\mathcal{F}$ are compact orbits of an isometric action, we can equip the space of leaves $B^{m+1} := Q^{m+1}/\mathcal{F}$ with the structure of a Riemannian manifold so that the canonical projection $\pi : Q^{m+1} \to B^{m+1}$ becomes a Riemannian submersion. As Q^{m+1} is complete and simply connected and the fibers of π are connected, it follows that B^{m+1} is also complete and simply connected. The covariant derivative of the unit Killing vector field U, restricted to the horizontal subspaces of the submersion, projects to a Kähler structure on B^{m+1} with respect to which it becomes a Hermitian symmetric space. Any isometry in $SU(m+2)$ projects to an isometry of B^{m+1}, the induced action of $SU(m+2)$ on B^{m+1} is transitive, and B^{m+1} is in fact isometric to $SU(m+2)/S(U(m) \times U(2)) = G_2(\mathbb{C}^{m+2})$ with some symmetric metric. From now on we identify the orbit space B^{m+1} with $G_2(\mathbb{C}^{m+2})$. The quaternionic Kähler structure on $G_2(\mathbb{C}^{m+2})$ is obtained by restricting the quaternionic Kähler structure of $\mathbb{H}P^{m+1}$ to the horizontal subspaces and projecting down to the Grassmannian.

3. Curvature of $G_2(\mathbb{C}^{m+2})$

We denote by g the Riemannian metric, by J the Kähler structure and by $\mathcal{J}$ the quaternionic Kähler structure of $G_2(\mathbb{C}^{m+2})$. If J_1 is an almost Hermitian structure in $\mathcal{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\operatorname{tr} JJ_1 = 0$. Thus, at each point p of $G_2(\mathbb{C}^{m+2})$, the almost Hermitian structure J_p is orthogonal to each almost Hermitian structure in $\mathcal{J}_p$.

Since we know the second fundamental form of Q^{m+1} and the Riemannian curvature tensor of $\mathbb{H}P^{m+1}$ explicitly, the Gauss equation gives us an explicit

expression for the Riemannian curvature tensor of Q^{m+1}. General theory about Riemannian submersions then gives us the Riemannian curvature tensor of the base space, which is the complex Grassmannian in the present situation.

Theorem 1. *The Riemannian curvature tensor R of $G_2(\mathbb{C}^{m+2})$ is locally given by*

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\ &\quad + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}(g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z) \\ &\quad + \sum_{\nu=1}^{3}(g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY)\ , \end{aligned}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathcal{J}$.

Notice how this expression involves the Riemannian curvature tensors of S^{4m}, $\mathbb{C}P^{2m}$ and $\mathbb{H}P^m$. Contracting R we get for the Ricci tensor S and the scalar curvature s of $G_2(\mathbb{C}^{m+2})$ the expressions

$$SX = 4(m+2)X \quad \text{and} \quad s = 16m(m+2)\ .$$

In particular, $G_2(\mathbb{C}^{m+2})$ is an Einstein manifold. We equip $\Lambda^2 TG_2(\mathbb{C}^{m+2})$ with the induced bundle metric, which we also denote by g. Using Theorem 1 we may easily calculate the sectional curvature function K of $G_2(\mathbb{C}^{m+2})$. Let $X, Y \in T_pG_2(\mathbb{C}^{m+2})$, $p \in G_2(\mathbb{C}^{m+2})$, be orthonormal and $\sigma = \operatorname{span}\{X,Y\}$. Then the sectional curvature $K(\sigma)$ of $G_2(\mathbb{C}^{m+2})$ with respect to σ is given by

$$\begin{aligned} K(\sigma) &= 1 + 3g(X\wedge Y, JX\wedge JY) + 3\sum_{\nu=1}^{3} g(X\wedge Y, J_\nu X\wedge J_\nu Y) \\ &\quad + \sum_{\nu=1}^{3} g(JX\wedge JY, J_\nu X\wedge J_\nu Y)\ . \end{aligned}$$

Note that

$$g(X\wedge Y. JX\wedge JY) = \cos^2\angle(Y,JX)\ ,$$

where $\angle(Y,JX)$ is the angle between Y and JX, which is just the Kähler angle of σ, and

$$\sum_{\nu=1}^{3} g(X\wedge Y. J_\nu X\wedge J_\nu Y) = \cos^2\angle(Y,\mathcal{J}X)\ ,$$

where $\angle(Y, \mathcal{J}X)$ is the angle between Y and $\mathcal{J}X$. From this expression for the sectional curvature we easily derive that $G_2(\mathbb{C}^3)$ is isometric to $\mathbb{C}P^2$ equipped with the Fubini Study metric of constant holomorphic sectional curvature eight. Therefore, we assume $m \geq 2$ from now on.

In order to get more information about the sectional curvature we compute the eigenvalues and eigenspaces of the Jacobi operators $R_X := R(.,X)X$. From Theorem 1 we get

$$\begin{aligned} R_X Y &= Y - g(Y,X)X + 3g(Y,JX)JX + 3\sum_{\nu=1}^{3} g(Y,J_\nu X)J_\nu X \\ &\quad + \sum_{\nu=1}^{3} g(X,J_\nu JX)J_\nu JY - \sum_{\nu=1}^{3} g(Y,J_\nu JX)J_\nu JX \end{aligned}$$

for any unit vector $X \in TG_2(\mathbb{C}^{m+2})$. A lengthy but straightforward calculation gives the following spectral data, where we denote by $\mathbb{C}X$ the real span of X and JX, by $\mathbb{H}X$ the real span of X and $\mathcal{J}X$, by $\mathbb{C}^\perp X$ the orthogonal complement of $\mathbb{C}X$ in $\mathbb{H}X$, and by $\mathbb{H}\mathbb{C}X$ the real span of $\mathbb{H}X$ and $\mathbb{H}JX$.

Theorem 2. *Let X be a unit vector tangent to $G_2(\mathbb{C}^{m+2})$. In each of the following cases we list all eigenvalues κ of R_X and the corresponding eigenspaces T_κ with its dimensions.*

(i) *$\underline{JX \perp \mathcal{J}X}$. Then we have*

κ	T_κ	$\dim T_\kappa$
0	$\mathbb{R}X \oplus \mathcal{J}JX$	4
1	$(\mathbb{H}\mathbb{C}X)^\perp$	$4m-8$
4	$\mathbb{R}JX \oplus \mathcal{J}X$	4 ;

(ii) *$\underline{JX \in \mathcal{J}X}$. Let J_1 be the almost Hermitian structure in $\mathcal{J}$ so that $JX = J_1X$. Then we have*

κ	T_κ	$\dim T_\kappa$
0	$\mathbb{R}X \oplus \{Y \mid Y \perp \mathbb{H}X,\ JY = -J_1Y\}$	$2m-1$
2	$\mathbb{C}^\perp X \oplus \{Y \mid Y \perp \mathbb{H}X,\ JY = J_1Y\}$	$2m$
8	$\mathbb{R}JX$	1 ;

(iii) *otherwise. There exist an almost Hermitian structure $J_1 \in \mathcal{J}$ and a unit vector Z orthogonal to $\mathbb{H}X$ so that*

$$JX = \cos\alpha J_1 X + \sin\alpha J_1 Z \ ,$$

where $\alpha = \angle(JX, \mathcal{J}X) \in]0, \pi/2[$. Then we have

κ	T_κ	$\dim T_\kappa$
0	$\mathbb{R}X \oplus \mathbb{R}Z$	2
$1-\cos\alpha$	$\{Y \mid Y \perp (\mathbb{H}X \oplus \mathbb{H}Z),\ JY = -J_1 Y\}$	$2m-4$
$1+\cos\alpha$	$\{Y \mid Y \perp (\mathbb{H}X \oplus \mathbb{H}Z),\ JY = J_1 Y\}$	$2m-4$
$2(1-\sin\alpha)$	$\{\cos(\frac{\alpha}{2}+\frac{\pi}{4})\, J_2 X + \sin(\frac{\alpha}{2}+\frac{\pi}{4})\, J_2 Z \mid J_2 \in (\mathbb{R}J_1)^\perp\}$	2
$2(1+\sin\alpha)$	$\{\sin(\frac{\alpha}{2}+\frac{\pi}{4})\, J_2 X - \cos(\frac{\alpha}{2}+\frac{\pi}{4})\, J_2 Z \mid J_2 \in (\mathbb{R}J_1)^\perp\}$	2
$4(1-\cos\alpha)$	$\mathbb{R}(\sin(\alpha/2)J_1 X - \cos(\alpha/2)J_1 Z)$	1
$4(1+\cos\alpha)$	$\mathbb{R}(\cos(\alpha/2)J_1 X + \sin(\alpha/2)J_1 Z)$	1

The number of distinct eigenvalues κ is seven unless $\cos(\alpha) = 4/5$ or $3/5$, in which case there are six or five distinct eigenvalues, respectively.

We now continue the discussion about the sectional curvature of $G_2(\mathbb{C}^{m+2})$. For a unit vector X tangent to $G_2(\mathbb{C}^{m+2})$ we denote by K_X the sectional curvature function with respect to all two-planes containing X, and by $K_{hol}(X)$ the (holomorphic) sectional curvature with respect to $\mathbb{C}X$. From Theorem 2 we get

$$0 \leq K_X \leq 4(1+\cos\alpha)$$

and

$$K_{hol}(X) = 4(1+\cos^2\alpha) \ ,$$

where $\alpha = \angle(JX, \mathcal{J}X)$. Moreover, both inequalities are sharp. By varying with X we obtain for the sectional curvature K and the holomorphic sectional curvature K_{hol} of $G_2(\mathbb{C}^{m+2})$ the estimates

$$0 \leq K \leq 8$$

and

$$4 \leq K_{hol} \leq 8 \ .$$

Moreover, all inequalities are sharp, and we have

$$\begin{aligned} K(\sigma) = 8 &\iff \sigma = J\sigma \subset \mathcal{J}\sigma \ , \\ K_{hol}(X) = 4 &\iff JX \perp \mathcal{J}X \ , \\ K_{hol}(X) = 8 &\iff JX \in \mathcal{J}X \ . \end{aligned}$$

A basic role in the geometric theory of Riemannian symmetric spaces is played by the so-called maximal flats. In the case of $G_2(\mathbb{C}^{m+2})$, a maximal flat is a two-dimensional totally geodesic submanifold isometric to some flat two-dimensional torus. From general theory of symmetric spaces it is known that if the sectional curvature with respect to some two-plane $\sigma \subset T_pG_2(\mathbb{C}^{m+2})$ vanishes, then the exponential map of $G_2(\mathbb{C}^{m+2})$ at p maps σ onto a maximal flat of $G_2(\mathbb{C}^{m+2})$, and every maximal flat arises in this way. A non-zero tangent vector X is called *singular* if it is contained in more than one maximal flat. From the tables in Theorem 2 it follows that X is singular if and only if $JX \perp \mathcal{J}X$ or $JX \in \mathcal{J}X$. Equivalently, as the above discussion of the holomorphic sectional curvature shows, X is singular if and only if the holomorphic sectional curvature determined by X is minimal or maximal.

So there are two types of singular tangent vectors, those X for which $JX \perp \mathcal{J}X$, and those for which $JX \in \mathcal{J}X$. The set of maximal flats to which a singular tangent vector X is tangent can be identified in an obvious manner with the real projective space induced from the sphere of all unit vectors Y orthogonal to X with $R_XY = 0$. Let X be a non-zero tangent vector of $G_2(\mathbb{C}^{m+2})$ and σ be a two-plane spanned by X and some tangent vector Z orthogonal to X. From the tables in Theorem 2 we derive

(i) If $JX \perp \mathcal{J}X$, then σ determines a maximal flat if and only if $Z \in \mathcal{J}JX$. In particular, the set of maximal flats to which X is tangent is an $\mathbb{R}P^2$.

(ii) If $JX \in \mathcal{J}X$, say $JX = J_1X$, then σ determines a maximal flat if and only if $Z \perp \mathbb{H}X$ and $JZ = -J_1Z$. In particular, the set of maximal flats to which X is tangent is an $\mathbb{R}P^{2m-3}$.

(iii) If X is non-singular, then there exist an almost Hermitian structure $J_1 \in \mathcal{J}$ and a tangent vector Z orthogonal to $\mathbb{H}X$ so that $JX = \cos\alpha J_1X + \sin\alpha J_1Z$, where $\alpha = \angle(JX, \mathcal{J}X)$. Then the span of X and Z determines the unique maximal flat to which X is tangent.

4. Maximal totally geodesic submanifolds

We shall now discuss the classification of maximal totally geodesic submanifolds in $G_2(\mathbb{C}^{m+2})$. The isomorphism $SU(4) = Spin(6)$ gives rise to an isometry from $G_2(\mathbb{C}^4)$ onto the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-planes in $\mathbb{R}^6$. The maximal totally geodesic submanifolds in oriented real two-plane Grassmannians have been determined explicitly by B.Y. Chen and T. Nagano in [7]. There are three types of maximal totally geodesic submanifolds in $G_2^+(\mathbb{R}^6)$, namely $G_2^+(\mathbb{R}^5)$, $(S^a \times S^b)/\mathbb{Z}_2$ with $a+b=4$, and $\mathbb{C}P^2$. Any maximal totally geodesic submanifold of $G_2(\mathbb{C}^4)$ is congruent to one of these.

We assume from now on $m \geq 3$. We start by describing the so-called classical totally geodesic embeddings of certain manifolds into $G_2(\mathbb{C}^{m+2})$.

(1) Let ℓ be a complex linear line in $\mathbb{C}^{m+2}$. Then the orthogonal complement $\ell^\perp$ of ℓ in $\mathbb{C}^{m+2}$ determines an m-dimensional complex projective space $\mathbb{C}P^m(\ell^\perp)$. The image of the embedding

$$F_\ell : \mathbb{C}P^m(\ell^\perp) \to G_2(\mathbb{C}^{m+2}) \ , \ \ell' \mapsto \ell \oplus \ell'$$

is a totally geodesic submanifold of $G_2(\mathbb{C}^{m+2})$. Two embeddings F_ℓ and $F_{\ell'}$ have the same image if and only if $\ell = \ell'$. Hence the set of all classical totally geodesic submanifolds of $G_2(\mathbb{C}^{m+2})$ attached to $\mathbb{C}P^m$ corresponds in a natural way to a $\mathbb{C}P^{m+1}$.

(2) Let V be an $(a+1)$-dimensional complex linear subspace of $\mathbb{C}^{m+2}$ and $V^\perp$ its orthogonal complement in $\mathbb{C}^{m+2}$, which is a $(b+1)$-dimensional complex linear subspace of $\mathbb{C}^{m+2}$, $a+b=m$, $0<a<m$. The image of the embedding

$$F_V : \mathbb{C}P^a(V) \oplus \mathbb{C}P^b(V^\perp) \to G_2(\mathbb{C}^{m+2}) \ , \ (\ell, \ell') \mapsto \ell \oplus \ell'$$

is a totally geodesic submanifold of $G_2(\mathbb{C}^{m+2})$. If $a \neq b$, all these submanifolds are distinct and the set of all such submanifolds corresponds in a natural way to the complex Grassmann manifold $G_{a+1}(\mathbb{C}^{m+2})$ of all $(a+1)$-dimensional complex linear subspaces in $\mathbb{C}^{m+2}$. If $a=b$, the maps F_V and $F_{V'}$ determine the same submanifold if and only if V' is either V or $V^\perp$.

(3) Let V be an $(m+1)$-dimensional complex linear subspace of $\mathbb{C}^{m+2}$. Then the image of the embedding

$$F_V : G_2(\mathbb{C}^{m+1})(V) \to G_2(\mathbb{C}^{m+2}) \ , \ \ell \oplus \ell' \mapsto \ell \oplus \ell'$$

is a totally geodesic submanifold of $G_2(\mathbb{C}^{m+2})$. The set of all such submanifolds corresponds in a natural way to a $\mathbb{C}P^{m+1}$.

(4) Let V be an $(m+2)$-dimensional real linear subspace of $\mathbb{C}^{m+2}$. Then the image of the embedding

$$F_V : G_2(\mathbb{R}^{m+2})(V) \to G_2(\mathbb{C}^{m+2}) \ , \ \ell \oplus \ell' \mapsto \ell_{\mathbb{C}} \oplus \ell'_{\mathbb{C}} \ ,$$

where $\ell_{\mathbb{C}}$ denotes the complex linear line in $\mathbb{C}^{m+2}$ determined by ℓ, is a totally geodesic submanifold of $G_2(\mathbb{C}^{m+2})$. Two embeddings F_V and $F_{V'}$ have the same image if and only if $V' = e^{it}V$ for some $t \in \mathbb{R}$. The set of all such submanifolds corresponds in a natural way to $SU(m+2)/SO(m+2)$.

(5) Let $\jmath$ be a quaternionic structure on $\mathbb{C}^{2n-2}$ compatible with its Hermitian structure i. Thus i, $\jmath$ and $i\jmath$ turn $\mathbb{C}^{2n+2}$ into a right quaternionic vector space $\mathbb{H}^{n+1}(\jmath)$. Then the image of the embedding

$$F_\jmath : \mathbb{H}P^n(\jmath) \to G_2(\mathbb{C}^{2n+2}) \ , \ z\mathbb{H} \mapsto \mathbb{C}z \oplus \mathbb{C}z\jmath$$

is a totally geodesic submanifold of $G_2(\mathbb{C}^{2n+2})$. Two embeddings $F_\jmath$ and $F_{\jmath'}$ have the same image if and only if $\jmath' = e^{it}\jmath$ for some $t \in \mathbb{R}$. The set of all such submanifolds corresponds in a natural way to $SU(2n+2)/Sp(n+1)$.

The following result says that these classical totally geodesic submanifolds are precisely the maximal totally geodesic submanifolds of $G_2(\mathbb{C}^{m+2})$.

Theorem 3. *Any maximal totally geodesic submanifold of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, is one of the above classical embeddings.*

The basic idea for the proof is as follows. To start with, a complete totally geodesic submanifold of a Riemannian symmetric space is again a Riemannian symmetric space. This follows readily from the very definition of a symmetric space in terms of geodesic symmetries. Thus the first problem we have to deal with is to determine all symmetric spaces which can be embedded in $G_2(\mathbb{C}^{m+2})$ as a maximal totally geodesic submanifold. This problem was settled by B.Y. Chen and T. Nagano in [8] using the so-called (M_+, M_-)-method. They obtained the following list:

(1) the m-dimensional complex projective space $\mathbb{C}P^m$;

(2) the Riemannian product $\mathbb{C}P^a \times \mathbb{C}P^b$ of an a-dimensional complex projective space and a b-dimensional complex projective space, where a, b are any positive integers with $a + b = m$;

(3) the complex Grassmann manifold $G_2(\mathbb{C}^{m+1})$;

(4) the real Grassmann manifold $G_2(\mathbb{R}^{m+2})$ of all two-dimensional linear subspaces in $\mathbb{R}^{m+2}$;

(5) the quaternionic projective space $\mathbb{H}P^n$, in case m is even and $m = 2n$.

It therefore remains to prove that any totally geodesic embedding of each of the spaces in this list is already a classical embedding as described above. We discuss this in more detail for the complex projective space $\mathbb{C}P^m$, for the other spaces the argument is along the same lines. At first one determines more details about the geometrical structure of the classical totally geodesic embeddings of $\mathbb{C}P^m$. It turns out that they give embeddings of $\mathbb{C}P^m$ as complex and totally complex submanifolds in $G_2(\mathbb{C}^{m+2})$. Recall that a submanifold of $G_2(\mathbb{C}^{m+2})$ is complex if any of its tangent spaces is invariant under the Kähler structure J. And a submanifold M is called totally complex if at each point $p \in M$ there exists an almost Hermitian structure J_1 in $\mathcal{J}$ leaving T_pM invariant and so that each almost Hermitian structure in $\mathcal{J}$ orthogonal to J_1 maps T_pM into the normal space of M at p. The standard models for the concept of a complex submanifold is a $\mathbb{C}^k$ in $\mathbb{C}^m$ and for a totally complex submanifold a $\mathbb{C}^k$ in $\mathbb{H}^m$. Another geometric fact one needs is that for any $p \in \mathbb{C}P^m(\ell^\perp)$ there exists an almost Hermitian structure $J_1 \in \mathcal{J}_p$, such that $JX = J_1X$ for all $X \in T_p\mathbb{C}P^m(\ell^\perp)$. Hence each non-zero tangent vector of $\mathbb{C}P^m(\ell^\perp)$ is a singular tangent vector of $G_2(\mathbb{C}^{m+2})$.

Next, consider any totally geodesic embedding of $\mathbb{C}P^m$ equipped with some Fubini Study metric into $G_2(\mathbb{C}^{m+2})$. Let X be any tangent vector of $\mathbb{C}P^m$. It follows from the Gauss equation that the Jacobi operator of $\mathbb{C}P^m$ with respect to X is the restriction to the corresponding tangent space of $\mathbb{C}P^m$ of the Jacobi operator of $G_2(\mathbb{C}^{m+2})$ with respect to X. In particular, the spectrum of the first Jacobi operator must be contained in the spectrum of the latter one. The spectrum of the Jacobi operator of $\mathbb{C}P^m$ is $\{0, c, 4c\}$, where $4c$ is the value of the holomorphic sectional curvature and the eigenvalues 0 and $4c$ have multiplicity one each. Comparing this spectrum with the one of the Jacobi operator of the

ambient Grassmannian according to Theorem 2 eventually gives that at some point the tangent space of $\mathbb{C}P^m$ coincides with the tangent space of one of the classical totally geodesic embeddings. Rigidity of totally geodesic submanifolds then implies that the embedding of $\mathbb{C}P^m$ is in fact a classical one.

5. Homogeneous real hypersurfaces

In this section we discuss homogeneous real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. The classification of the homogeneous real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is a consequence of the more general classification of cohomogeneity one actions (up to orbit equivalence) on irreducible, simply connected, Riemannian symmetric spaces of compact type by Kollross [10]. Recall that a cohomogeneity one isometric action on a compact, complete, simply connected Riemannian manifold has two singular orbits, each of which has codimension greater than one and is the focal set of the other one. Moreover, each principal orbit can be viewed as a tube around any of these two singular orbits.

Theorem 4. *A real hypersurface of $G_2(\mathbb{C}^{m+2}) = SU(m+2)/S(U(m) \times U(2))$, $m \geq 3$, is homogeneous if and only if it is congruent to*

(i) a principal orbit of the action of $S(U(m+1) \times U(1)) \subset SU(m+2)$. The two singular orbits are totally geodesically embedded $\mathbb{C}P^m$ and $G_2(\mathbb{C}^{m+1})$.

(ii) (if m is even, say $m = 2n$) a principal orbit of the action of $Sp(n+1) \subset SU(2n+2)$. One of the two singular orbits is a totally geodesically embedded $\mathbb{H}P^n$, the other one is the homogeneous complex Einstein hypersurface $Sp(n+1)/Sp(n-1)U(2)$.

Starting from a totally geodesic singular orbit, the principal curvatures and their eigenspaces of the tubes around them can be calculated explicitly by using Jacobi vector fields. Using the classical totally geodesic embeddings described in the previous section, we get the following description.

Let V be an $(m+1)$-dimensional complex linear subspace of $\mathbb{C}^{m+2}$ and ℓ the complex linear line in $\mathbb{C}^{m+2}$ perpendicular to V. The focal set of $G_2(\mathbb{C}^{m+1})(V)$ in $G_2(\mathbb{C}^{m+2})$ is $\mathbb{C}P^m(\ell^\perp)$ and consists precisely of all points in $G_2(\mathbb{C}^{m+2})$ at distance $\pi/2\sqrt{2}$ from $G_2(\mathbb{C}^{m+1})(V)$, or equivalently, of all antipodal points of $G_2(\mathbb{C}^{m+1})(V)$ in $G_2(\mathbb{C}^{m+2})$. Any tube M_r of radius $0 < r < \pi/2\sqrt{2}$ around $G_2(\mathbb{C}^{m+1})(V)$ is a homogeneous real hypersurface of $G_2(\mathbb{C}^{m+2})$ with four (resp. three for $r = \pi/4\sqrt{2}$) distinct principal curvatures. More precisely, denote by ξ the "outward" unit normal field of M_r for some fixed r. Then there exists an

almost Hermitian structure J_1 in $\mathcal{J}$, which is uniquely defined at each point of M_r, such that $J\xi = J_1\xi$. The principal curvatures of M_r with respect to ξ are

$$\lambda_1 := -2\sqrt{2}\cot(2\sqrt{2}r)\ ,\ \lambda_2 := -\sqrt{2}\cot(\sqrt{2}r)\ ,\ \lambda_3 := \sqrt{2}\tan(\sqrt{2}r)\ ,\ \lambda_4 := 0\ ,$$

with corresponding multiplicities

$$m(\lambda_1) = 1\ ,\ m(\lambda_2) = 2\ ,\ m(\lambda_3) = m(\lambda_4) = 2m-2\ .$$

The corresponding eigenspaces are

$$\begin{aligned} E(\lambda_1) &:= \mathbb{R}J\xi = \mathbb{R}J_1\xi\ , \\ E(\lambda_2) &:= \mathbb{C}^{\perp}\xi\ , \\ E(\lambda_3) &:= \{X \in TM_r \mid X \perp \mathbb{H}\xi\ ,\ JX = J_1X\}\ , \\ E(\lambda_4) &:= \{X \in TM_r \mid X \perp \mathbb{H}\xi\ ,\ JX = -J_1X\}\ . \end{aligned}$$

Next, consider the classical totally geodesic embedding $\mathbb{H}P^n(\jmath) \to G_2(\mathbb{C}^{2n+2})$. The focal set $F^n(\jmath)$ of $\mathbb{H}P^n(\jmath)$ in $G_2(\mathbb{C}^{2n+2})$ is a complex hypersurface and consists of all points in $G_2(\mathbb{C}^{2n-2})$ at distance $\pi/4$ from $\mathbb{H}P^n(\jmath)$. Moreover, it is congruent to the complex Einstein hypersurface $Sp(n+1)/Sp(n-1)U(2)$. The principal curvatures of $F^n(\jmath)$ with respect to any unit normal vector are $+1, 0, -1$. At each point $q \in F^n(\jmath)$, the null space of the shape operator is the 6-dimensional vector space $\mathcal{J}\perp_q F^n(\jmath)$ and independent of the choice of the unit normal at q. Here, $\perp_q F^n(\jmath)$ denotes the normal space of $F^n(\jmath)$ at q. The other two eigenspaces are real subspaces and mapped into each other by J. Any tube M_r of radius $0 < r < \pi/4$ around $\mathbb{H}P^n(\jmath)$ is a homogeneous real hypersurface of $G_2(\mathbb{C}^{2n+2})$ with five distinct principal curvatures. More precisely, the principal curvatures with respect to the "outward" unit normal field ξ of M_r are

$$\lambda_1 := -\cot r\ ,\ \lambda_2 := -2\cot 2r\ ,\ \lambda_3 := \tan r\ ,\ \lambda_4 := 2\tan 2r\ ,\ \lambda_5 := 0\ ,$$

and the respective multiplicities are

$$m(\lambda_1) = m(\lambda_3) = 4n-4\ ,\ m(\lambda_2) = m(\lambda_5) = 3\ ,\ m(\lambda_4) = 1\ .$$

For the corresponding spaces of principal curvature vectors we have

$$E(\lambda_4) = \mathbb{R}J\xi\ ,\ E(\lambda_2) = \mathcal{J}\xi\ ,\ E(\lambda_5) = \mathcal{J}J\xi\ ,$$

and the eigenspaces $E(\lambda_1)$ and $E(\lambda_3)$ are mapped into each other by J.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then its normal bundle $\perp M$ has rank one. Applying the Kähler structure J of the Grassmannian to the normal bundle gives a one-dimensional subbundle $J(\perp M)$ of the tangent bundle TM of M. Similarily, applying the almost Hermitian structures in the quaternionic Kähler structure $\mathcal{J}$ of the Grassmannian to the normal bundle gives a three-dimensional subbundle $\mathcal{J}(\perp M)$ of TM. If M is a homogeneous real hypersurface in $G_2(\mathbb{C}^{m+2})$, the above expression for the shape operator shows us that both $J(\perp M)$ and $\mathcal{J}(\perp M)$ are invariant under the shape operator of M. In joint work with Y.J. Suh [4] we studied these two geometric properties of the shape operator in detail. In particular, we proved that these two conditions imply that M lies on a tube around some totally geodesic $\mathbb{C}P^m$ or $\mathbb{H}P^n$, the latter possibility occurs only when $m = 2n$ is even. Combining this with Theorem 4 we get a characterization of homogeneous real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ in terms of geometrical features of its shape operator.

Theorem 5. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $J(\perp M)$ and $\mathcal{J}(\perp M)$ are invariant under the shape operator of M if and only if M is an open part of a homogeneous real hypersurface in $G_2(\mathbb{C}^{m+2})$.*

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