Classifying Markets up to isomorphism

John Armstrong, King's College London

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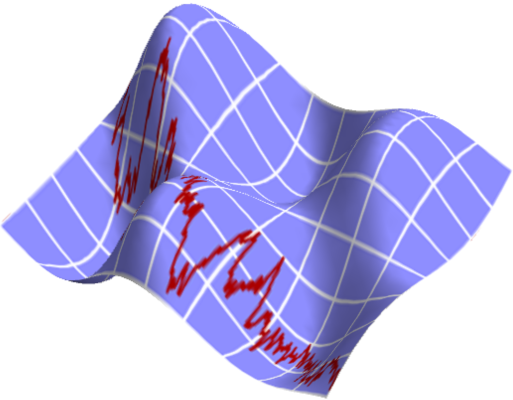


Overview

- Goals:
  - Explain what it means for two markets to be isomorphic.
  - Explain how automorphisms of markets correspond to mutual fund theorems.
  - Give some classification theorems for markets - i.e. identify the isomorphism classes.
- Part 1: Category theory - the abstract notion of isomorphism
- Part 2: One period markets - classifications of Markowitz markets and complete markets.
- Part 3: Continuous time markets.
- Example consequence: Given a diffusion model you can change the drift to obtain a market isomorphic to a Black–Scholes–Merton model.

Part I

Category Theory

Category theory

Definition

A category C consists of the following data:

(i) a class $\text{ob}(C)$ of objects.

(ii) a class $\text{hom}(C)$ of morphisms. To each morphism f are associated a source $a \in \text{ob}(C)$ and target $b \in \text{ob}(C)$. We write

$$f : a \to b$$

$\text{hom}(a, b)$ is the class of all morphisms from a to b.

(iii) for all $a, b, c \in \text{ob}\, C$ a binary operation called composition

$$\text{hom}(a, b) \times \text{hom}(b, c) \to \text{hom}(a, c).$$

If $f : a \to b$, $g : b \to c$ we write $g \circ f$ or just gf for the composition.

The composition satisfies the properties associativity and the existence of an *identity* morphism $\mathbf{1}_x : x \to x$ for all $x \in \text{ob}(C)$.

Examples of categories

Object	Morphisms
Vector Space	Linear Transformations
Group	Homomorphisms
Topological Space	Homeomorphism
Metric Space	Isometry
Banach Space	Bounded Linear Transformation
Markowitz Market	Markowitz isomorphism

- Two objects are isomorphic if they are "identical as far as your category is concerned".
- Example: A sphere and a cube are isomorphic topologically, but not as metric spaces.
- "Interesting" properties of an object should be invariant under isomorphisms
- Example: Two five pound notes are isomorphic. Their serial numbers are not interesting, only their purchasing power. (A five pound note is also isomorphic to five pound coins.)

Duality

- The definition of a category does not require that morphisms $f : a \to b$ are represented by functions. They are simply "arrows" starting at a and ending at b.
- Given a category you can obtain a new category by reversing the arrows. This is called the opposite category.

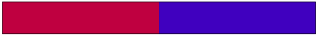
Classification Theorems

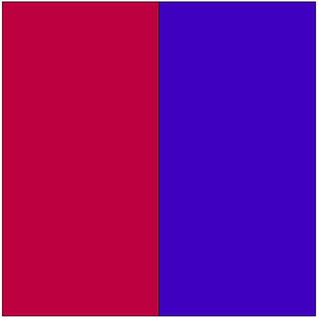
- Isomorphism = morphism with two-sided inverse
- Automorphism = isomorphism of an object two itself
- Classification = identify the isomorphism classes

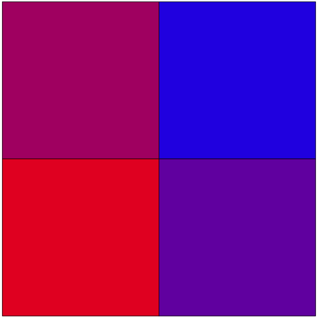
1. Finite dimensional real vector spaces are classified by their dimension.
2. Matrices are classified up to similarity by Jordan normal form.
3. Möbius (1861)–Brahana (1921): Closed surfaces are classified topologically by their Euler characteristic and whether they are orientable.
4. Finite simple groups: "The proof consists of tens of thousands of pages in several hundred journal articles written by about 100 authors, published mostly between 1955 and 2004."
5. Perelman (2006): Completed the classification of compact 3-manifolds.

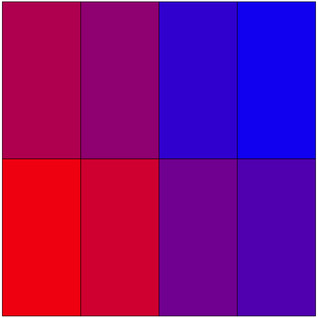
Example: the category of probability spaces

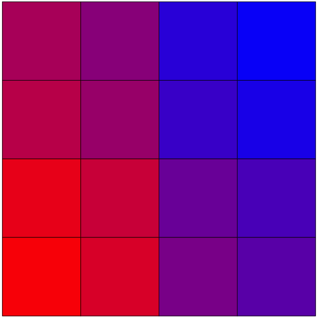
- The objects consist of probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$.
- The morphisms consist of almost-sure equivalence classes of measurable functions, f which preserve the measure $\mathbb{P}$, i.e. $\mathbb{P}(U) = \mathbb{P}(f^{-1}(U))$ for measurable U.
- Discrete probability spaces are classified by a decreasing sequence of non-negative numbers $p_1 \geq p_2 \geq p_3 \geq \dots$ with $\sum p_i = 1$.
- The probability spaces $[0, 1]$ and $[0, 1] \times [0, 1]$ equipped with the Lebesgue measure are isomorphic, and they are isomorphic to the probablity space generated by Brownian motion!
- A standard probability space is isomorphic to the union of $[0, 1]$ and a discrete probability space. Henceforth all probability spaces are assumed standard.

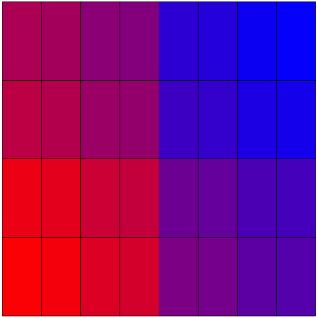


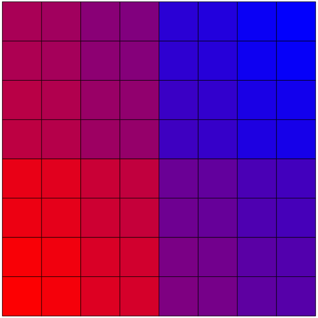












Invariants

A first step to classification theorems is to define invariants

1. The dimension of a vector space is invariant under linear bijections.
2. The characteristic polynomial of a matrix is invariant under similarities.
3. The Euler characteristic of a surface is invariant under homeomorphisms.
4. The Gaussian curvature of a surface is an invariant under isometries. But what does invariant mean exactly here?

Covariant Functor

A covariant functor is a mapping between categories and their morphisms that respects composition and identities.

Definition

A covariant functor F from a category C to a category D is a mapping which

(i) associates to each object $x \in \text{ob}(C)$ an object in $F(x) \in \text{ob}(D)$.

(ii) associates to a morphism $f : x \to y$ in $\text{hom}(C)$ a morphism $F(f) : F(x) \to F(y)$ in $\text{hom}(D)$.

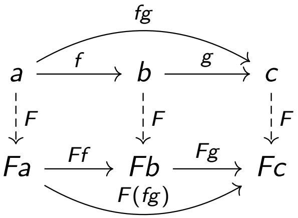
and which satisfies

(i) For all $x \in \text{ob}(C)$, $F(\mathbf{1}_x) = \mathbf{1}_{F(x)}$

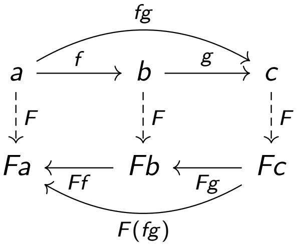
(ii) If $f : a \to b$ and $g : b \to c$ then $F(g \circ f) = F(g) \circ F(f)$.

Covariant and contravariant functors

Covariant functor



Contravariant functor



Example

- Vector space duality defines a contravariant functor, F from the category of vector spaces to itself.

$$F(V) = V^*$$

if $T : V \to W$ then

$$F(T) : W^* \to V^*$$

by

$$F(T) = T^*$$

- Vector space double duality defines a covariant functor F from the category of vector spaces to itself.

Example: L^0

Let L^0 be the functor mapping the category of probability spaces to the category of vector spaces by:

$$L^0(\Omega) = L^0(\Omega; \mathbb{R})$$

If $f : \Omega_1 \to \Omega_2$ is measurable, define

$$L^0(f) : L^0(\Omega_2; \mathbb{R}) \to L^0(\Omega_1; \mathbb{R})$$

by

$$L^0(f)(X) = X \circ f.$$

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{\quad f \quad} & \Omega_2 \\ \downarrow{\scriptstyle L^0} & & \downarrow{\scriptstyle L^0} \\ L^0(\Omega_1; \mathbb{R}) & \xleftarrow[L^0(f)]{} & L^0(\Omega_2; \mathbb{R}) \end{array}$$

Invariantly defined element

Let $F : C \to D$ be a covariant functor and let D be a category of sets. A function

$$\phi : C \to \mathrm{Set}$$

is an *invariantly defined element for F* if

$$\phi(x) \in F(x)$$

for all $x \in \mathrm{ob}(C)$ and $\phi(fx) = F(f)\phi(x)$ for all isomorphisms f.
If F is a contravariant functor we instead require
$\phi(fx) = F(f^{-1})\phi(x)$.

Example: the Theorema Egregium

Gauss proved that the Gaussian curvature of a surface is an invariantly defined element for C^∞ the contravariant functor mapping the category of surfaces up to isometry to the category of rings by sending a surface to the ring of smooth functions on that surface.

We will see that the absolute value of the market price of risk is an invariantly defined element for the functor L^0 acting on continuous time markets.

Part II

One period markets

Definition

A one period financial market $((\Omega, \mathcal{F}, \mathbb{P}), c)$ consists of: a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; a function $c : L^0(\Omega; \mathbb{R}) \to \mathbb{R} \cup \{\pm\infty\}$. We call $c^{-1}(\mathbb{R} \cup \{-\infty\})$ the domain of c, denoted $\operatorname{dom} c$.

- *A random variable $X \in L^0(\Omega; \mathbb{R})$ represents the final payout of a financial instrument X. $c(X)$ is the cost of X. If its cost is infinite, it cannot be purchased.*

Definition

A morphism of markets $M_1 = ((\Omega_1, \mathcal{F}_1, \mathbb{P}_1), c_1)$ and $M_2 = ((\Omega_2, \mathcal{F}_2, \mathbb{P}_2), c_2)$ is a Prob morphism $\phi : \Omega_1 \to \Omega_2$ satisfying $c_2(X) \geq c_1(X \circ \phi)$ for all $X \in L^0(\Omega_2; \mathbb{R})$.

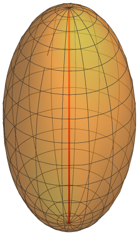
- *A morphism $\phi : M_1 \to M_2$ represents an inclusion of M_2 in M_1. Any financial product can be purchased for the same or a lower price in market M_1.*

Mutual fund theorems and automorphisms

Suppose we have a financially interesting problem whose solution is a set of financial products $\mathcal{X} \subseteq L^0(\Omega, \mathbb{R})$.

If $f : \Omega \to \Omega$ is an automorphism, then the problem will remain unchanged if we apply f. Hence $L^0(f)(\mathcal{X}) = \mathcal{X}$. So $\mathcal{X}$ is invariant under automorphisms.

If $\mathcal{X}$ is non-empty and convex, G is a compact group of automorphisms, then $\mathcal{X}$ must contain an invariant element for G. The solution to our problem can be taken to lie in $\mathrm{dom}\, c \cup L^0_G(\Omega; \mathbb{R})$ which may be very small.



Example: Markowitz markets

Definition

We define the category Markowitz1 to have objects given by markets where

- c is linear — i.e. prices are linear so no bid-ask spread are quantity constraints
- $\operatorname{dom} c$ is finite dimensional — i.e. there are only a finite number of basic assets
- The market is arbitrage free
- All assets X with finite cost have a Gaussian distribution

the morphisms are given by market morphisms.

Duality theorem for Markowitz markets

Definition

We define the category Markowitz2 to have objects (V, b, p, C) where V is a finite dimensional vector space, b is a positive definite bilinear form on V, p and C are linearly independent linear functionals. Morphisms are linear maps $\phi : V_1 \to V_2$ satisfying

$$b_2(\phi(v_1), \phi(v_2) = b_1(v1, v2)$$

$$p_2(\phi(v_1)) = p_1(v1)$$

$$C_2(\phi(v_1)) = C_1(v1)$$

- Elements of V represent portfolios
- b measures the covariance of two portfolios
- p measures the expected payoff of a portfolio
- C measures the initial cost

We say that two categories C and D are dual if C is equivalent to the opposite category of D.

Duality theorem for Markowitz markets

Theorem

Markowitz1 and Markowitz2 are dual.

Sketch.

Given an object $M = (\Omega, \mathcal{F}, \mathbb{P}, c)$ in Markowitz1 define:

$$V(M) = \operatorname{dom} c, \quad C(M)(X) = c(X)$$

$$p(M)(X) = \mathbb{E}(X), \quad b(X, Y) = \operatorname{Cov}(X, Y).$$

Given an object $M' = (V, b, p, C)$ in Markowitz2 define

$$\Omega(M') = V^*, c(M')(X) = \begin{cases} C(X) & X \in V^{**} \\ \infty & \text{otherwise} \end{cases}$$

Let $** : V \to V^{**}$ be the double duality isomorphism. $\mathbb{P}(M')$ is defined to be the multivariate normal distribution on V^* with mean $p \in V^*$ and covariance

$$\operatorname{Cov}(X^{**}, Y^{**}) = b(X, Y)$$

Equivalence and duality

We must also map morphisms in one category to morphisms in the other. This is a duality rather than an equivalence because a morphism

$$f : \Omega_2 \to \Omega_1$$

is mapped to the dual map

$$L^0(f) : \operatorname{dom} c_1 \subseteq L^0(\mathbb{R}) \to \operatorname{dom} c_2 \subseteq L^0(\mathbb{R})$$

The maps

$$M = (\Omega, \mathbb{P}, c) \xrightarrow{\phi} (V(M), b(M), p(M), C(M))$$

and

$$M' = (V, b, p, C) \xrightarrow{\psi} (\Omega(M'), \mathbb{P}(M'), c(M'))$$

are not inverses. However $\psi \circ \phi(M')$ is the double dual of M' and so is naturally isomorphic to it. Similarly $\phi \circ \psi$. The definition of equivalence of categories is designed to ensure that this is enough to prove equivalence of Markowitz1 and op(Markowitz1).

Classification of non-degenerate Markowitz markets

The classification of Markowitz markets is now easy

- All positive-definite bilinear forms are isomorphic to the Euclidean inner product on $\mathbb{R}^n$ via the Gram-Schmidt process.
- Hence without loss of generality we may assume a Markowitz market is given by $(\mathbb{R}^n, g^E, v_1^*, v_2^*)$ where g_E is the Euclidean inner product and $* : V \to V^*$ by

$$v^*(x) = g_E(v, x) \quad \forall v \in V.$$

- A Markowitz market is therefore determined by two vectors v_1 and v_2 in Euclidean space.
- We may apply a rotation to ensure that $v_1 = (\alpha, 0, 0, \ldots, 0)$ and $v_2 = (\beta, \gamma, 0, 0, ..., 0)$.

Corollary

Non degenerate Markowitz markets are classified by their efficient frontier.

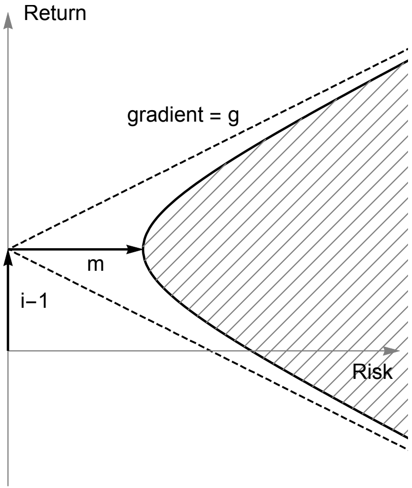


Figure:

The two-mutual-fund theorem

Corollary

The only invariant portfolios lie in the span of v_1, v_2.

Proof.

$(x_1, x_2, x_3, \ldots, x_n) \to (x_1, x_2, -x_3, \ldots, -x_n)$ is a market automorphism. □

- The solution of any convex, financially interesting problem in the Markowitz model can be assumed to lie in the span of v_1, v_2.
- The solution of any convex problem using no data about the market other than the covariance b, cost C and payoff p can be assumed to lie in the span of v_1, v_2.

This is a substantial generalization of the classical two mutual fund theorem. It applies to problems you haven't thought of yet!

Example

$$\text{Invariant input} \xrightarrow{\substack{\text{Mathematical operations} \\ \text{without arbitrary choices}}} \text{Invariant output}$$

- Let $M = (V, b, p, C)$ be a Markowitz market. Let u be a concave increasing utility function.
- Let $\mathbb{P}(M)$ be the Gaussian measure on V^* found in the previous theorem. It is invariantly defined.
- Let W denote all measures on V^* within a given Wasserstein distance of the measure $\mathbb{P}(M)$. W is invariantly defined because $\mathbb{P}(M)$ is invariantly defined.
- Consider the robust optimization problem

$$\mathcal{R} = \operatorname*{argmax}_{v \in V} \inf_{w \in W} \mathbb{E}_w(u(v^{**}))$$

- $\mathcal{R}$ is invariantly defined and convex. Hence it contains an invariant element.

The limits of mean-variance analysis

- How can we find a low-dimensional representation of the bond market?
- Principal component analysis?

One period complete markets

Definition

A one period complete market has a cost function of the form

$$c(X) = \beta \mathbb{E}(QX)$$

for some pricing kernel $Q \in L^0(\Omega)$ with $Q > 0$, $E_{\mathbb{P}}(Q) = 1$ and a where $\beta \in R_{>0}$ is a discount factor.

Example

The market obtained by pursuing an self-financing trading strategy in the Black–Scholes market until a terminal time T.

Example

The market $\mathcal{C}$ with $\Omega = [0, 1]$ with the Lebesgue measure and $\beta = Q = 1$ is called a Casino.

It is easy to check that Q may be recovered from c, hence Q is invariantly defined for L^0.

Classification of one period complete markets

Theorem

Let M_1 and M_2 be complete one period markets then $M_1 \times \mathcal{C}$ is isomorphic to $M_2 \times \mathcal{C}$ if and only if the discount factors of M_1 and M_2 are equal and their pricing kernels are equal in distribution.

Proof.

Apply Rohklin's classification of homomorphisms between probability spaces to Q. □

(The full classification theorem without the casino is a little more tedious to state)

If Q is absolutely continuous then we may take $M_1 = [0, 1]$ and $Q \in L^0([0, 1]; \mathbb{R})$ to be a decreasing function with integral 1. Optimization problems can now be solved by calculus of variations.

Applications: Pensions, S-Shaped utility...

Part III

Continuous Time Markets

Multiperiod markets

Definition

A multi-period market consists of

(i) A filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ where $t \in \mathcal{T} \subseteq [0, T]$ for some index set $\mathcal{T}$ containing both 0 and T. We write $\mathcal{F} = \mathcal{F}_T$. We require $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(ii) For each $X \in L^0(\Omega; \mathbb{R})$, an $\mathcal{F}_t$ adapted process $c_t(X)$ defined for t in $\mathcal{T} \setminus T$.

Random variables $X \in L^0(\Omega, \mathcal{F}_T; \mathbb{R})$ are interpreted as contracts which have payoff X at time T. The cost of this contract at time t is $c_t(X)$.

Definition

A filtration isomorphism of filtered spaces $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $t \in \mathcal{T}$ for some index set $\mathcal{T}$ is a mod 0 isomorphism for $\mathcal{F}$ which is also a mod 0 isomorphism for each $\mathcal{F}_p$. An isomorphism of multi-period markets is a filtration isomorphism that preserves the cost functions.

Continuous time complete markets

Definition

A continuous time market $(\Omega, \mathcal{F}_t, \mathbb{P}), c_t)$ on $[0, T]$ is called a continuous time complete market with risk free rate r if there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ with

$$c_t(X) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{F}_t)$$

for $\mathbb{Q}$-integrable random variables X and equal to ∞ otherwise.

Example: Diffusion models

Consider a multi-dimensional diffusion model

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{\mu}(\boldsymbol{X}_t, t)\,\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\,\mathrm{d}\boldsymbol{W}_t.$$

subject to modest conditions, we can find a unique equivalent Martingale measure $\mathbb{Q}$ such that

$$c_t(X) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{F}_t).$$

for any contingent claim X.

Theorema Egregium?

Definition

The absolute market price of risk in a diffusion market is the element of $L^0(\Omega \times [0, T], \mathbb{P})$ defined by

$$\mathrm{AMPR}_t = |\boldsymbol{\sigma}^{-1}(r\boldsymbol{X}_t - \boldsymbol{\mu})|.$$

Theorem

The absolute market price of risk is an invariantly defined element for L^0.

Proof.

Let $q = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ then one can show

$$\int_0^t \frac{1}{q_s^2}\,\mathrm{d}[q,q]_s = \int_0^t \mathrm{AMPR}_s^2\,\mathrm{d}s.$$

□

The Test Case

Theorem

Let M be a continuous time complete market with risk free rate r, time period T based on a Wiener space of dimension n and with AMPR *given by*

$$\mathrm{AMPR}_t = A(t) \geq 0$$

for a bounded measurable function of time $A(t)$. Suppose that the process q_t is continuous. In these circumstances M is isomorphic to the diffusion market with

$$\boldsymbol{\mu} = r\boldsymbol{X}_t + A(t)\, e_1$$

$$\boldsymbol{\sigma} = \mathrm{id}_n$$

and $\boldsymbol{X}_0 = 0$ where $\{e_i\}$ is the standard basis for $\mathbb{R}^i$ and id_n is the identity matrix.

Sketch proof

Define

$$\tilde{Z}_t = \log q_t + \frac{1}{2}\int_0^t A(s)^2 \mathrm{d}s.$$

Then define

$$\tilde{W}_t^1 = -\int_0^t \frac{1}{A(s)}\,\mathrm{d}\tilde{Z}_s.$$

Show that $\tilde{W}_t^1$ is a Martingale and then compute

$$[\tilde{W}^1, \tilde{W}^1]_t = t.$$

Apply Levy's characterisation of Brownian motion to deduce that $\tilde{W}^1$ is a Brownian motion. Now extend W^1 to an n-dimensional Brownian motion.

Financial consequences

- All Black–Scholes–Merton models are isomorphic to an "essentially 1-dimensional" Bachelier market
- The equation
$$|\boldsymbol{\sigma}^{-1}(r\boldsymbol{X}_t - \boldsymbol{\mu})| = A(t)$$
is underdetermined. Hence any diffusion model is equivalent to a Black–Scholes–Merton model after a change of drift.
- Note that the drift is hard to measure from statistics and the form of the drift is chosen for parsimony.
- Rather than attempt to model assets, should we model the invariants such as AMPR?

Mutual fund theorem

Corollary (Continuous time one-mutual-fund theorem)

In diffusion markets with absolute market price of risk, any invariant, non-empty, convex set of martingales contains an element which can be replicated using only the risk-free-asset and the portfolio with components given by

$$(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}(r\boldsymbol{X}_t - \boldsymbol{\mu}).$$

Example

Consider optimal pension investment for a collective of individuals whose mortality is independent of the market. Even though you don't know the meaning of optimal investment for a collective, you know that however this is operationalized the optimal investment strategy will follow the one-mutual-fund theorem.

Selected references

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