Symmetry of Markets

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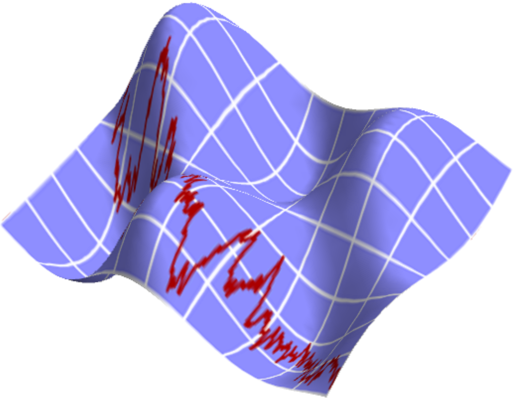


Overview

- Goal: explain how symmetry of markets is related to mutual fund theorems.
- Part 1: A geometric account of Markowitz's Theory and the classical two fund theorem.
- Part 2: Symmetry and one period complete markets
- Part 3: Symmetry and continuous time markets
- Example consequence: The n-dimensional Black–Scholes–Merton can always be simplified to a model with a single traded asset.

Part I

Markowitz Markets

Algebraic Definition

Definition

A non-degenerate Markowitz market is:

- a vector space V of portfolios
- a positive definite symmetric bilinear form $r : V \times V \to \mathbb{R}$ representing the covariance
- two linearly independent linear functionals
  - c representing the cost of a portfolio
  - p representing the mean payoff of a portfolio

Degenerate markets don't satisfy the independence and definiteness assumptions. I'll skip the words non-degenerate from now on.

Definition

Two Markowitz markets are isomorphic if their is a vector space isomorphism preserving r, c and p.

Category Theory

- Category theory formalizes the concept of "isomorphism" and "homomorphism".
- A category consists of
  - objects: in this case Markowitz Markets
  - morphisms with a source object and target object: in this case Markowtiz isomorphisms
  - a composition operation: in this case composition of isomorphisms
- It must satisfy various axioms, for example the existence of identity morphisms.
- Two objects are isomorphic if there are invertible morphisms between the two.
- The theory is ultra general. In most examples morphisms will be functions and composition will be function composition.

Examples of categories

Object	Morphisms
Vector Space	Linear Transformations
Group	Homomorphisms
Topological Space	Homeomorphism
Metric Space	Isometry
Banach Space	Bounded Linear Transformation
Markowitz Market	Markowitz isomorphism

- Two objects are isomorphic if they are "identical as far as your category is concerned".
- Example: A sphere and a cube are isomorphic topologically, but not as metric spaces.
- "Interesting" properties of an object should be invariant under isomorphisms
- Example: Two five pound notes are isomorphic. Their serial numbers are not interesting, only their purchasing power. (A five pound note is also isomorphic to five pound coins.)

The Markowitz Category

- We have defined our category in terms of a vector space V, a bilinear form r and two linear functions c and p.
- We are effectively saying that any features of the market that cannot be expressed invariantly in terms of V, r, c and p are not "financially interesting".
- We have not specified a basis of n special vectors representing the assets traded. This implies that we don't consider the distinction between a traded asset and a portfolio of assets to be financially interesting.
- We have not specified the payoff distribution of our assets, only the expected value and covariance structure. We are asserting that anything "financially interesting" can be discovered by mean variance analysis. Hence the quotes.
- The Markowitz category encapsulates mathematically what can be understood about a market through mean variance analysis alone.

Classification Theorems

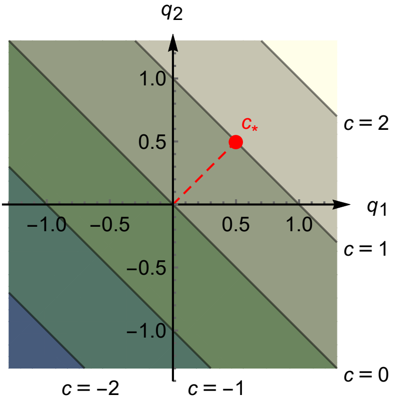
- Once you have defined a category, you can try to classify objects up to isomorphism.
- All finite dimensional real vector spaces are isomorphic to $\mathbb{R}^n$.
- All finite dimensional real vector spaces with a positive definite symmetric bilinear form are isomorphic to $\mathbb{R}^n$ with bilinear form $r(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$.
  - Equivalently, finite dimensional inner product spaces are isomorphic.
  - Equivalently, Euclid's axioms completely determine the geometry of n-dimensional space.
  - Proof: Using the Gram–Schmidt process you can find an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$. Writing r with respect to this basis it is isomorphic to the dot product.
- In particular, as an inner product space, any Markowitz market is isomorphic to Euclidean space.

Financial consequence

- In any Markowitz market we can find n uncorrelated portfolios of standard deviation 1.
- Using these portfolios as a basis for the market, we can represent any portfolio as a point in $\mathbb{R}^n$ with the standard deviation corresponding to the distance from the origin.
- We can solve portfolio optimization problems using Euclidean geometry.

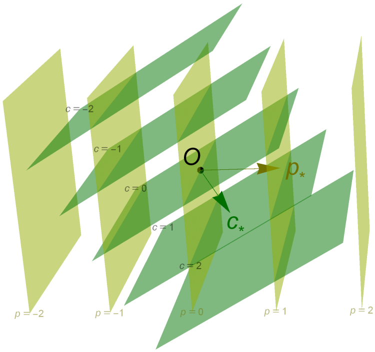
Geometry of linear functionals

The cost c of a portfolio can be represented geometrically by its contours. These are the hyperplanes of constant cost. There is a portfolio c_* that minimizes the risk among portfolios of cost 1. The vector c_* is a normal to the hyperplane of cost 1.



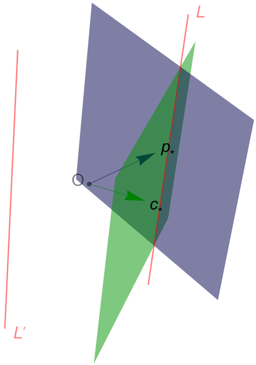
Geometry of linear functionals

The mean payoff p of a portfolio can be represented geometrically by its contours too. There is a portfolio p_* that minimizes the risk among portfolios of mean payoff 1.



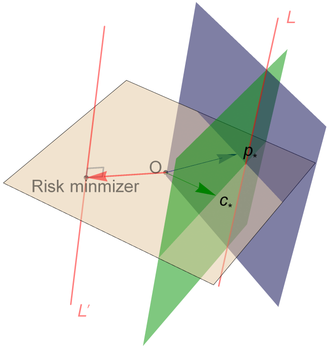
The two fund theorem: 1

The portfolios of cost 1 and mean payoff 1 lie in the intersection, L, of the hyperplanes orthogonal to p_* and c_*. Portfolios of another given cost and given mean payoff lie in a parallel hyperplane, L'.



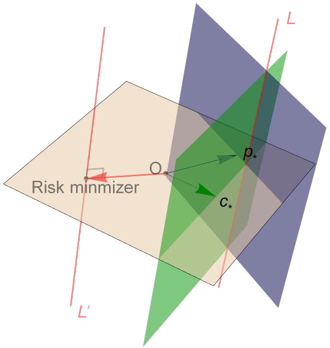
The two fund theorem: 2

Any linear combination of c_* and p_* is orthogonal to all vector in L'. In particular the intersection of the plane Oc_*p_* and L' is orthogonal to L' and so minimizes risk among all points on L'.



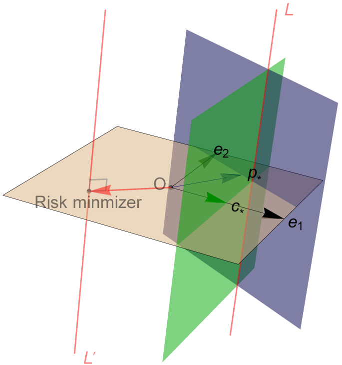
The two fund theorem: 3

The conclusion financially is that any risk minimizing portfolio of given cost and mean payoff can be expressed as a linear combination of any two "mutual funds" given by linearly independent portfolios in the plane $Oc_* p_*$.



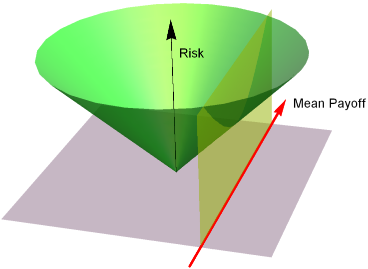
Classification of markets

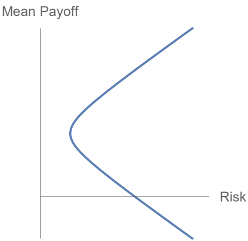
We can rotate the plane in $\mathbb{R}^n$ so it is spanned by e_1 and e_2. We can further rotate so c^* lies along the e_1 axis. The isomorphism class of the market is now uniquely determined by the length of c^* and the coordinates of p^*.



The efficient frontier

- The base plane is the plane of efficient portfolios.
- The cone is a graph of the risk of each portfolio, which is given by the distant from the origin.
- The thick red line contains the portfolios of cost 1.



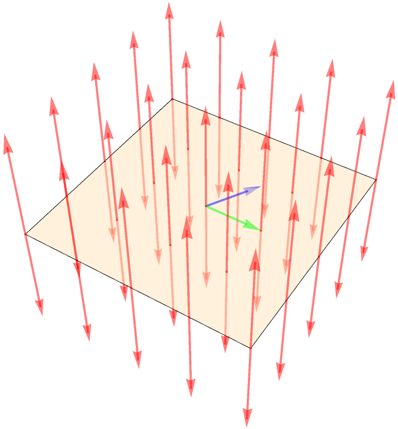


Symmetry

The reflection in the plane of efficient portfolios

$$(x_1, x_2, x_3, \ldots, x_n) \to (x_1, x_2, -x_3, \ldots, -x_n)$$

is an automorphism of the market. Therefore all invariant portfolios lie in the plane of efficient portfolios.



Invariantly defined objects

- An "invariantly defined" object is something you can describe using the data in your category without making an arbitrary choice.
- Example: The origin is an invariantly defined element in the category of vector spaces.
- Example: The origin is not an invariantly defined element in the category of topological spaces.
- Example: The serial number of a five pound note is not invariantly defined.
- The formal language of category theory can be used to give a rigorous definition of an "invariantly defined object" and then prove these statements.

A complex invariantly defined object

- Given a Markowitz market, consider the space $\mathcal{D}$ of probability densities on V that have a Wasserstein distance less than 0.01 from the Gaussian distribution on V^* with mean p and covariance matrix given by r. This set of densities is "invariantly defined".
- The optimal portfolios for a robust utility optimization problem when considering all measures in $\mathcal{D}$ will also be "invariantly defined"
- This is proved formally by showing if one combines invariantly defined sets using any of the constructions of set theory other than the axiom of choice, one will obtain an invariantly defined set.

Relation to Symmetry

Lemma

An object is invariantly defined only if it invariant under the appropriate action of automorphism group.

Corollary

Any invariantly defined portfolio must lie in the plane of efficient portfolios.

Corollary

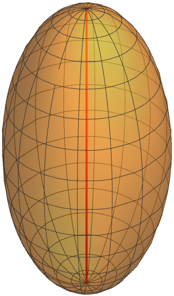
If the complex robust optimization problem has a unique solution, it lies in the plane of efficient portfolios.

Relation to Convexity

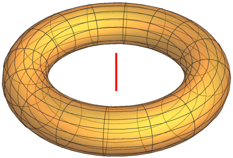
Lemma

Any convex subset C of a vector space V which is invariant under a compact Lie group, G, acting linearly on V contains an invariant element.

Convex, S^1-invariant



Non-convex, S^1-invariant



Proof.

Pick a point v in the set. Compute the average point under the action of G (the required measure exists by compactness of G). For $h \in G$

$$h\frac{1}{|G|}\int_G g\, v\, \mathrm{d}g = \frac{1}{|G|}\int_G hg\, v\, \mathrm{d}g = \frac{1}{|G|}\int_G g\, v\, \mathrm{d}g$$

by substitution. So

$$\frac{1}{|G|}\int_G g\, v\, \mathrm{d}g$$

is an invariant element of C. $\square$

Two fund theorem revisited

Corollary

We have found an action of $\mathbb{Z}_2$ *on any Markowitz market. Hence any invariant convex set of portfolios contains an efficient portfolio. Hence the solution to any convex portfolio optimization problem in a Markowitz market can be taken to be an efficient portfolio.*

- This argument is extremely general.
- If we consider other categories of markets and can find automorphisms of our markets we will find more mutual fund theorems.

Part II

General one period markets

Category: Probability Spaces

- Objects: Probability spaces $(\Omega, \sigma, \mathbb{P})$
- Morphisms: mod 0 isomorphisms.

$$\phi : \Omega^1 \setminus N_1 \to \Omega^2 \setminus N_2$$

  - N_1 and N_2 are null sets
  - ϕ is a bijection
  - ϕ is measurable
  - ϕ^{-1} is measurable

Definition

A probability space $(\Omega, \sigma, \mathbb{P})$ is *standard* if it is isomorphic mod 0 to either: the Lebesgue measure on $[0, 1]$; a probability space on a finite or countable number of atoms; a convex combination of both.

Properties of standard probability spaces

- Itô: "all probability spaces appearing in practical applications are standard"
- Kolmogorov: Perhaps we should add an axiom.
- Regular measures on $\mathbb{R}^n$ are standard
- The Wiener measure on $C^0[0, \infty)$ is standard
- Countable products of standard probability spaces are standard
- Non-null measurable subsets of standard probability spaces are standard when given the conditional measure.
- Non standard probability spaces require either very large Ω or the axiom of choice to construct.
- History: von Neumann, Rohklin.

Category: One period financial market

Definition

A one period financial market $((\Omega, \sigma, \mathbb{P}), c)$ consists of: a probability space $(\Omega, \sigma, \mathbb{P})$; a function $c : L^0(\Omega; \mathbb{R}) \to \mathbb{R} \cup \{\pm\infty\}$. We will call $c^{-1}(\mathbb{R} \cup \{-\infty\})$ the domain of c, denoted $\operatorname{dom} c$. Isomorphisms of markets are mod 0 isomorphisms that preserve c.

- Random variables represent possible asset payoffs
- Random variables in $\operatorname{dom} c$ are traded assets.
- c needn't be linear.
- Arbitrage may occur.
- Typically interested in restricted classes of markets, e.g. $\mathbb{P}$ standard.
- Dual to our algebraic definition. Portfolios, V, correspond to a subspace of $V \subseteq L^0(\Omega; \mathbb{R})$. So Ω corresponds to V^*.

Complete markets

Definition

A complete market is a market where

- $\mathbb{P}$ is standard.
- There exists $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

$$c(X) = \gamma E_{\mathbb{Q}}(X)$$

  for some discount factor $\gamma > 0$.

- Example: A casino S^1 has $\Omega = S^1$ (the circle) and $\mathbb{P} = \mathbb{Q} =$ Lebesgue.
- $F_{\frac{\mathrm{d}Q}{\mathrm{d}P}}$ is an invariant of the market.
- Given a F_X c.d.f. of a positive random variable X of mean 1 define a market M_{F_X} by $\Omega = [0,1]$, $\mathbb{P} =$ Lebesgue and

$$\mathbb{Q}(A) = \int_A F_X^{-1}(x)\mathrm{d}x \text{ for } A \subset [0,1].$$

- Mean 1 ensures $\int_{[0,1]} \mathrm{d}\mathbb{Q} = 1$

Classification of complete markets

Theorem

Let M be a complete market then

$$M \times S^1 \cong M_{F_X} \times S^1$$

for some positive random variable X of mean 1

Theorem

The solution of any convex optimization problem for a random variable on a complete market can be taken to be of the form $X = f(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}})$.

Proof.

- The result is true on $M_{F_X} \times C$ as S^1 is a compact Lie group.
- The resulting investment depends only on $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$, so the asset can be defined in M. Hence the flexibility of investing in a casino provides no advantage.

Part III

Continuous time markets

Category: Multi-period markets

Definition

A multi-period market consists of

(i) A filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ where $t \in \mathcal{T} \subseteq [0, T]$ for some index set $\mathcal{T}$ containing both 0 and T. We write $\mathcal{F} = \mathcal{F}_T$. We require $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(ii) For each $X \in L^0(\Omega; \mathbb{R})$, an $\mathcal{F}_t$ adapted process $c_t(X)$ defined for t in $\mathcal{T} \setminus T$.

Random variables $X \in L^0(\Omega, \mathcal{F}_T; \mathbb{R})$ are interpreted as contracts which have payoff X at time T. The cost of this contract at time t is $c_t(X)$.

Definition

A filtration isomorphism of filtered spaces $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $t \in \mathcal{T}$ for some index set $\mathcal{T}$ is a mod 0 isomorphism for $\mathcal{F}$ which is also a mod 0 isomorphism for each $\mathcal{F}_p$. An isomorphism of multi-period markets is a filtration isomorphism that preserves the cost functions.

Complete Continuous Time Markets

Harrison and Pliska show we can associate a complete market to an SDE of the form

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{\mu}(\boldsymbol{X}_t, t)\,\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\,\mathrm{d}\boldsymbol{W}_t. \tag{1}$$

Together with a risk-free rate r.

Example

The Black–Scholes–Merton market is given by

$$\boldsymbol{\mu}(\boldsymbol{X}_t, t) = \mathrm{diag}(\boldsymbol{X}_t)\tilde{\boldsymbol{\mu}}$$

$$\boldsymbol{\sigma}(\boldsymbol{X}_t, t) = \mathrm{diag}(\boldsymbol{X}_t)\tilde{\boldsymbol{\sigma}}$$

Example

The Bachelier market is given by

$$\boldsymbol{\mu}(\boldsymbol{X}_t, t) = r\boldsymbol{X}_t + \tilde{\boldsymbol{\mu}}(t)$$

$$\boldsymbol{\sigma}(\boldsymbol{X}_t, t) = \tilde{\boldsymbol{\sigma}}$$

Invariants of Continuous Time Markets

- The diameter of a Riemannian manifold is an invariant.
- The curvature of a Riemannian manifold is a local invariant. It is easily calculated in terms of the derivatives of the metric at a point.
- The distribution of $\frac{d\mathbb{Q}}{d\mathbb{P}}$ at time t is an invariant.
- Can we find local invariants in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ and their derivatives analagous to curvature?

Absolute Market Price of Risk

Definition

If X_t is an adapted process

$$\mathrm{drift}_{\mathbb{P}}(X)_t := \lim_{h \to 0} \mathbb{E}\left(\frac{X_{t+h} - X_t}{h}\right).$$

if this exists.

Definition

$$Q_t := \mathbb{E}_{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \mid \mathcal{F}_t\right).$$

$$\mathrm{AMPR}_t = \sqrt{-2\,\mathrm{drift}_{\mathbb{P}}(\log Q_t)}. \tag{2}$$

Lemma

$$\mathrm{AMPR}_t = |\boldsymbol{\sigma}^{-1}(r\boldsymbol{X}_t - \boldsymbol{\mu})|$$

Theorem (The test case)

Let M be a continuous time complete market with risk free rate r, time period T based on a Wiener space of dimension n and with AMPR *given by*

$$\mathrm{AMPR}_t = A(t) \geq 0$$

for a bounded measurable function of time $A(t)$. Suppose that the process Q_t is continuous. In these circumstances M is isomorphic to the Bachelier market with

$$\mathrm{d}\boldsymbol{X}_t = (r\boldsymbol{X}_t + A(t)\, e_1)\, \mathrm{d}t + \mathrm{d}\boldsymbol{W}_t$$

and $\boldsymbol{X}_0 = 0$. Here $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. We will call markets of this form canonical Bachelier markets.

Proof sketch - part 1

- **Motivation:** We expect the market to be essentially one dimensional, with a non-zero market price of risk for $\tilde{W}_t^1$. So any fluctuations in Q_t should be correlated with $\tilde{W}_t^1$ but not any of the $\tilde{W}_t^i$ for $t \geq 0$. We should therefore be able to express $\tilde{W}_t^1$ in terms of Q_t.
- Required expressions are:

$$\tilde{Z}_t = \log Q_t + \frac{1}{2}\int_0^t A(s)^2 \mathrm{d}s.$$

$$\tilde{W}_t^1 = -\int_0^t \frac{1}{A(s)}\,\mathrm{d}\tilde{Z}_s.$$

- Compute the quadratic variation of $\tilde{W}_t^1$ and simplify using the expression

$$\mathrm{AMPR}_t = A(t) \geq 0.$$

$$\implies [\tilde{W}^1, \tilde{W}^1]_t = t$$

- By Levy's characterisation of Brownian motion, $\tilde{W}_t^1$ is Brownian motion.

Proof sketch - part 2

- Combine the idea of the Gram–Schmidt process with the Martingale representation to extend $\tilde{W}_t^1$ to an n-dimensional Brownian motion $\tilde{W}_t$ with covariance matrix id_n.
- Compute $\tilde{Q}_t = \mathbb{E}(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|\mathcal{F}_t)$ for the market

$$\mathrm{d}\boldsymbol{X}_t = (r\boldsymbol{X}_t + A(t)\,e_1)\,\mathrm{d}t + \mathrm{d}\tilde{\boldsymbol{W}}_t$$

  and show it coincides with Q_t.
- Therefore prices coincide in these markets, hence the markets are isomorphic.

Consequences

- Black–Scholes–Merton markets are Bachelier markets in disguise. This explains the surprising tractability of the Merton problem.
- We have the symmetry

$$(\tilde{W}_t^1, \tilde{W}_t^2, \tilde{W}_t^3, \dots \tilde{W}_t^n) \to (\tilde{W}_t^1, -\tilde{W}_t^2, -\tilde{W}_t^3, \dots - \tilde{W}_t^n)$$

  in a canonical Bachelier market, so the only invariant asset is asset X^1.
- In a canonical Bachelier market only X^1 has a non-trivial market price of risk. Investing in the other n-assets gives risk without reward, so is clearly foolish.
- In markets that are merely isomorphic to this, X^1 may not an exchange traded asset, but can be replicated by a continuous time trading strategy.

Continuous Time Mutual Fund Theorem

Theorem

In continuous time markets with deterministic, bounded, absolute market price of risk, any non-empty convex set of martingales contains an element which can be replicated by a continuous time trading strategy in the risk-free asset and a portfolio with quantities α in each asset given by

$$\boldsymbol{\alpha} = (\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}(r\boldsymbol{X}_t - \boldsymbol{\mu}).$$

- Example: The classical Merton-problem
- Example: Optimal investment of a Collective Defined Contribution pension fund in a Black–Scholes–Merton market when other risk factors such as lifecycle events (birth, death, marriage etc.) are independent of the market.
- We can generally reduce an apparently intractable n-dimensional problem to a tractable 1-dimensional problem.