Itô Stochastic Differential Equations as 2-Jets

Drawing SDEs

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Outline

Aims:

- Give a coordinate free definition of a stochastic differential equation (SDE).
- Draw a picture of a stochastic differential equation

$$\begin{array}{ccc} \text{SDE for X} & \xrightarrow{\text{Itô's lemma}} & \text{SDE for } f(X) \\ \text{Draw}\downarrow & & \text{Draw}\downarrow \\ \text{Picture of SDE for X in } \mathbb{R}^n & \xrightarrow{f} & f(\text{Picture of SDE for X}) \end{array}$$

Analogy: Vector fields (ODEs) on manifolds can be defined in many ways: in coordinate charts; as first order approximations to curves; as operators satisfying certain algebraic properties.

Remarks

Other approaches to coordinate free definition of SDEs do exist:

- Use *Stratonovich calculus* to define SDEs on manifolds.
- Use *Schwartz-Morphisms* to define SDEs on manifolds.

Our apporoach has a number of applications:

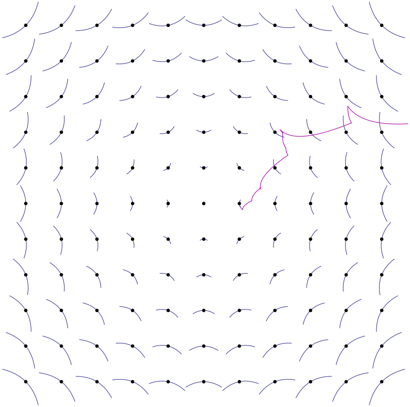
- Gives practical schemes for simulating SDEs on manifolds, which is useful in statistical estimation.
- Allows us to define a new notion of “projection” which can be used to obtain better low dimensional approximations to high dimensional SDEs.
- Visualisation of SDEs.

Stochastic Difference Equation

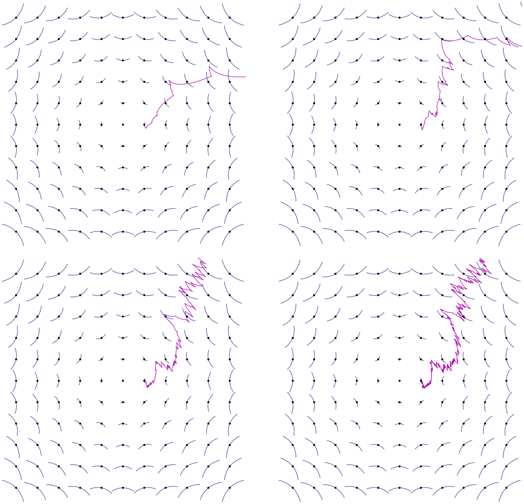
Let γ_x be a field of curves defined at each point of a manifold M. Define a process by

$$X_0 := x_0, \quad X_{t+\delta t} := \gamma_{X_t}(W_{t+\delta t} - W_t). \tag{1}$$

where W_t is Brownian motion.



Take the limit as δt tends to 0



A coordinate free notion of convergence

What metric should we use?

- Classical: $E(|X_t - \tilde{X}_t|^2)$
- Coordinate free: “Mean square on compacts”. Motivation is that all Riemannian metrics are equivalent on compact sets.

Definition:

- Let K be a compact set in our manifold, g a Riemannian metric.
- Define $\sim$ so that $M/\sim$ is the one point compactification of the interior K^0.
- Define a semi-metric

$$\tilde{d}^{g,K}([x],[y]) = \inf_{X\sim x, Y\sim y} d^g(X,Y).$$

- We have convergence in mean square on compacts if we have convergence in $E(\tilde{d}^{g,K}(X_t,\tilde{X}_t)^2)$ for all K and g.

Stochastic Differential Equation

If X_t is the limit as $\delta t \to 0$ then we write:

$$\text{Coordinate free SDE:} \quad X_t \curvearrowright \gamma_{X_t}(\mathrm{d}W_t), \quad X_0 = x_0.$$

Example

We take the manifold $\mathbb{R}$ and define curves each point in $\mathbb{R}$ by:

$$\gamma_x^\alpha(s) = x + s^\alpha$$

$$X_{n\delta t} = x_0 + \sum_{i=1}^{n} (W_{(i+1)\delta t} - W_{i\delta t})^\alpha = x_0 + (\delta t)^{\frac{\alpha}{2}} \sum_{i=1}^{n} \epsilon_i^\alpha$$

- When α=1, $X_t = x_0 + W_t$
- When α=2, $X_t = x_0 + t$
- When $\alpha \geq 3$, $X_t = x_0$

2-jets

We now break coordinate invariance and work in local coordinates.

- Taylor expansion

$$\delta X_t = \gamma'_{X_t}(0)\delta W_t + \frac{1}{2}\gamma''_{X_t}(0)(\delta W_t)^2 + R_{X_t}(\delta W_t)^3, \quad X_0 = x_0.$$

- The limit as $\delta t \to 0$ depends upon the first two terms.
- We only need the 2-jet of the curve to define the SDE.

Definition

An n-jet is an equivalence class of smooth maps $f : M \to N$ between manifolds that are considered equal if the order n terms of their Taylor expansions are equal.

- ODE's correspond to one jets (vector fields)
- SDE's correspond to 2-jets of maps from $\mathbb{R}^k \to M$.

$$X_t \smile\!\!\!\to j_2(\gamma_{X_t}(\mathrm{d}W_t)), \quad X_0 = x_0.$$

Itô calculus

We have the Taylor expansion

$$\delta X_t = \gamma'_{X_t}(0)\delta W_t + \frac{1}{2}\gamma''_{X_t}(0)(\delta W_t)^2 + R_{X_t}(\delta W_t)^3, \quad X_0 = x_0.$$

Our example calculation suggests $(\delta W_t)^2 \approx \delta t$

$$\delta X_t \approx \gamma'_{X_t}(0)\delta W_t + \frac{1}{2}\gamma''_{X_t}(0)\delta t \quad X_0 = x_0.$$

Defining $a(X) := \gamma''_X(0)/2$ and $b(X) := \gamma'_X(0)$:

$$\delta X_t = a(X_t)\delta t + b(X_t)\delta W_t.$$

it is well known that this Euler scheme converges to the solution of the Itô SDE

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + b(X_t)\,\mathrm{d}W_t.$$

Note the use of the vector space structure of $\mathbb{R}^n$ in this formulation.

Itô's lemma

Lemma

If $f : M \to N$ is smooth and the process X_t satisfies

$$X_t \curvearrowright j_2(\gamma_{X_t})(\mathrm{d}W_t)$$

then $f(X_t)$ satisfies

$$f(X)_t \curvearrowright j_2(f \circ \gamma_{X_t})(\mathrm{d}W_t).$$

Lemma

If $f : M \to N$ is smooth and the process X_t satisfies

$$\mathrm{d}X_t^i = a^i(X_t)\mathrm{d}t + b^i(X_t)\,\mathrm{d}W_t$$

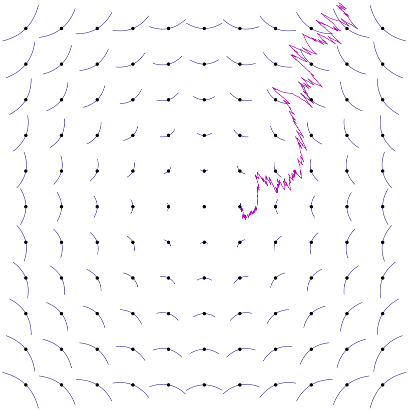
then $f(X_t)$ satisfies

$$\mathrm{d}f^i(X)_t = \left(\sum_j \frac{\partial f^i}{\partial x^j} a^j(X,t) + \frac{1}{2} \sum_i \frac{\partial^2 f^i}{\partial x^j \partial x^j} (b^j(X,t))^2 \right) \mathrm{d}t + \left(\sum_j \frac{\partial f^i}{\partial x^j} b^j(X,t) \right) \mathrm{d}W_t$$

(Classical formulation also requires technical bounds and Lipschitz conditions)

Itô's lemma example

$$\gamma^{E}_{(x_1,x_2)}(t) = (x_1, x_2) + t(-x_2, x_1) + 3t^2(x_1, x_2).$$



$$\mathrm{d}(x_1, x_2) = \frac{3}{2}(x_1, x_2)\mathrm{d}t + (-x_2, x_1)\mathrm{d}W_t.$$

Itô's lemma example

If

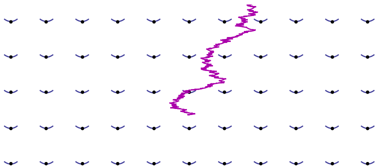
$$\mathrm{d}(x_1, x_2) = 3(x_1, x_2)\mathrm{d}t + (-x_2, x_1)\mathrm{d}W_t.$$

and

$$f(\exp(s)\cos(\theta), \exp(s)\sin(\theta)) = (\theta, s),$$

by Itô's lemma $(\theta_t, s_t) = f((x_1, x_2))$ satisfies

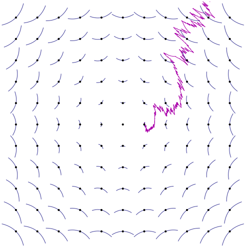
$$d(\theta, s) = (0, \frac{7}{2})dt + (1, 0)dW_t. \tag{2}$$



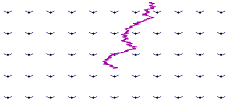
Itô's lemma graphically

$d(x_1, x_2) = 3(x_1, x_2)dt + (-x_2, x_1)dW_t$ Itô's Lemma $\rightarrow$ $d(\theta, s) = (0, \frac{7}{2})dt + (1, 0)dW_t$

Draw $\downarrow$ Draw $\downarrow$



Transform $\rightarrow$



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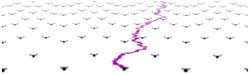
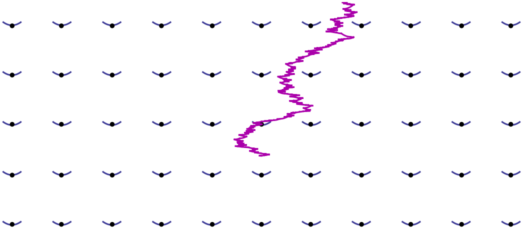
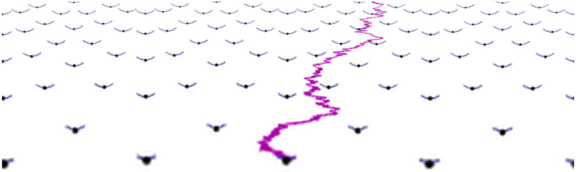


Transforming algebraically and graphically

Algebraically



Graphically

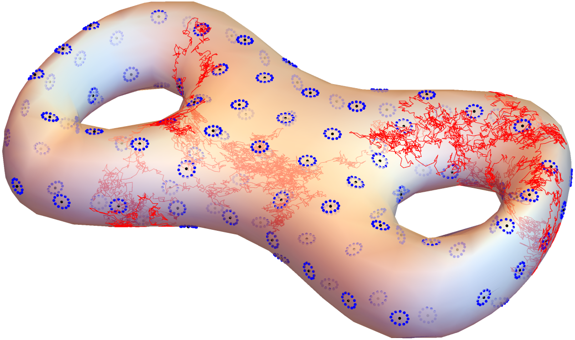


Coordinate free stochastic differential geometry

- If
$$\gamma_x : \mathbb{R}^k \to M$$
is an SDE and $f : M \to \mathbb{R}$ is a smooth function define
$$\mathcal{L}_x f = \frac{1}{2}\Delta(f \circ \gamma_x).$$
This is the *backward diffusion operator*. It acts on functions.
- For compact M, we have a pairing between densities ρ and functions f given by
$$\int_M f\rho.$$
We can define $\mathcal{L}^*$ to be the formal adjoint of $\mathcal{L}$ with respect to this pairing. This is the *forward diffusion operator*, it acts on densities.
- γ_x defines *Brownian motion* on a Riemannian manifold if $\mathcal{L} = \frac{1}{2}\Delta_g$.

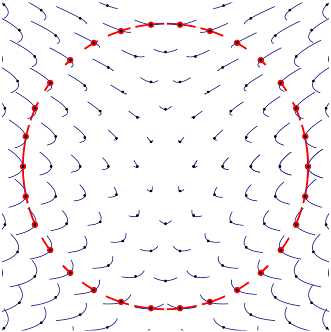
Applications - Simulating SDEs on manifolds

The classical Euler scheme is a poor choice for simulating SDEs on manifolds as one rapidly leaves the manifold. By choosing the curves γ to closely follow a manifold, one can obtain simulations that stay close to the manifold.



Applications - Projection

It is intuitively clear how to project an SDE onto a manifold in the jet formulation. This gives a different notion of projection than that obtained using Stratonovich calculus and which is in a clearly defined sense superior to Stratonovich projection as a method for approximating SDEs.



Stratonovich calculus and jets

- *Stratonovich calculus* is an alternative to Itô calculus which allows you to define SDEs using vector fields.
- Given a vector field A, let Φ_A^t be the family of diffeomorphisms obtained by flowing along A for time t.
- Given two vector fields A and B, define

$$\gamma_x(t) = \Phi_A^{t^2} \circ \Phi_B^t(x).$$

- This SDE is equivalent to the classical Stratonovich SDE

$$\circ \, \mathrm{d}X_t = A(X_t)\mathrm{d}t + B(X_t) \circ \mathrm{d}W_t.$$

- Thus Stratonovich calculus can be interpreted simply as giving an alternative parameterization of a field of curves to the parameterization of Itô calculus.

Jets and fan diagrams

Another way of drawing an SDE is a fan diagram

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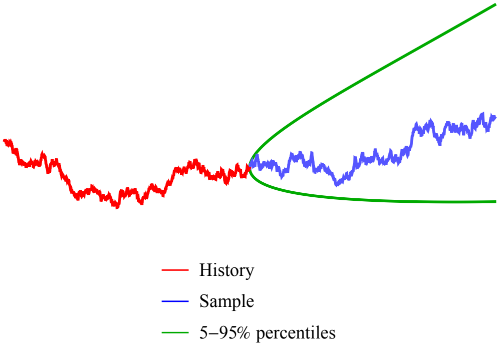
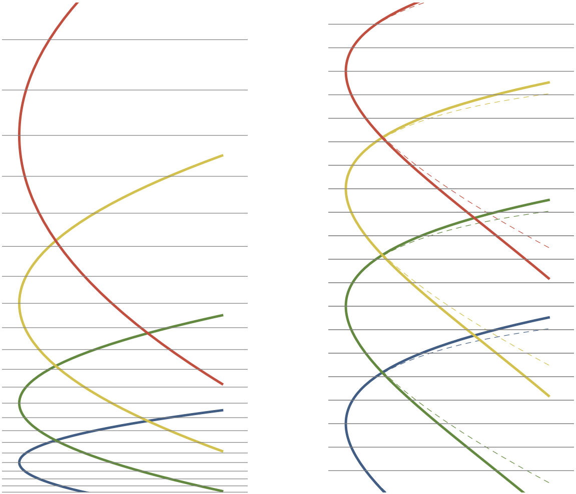


Brownian motion and Geometric Brownian motion



Geometric Brownian Motion Brownian Motion

Fan diagrams at the percentiles $\pm\Phi(1)$

Mean, Median, Mode

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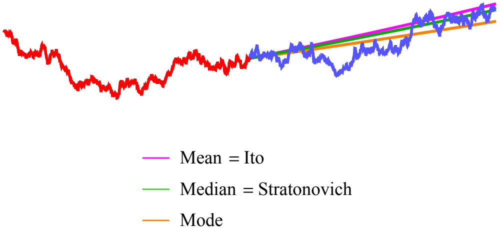

Summary

- The language of 2-jets allows us to formulate stochastic differential equations in a coordinate free manner.
- Itô's lemma has the simple interpretation of composition of functions
- One can visualize stochastic differential equations
- These visualization tools show statistically interesting features of the SDE