Optimizing S-shaped utility and risk management

Ineffectiveness of VaR and ES constraints

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Are ES constraints effective against rogue traders?

We will approach this question as follows

- Develop a mathematical model for how a “rogue trader” will behave.
- Use this to determine their behaviour in some standard market models, in particular the Black-Scholes model, when risk constraints are applied.
- Consider VaR, ES and constraints based on expected utility.
- Calculate the consequences of their behaviour and decide if it is desirable.

Complimentary to the “axiomatic” approach e.g.:

- von Neummann and Morgenstern gave an axiomatic approach to preferences over probability distributions that leads to *utility functions*.
- Artzner, Delbaen, Eber and Heath gave an axiomatic approach to *coherent risk measures* that suggests VaR is not a good risk measure, but ES (aka CVaR) is.
- etc. etc. There is a large literature.

Utility functions

Theorem

(von Neumann–Morgernstern) Let $\preceq$ be a preference relation defined on probability densities satisfying 3 relatively uncontroversial axioms plus the independence axiom

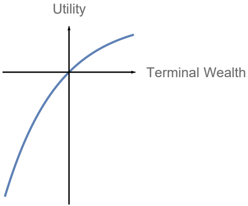
$$L \preceq M \implies pL + (1-p)N \preceq pM + (1-p)N$$

then $\preceq$ can be given in terms of a utility function $u : \mathbb{R} \to \mathbb{R}$ by

$$L \preceq M \text{ iff } E(u(L)) \leq E(u(M)).$$

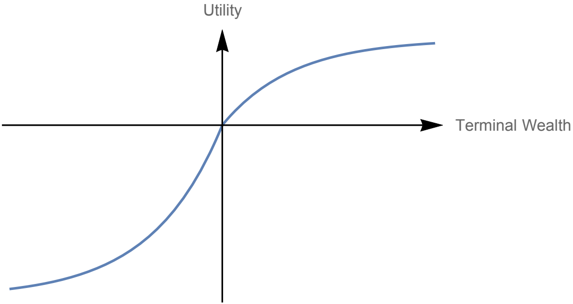
One might additionally expect

- u will be increasing (preference for profit);
- u will be concave (risk-aversion).



S-shaped utility functions

Kahneman and Tversky found in psychological experiments that most people appear to have S-shaped utility curves, so they are not always risk averse.



- A rogue trader loses their job and reputation but nothing more if they experience large losses.
- A limited liability company will have S-shaped utility.

We will model rogue traders as optimizing an S-shaped utility function.

Risk constraints

We will consider risk-constraints of the form $\rho(X) \leq L$ where ρ is a risk figure depending on the distribution of the portfolio payoff X and L is a risk limit. ρ could be:

- a Value at Risk (VaR) figure,
- an Expected Shortfall (ES also known as CVaR) figure
- an expected disutility $-E(u_R(X))$. u_R is the risk-manager's utility not the trader's utility.

Definitions

- The 5%-Value at Risk (VaR) of a portfolio over a given time horizon corresponds to maximum loss experienced in the 95% best-case scenarios.
- The 5%-Expected Shortfall (ES) over the same time horizon corresponds to the expected loss in the 5% worst-case scenarios.

More precisely if $\alpha \in [0, 1]$

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\alpha(X) \mathrm{d}\alpha.$$

Formal defintion

For this talk, an function $u : \mathbb{R} \to \mathbb{R}$ is S-shaped if:

- It is increasing.
- $u(x) \leq 0$ for $x \leq 0$.
- $u(x) \geq 0$ for $x \geq 0$.
- For sufficiently small x, $u(x) \geq C(-x)^\eta$ for some constant $C > 0$ and $\eta \in (0, 1)$. We say it is risk-seeking on the left.
- For sufficiently large x, $u(x) \leq Cx^\eta$ for some constant $C > 0$ and $\eta \in (0, 1)$. We say it is risk-averse on the right.

Modelling the Market

- We assume both trader and risk manager agree on the probability model $\mathbb{P}$ underlying the dynamics of the market.
- We assume that prices in the model are given by discounted expectations in a risk-neutral probability model $\mathbb{Q}$. We consider only the case of a constant risk-free rate r.
- We assume that the market is complete. That is we assume that arbitrary derivative securities can be purchased at the risk-neutral price (so long as this price exists).

Examples

- The Black–Scholes–Merton market in continuous time where one can trade in the stock and a risk free bond.
- A discrete time version of the Black–Scholes–Merton market where any derivative can be purchased at the Black–Scholes price so long as it has a fixed maturity T and European style exercise.

The optimization problem

Find a sequence of investments $X_1, X_2, \ldots$ achieving optimal trader utility

$$\lim_{i \to \infty} E(u_T(X_i)) = \sup_X E(u_T(X))$$

subject to a cost constraint

$$E_Q(X) \leq e^{rT} C' = C$$

and a risk-management constraint

$$\rho(X) \leq L.$$

Remark

We seek a sequence of investments because we cannot always expect the supremum to be achieved. For example it is obvious that in markets with no risk-constraints there will normally be no limit on the expected utility other than $\sup(u_T)$ itself.

Results

Subject to some additional requirements on the market which are all satisfied in the Black–Scholes cases we find:

- For ES constraints, the only limit on the expected utility that can be achieved is $\sup(u_T)$.
- Hence for VaR constraints, the only limit on the expected utility that can be achieved is also $\sup(u_T)$.
- Expected disutility constraints $-E(u_R(X)) \leq L$ written in terms of a risk-manager's concave increasing utility function u_R typically DO limit the utility that can be achieved. (This result requires some further assumptions on the risk-managers utility function).

The main step to proving these results is reducing the optimization problem to a 1-dimensional problem that is easy to solve.

Interpretation

Our interpretation is that

- Rogue traders will not be concerned if they are obliged to act under ES and VaR constraints. The utility they can achieve is unaffected.
- Moreover, for reasonable risk manager utility functions u_R, rogue traders will choose strategies that have unboundedly negative risk manager utilities.
- In brief: ES and VaR constraints don't work. Expected utility constraints do work.

Reduction to one dimension: the financial intuition

Recall the problem we wish to solve is
Maximize

$$E(u_T(X))$$

subject to a cost constraint

$$E_Q(X) \leq C$$

and a risk-management constraint

$$\rho(X) \leq L.$$

Remark
Note that all that matters are the $\mathbb{P}$ and $\mathbb{Q}$ measure distributions of X. Intuitively the trader can decide how much money to put on a specific event ω by just looking at the ratio of the $\mathbb{P}$ and $\mathbb{Q}$ measure probabilities.

Rigorous formulation

Theorem

(Subject to a very mild technical condition) We may restrict attention to X of the form

$$X = \tilde{f}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) = f\left(1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}P}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right)$$

where f is an increasing function and

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$$

is the Radon-Nikodym derivative.

In other words, go long on events you think are under-priced and go short on events you think are over-priced. Buy low, sell high. The proof relies on the Hardy–Littlewood theory of rearrangements. A general version of this result has been found independently by Xunyu Zhou.

Rewriting the optimization problem

Subject to very mild technical conditions, we may write our ES optimization problem as follows.

Find a payoff function $f : [0,1] \to \mathbb{R}$ depending only on $1 - F_{\frac{\mathrm{d}Q}{\mathrm{d}P}}$ maximizing

$$\int_0^1 u_T(x)\mathrm{d}x$$

subject to a cost constraint

$$\int_0^1 f(x)q(x)\mathrm{d}x \leq C$$

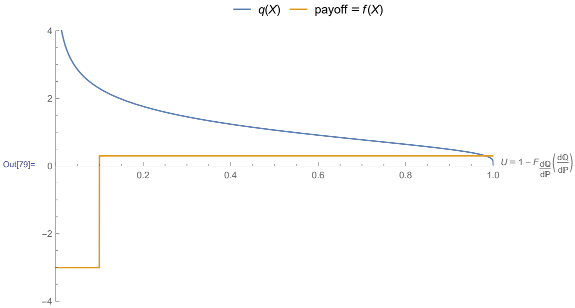
and an ES constraint

$$\frac{1}{p}\int_0^p f(x)\mathrm{d}x \geq L$$

where q is the probability density function of $X = 1 - F_{\frac{\mathrm{d}Q}{\mathrm{d}P}}(\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}})$. X is uniformly distributed.

Pictorial representation

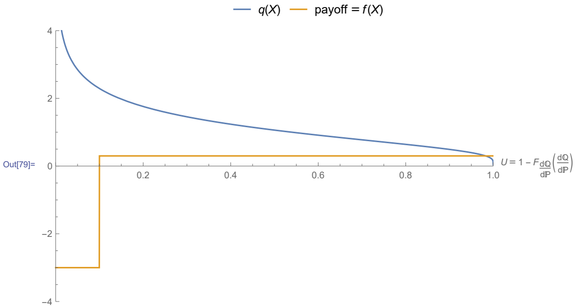
We must choose an increasing payoff function f to maximize $\int_0^1 u_T(f(x))\mathrm{d}x$ subject to $\int_0^p f(x)\mathrm{d}x \geq L$ and $\int_0^1 f(x)q(x)\mathrm{d}x \leq C$.



The density $q(x)$ shown is for the Black–Scholes model. If $q(x) \to \infty$ as $x \to 0$ then arbitrary expected trader utilities u_T can be achieved using step functions as shown.

Main negative result

- If $\sup q(x) = \infty$ then VaR and ES constraints are ineffective in constraining a trader with S-shaped utility.
- Digital payoffs of the form shown below can be used to achieve arbitrarily high trader utilities subject to the cost and risk constraints. The only limit is $\sup u_T$ itself.



The Black–Scholes case

In the $\mathbb{P}$ measure

$$z_T := \log S_T \sim N(\log S_0 + (\mu - \frac{1}{2}\sigma^2)T, \sigma\sqrt{T})$$

In the $\mathbb{Q}$ measure

$$z_T := \log S_T \sim N(\log S_0 + (r - \frac{1}{2}\sigma^2)T, \sigma\sqrt{T})$$

Write $p(z_T)$ for the pdf of z_T in the $\mathbb{P}$ measure. $q(z_T)$ for the $\mathbb{Q}$ measure pdf.

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(z_T) = \frac{q(z_T)}{p(z_T)} = \frac{\exp\left(-\frac{(z_T - \log S_0 - (r - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\right)}{\exp\left(-\frac{(z_T - \log S_0 - (\mu - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\right)}$$

$$= e^{\frac{(\mu - r)\left(T\left(\mu + r - \sigma^2\right) + 2\log(S_0) - 2z_T\right)}{2\sigma^2}}$$

$$\to \infty \text{ as } z_T \to -\infty \text{ if } \mu > r$$

Main positive result

- Suppose the risk-manager's utility function is given by

$$u_R(x) = \begin{cases} -(-x)^\gamma & x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

  for γ in $(1, \infty)$. Suppose they impose a limit $E_{\mathbb{P}}(u(X)) \geq L$.
- Suppose the trader has S-shaped utility and moreover is *difficult to satisfy* which means that if we prohibit short selling, they cannot achieve the supremum of their utility function.
- Suppose that

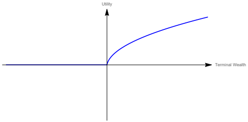
$$E_{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}^{\frac{\gamma_R}{\gamma_R-1}}\right) \tag{1}$$

  is finite.
- Then the risk manager's expected utility constraint is binding.

The requirement (1) is automatically satisfied in the Black–Scholes model.

Proof of result

We restrict attention to traders with limited-liability.



For any increasing f we can find p such that

$$f(X) = \begin{cases} f(X) \geq 0 & x > p \\ f(X) \leq 0 & x < p \end{cases}$$

For fixed choice of p the problem is then the convex problem

$$\begin{array}{cc} \text{minimize} & \int_p^1 -u_T(f(x))\mathrm{d}x \\ \text{subject to} & \int_0^p u_R(f(x))\mathrm{d}x \leq L \\ \text{and} & \int_0^1 q(x)f(x)\mathrm{d}x \leq C \end{array}$$

Example solutions in Black-Scholes model

Note payoff profiles drawn against S_T rather than uniform X.

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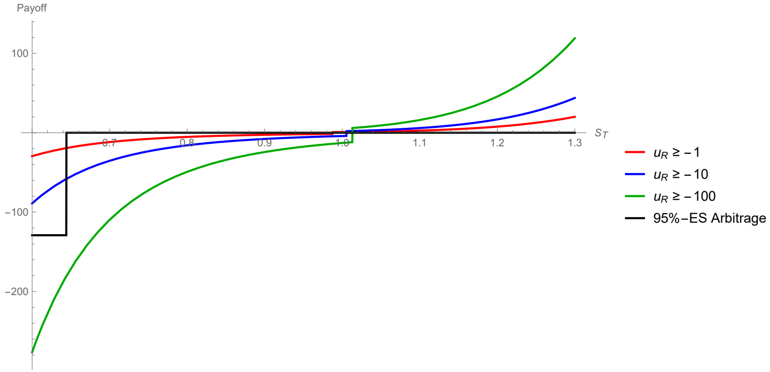

Incomplete markets

The most obvious criticism of our result is that we assume a complete market.

Definition

An α-ES arbitrage portfolio is a portfolio which has:

- a negative expected shortfall at confidence level α
- a non-positive cost

Justification:

- Since the expected shortfall is negative, the payoff must sometimes be positive.
- If such a portfolio exists, then a trader can buy arbitrarily large quantities without violating *ES* or cost constraints.
- If the trader has limited liability then their expected utility will only increase as they buy larger quantities of the portfolio.

If an α-ES arbitrage portfolio exists, α-ES limits will be ineffective.

Summary

- In general VaR and ES limits are not effective in curbing the risks taken by rogue traders.
- Limits set using concave increasing utility functions can be effective in reasonable market models.