Indifference pricing of index options with transaction costs

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- We study pricing and hedging of European options on the S&P500 index.
- Instead of risk neutral pricing, we develop a computational framework for hedging-based indifference pricing.
- In general, the indifference prices are nonlinear functions of an options cash-flows and they depend on an agent's
  - views on future development of the market,
  - risk preferences,
  - financial position.
- For replicable claims, indifference prices are independent of such subjective factors and they coincide with the classical risk neutral prices.

Optimal investment

Consider the asset-liability management problem

$$\begin{array}{lll} \text{minimize} & Ev(c + S_T(-x)) \quad \text{over} \quad x \in D \\ \text{subject to} & S_0(x) \leq w, \end{array} \tag{ALM}$$

where

- $w \in \mathbb{R}$ is the initial wealth,
- $c \in L^0(\Omega, \mathcal{F}, P)$ a random claim to be paid at time T,
- $D \subseteq \mathbb{R}^J$ is the set of feasible portfolios,
- S_t are convex functions giving the cost of buying a portfolio $x \in \mathbb{R}^J$ at time t. While S_0 is deterministic, S_T is random.
- v is a nondecreasing convex function on $\mathbb{R}$ describing the agent's disutility function from delivering cash at time T.

Optimal investment

Consider the asset-liability management problem

$$\begin{array}{lll} \text{minimize} & Ev\left(c + S_T(-x)\right) \quad \text{over} \quad x \in D \\ \text{subject to} & S_0(x) \le w. \end{array} \tag{ALM}$$

- The optimum value and optimal solutions depend on the agent's
  - views described by the probability measure P under which the expectation is taken,
  - risk preferences described by the disutility function v,
  - financial position described by $(w, c) \in \mathbb{R} \times L^0$.
- When trading, one is concerned on how the optimum value is affected by changes in the financial position (w, c).
- We denote the optimum value by $\varphi(w, c)$.

Indifference pricing

- Consider the problem of valuing a contingent claim $c \in L^0$ from the point of view of an agent whose current financial position is given by $(\bar{w}, \bar{c}) \in \mathbb{R} \times L^0$.
- The indifference selling price

  $$\pi_s(\bar{w}, \bar{c}; c) = \inf\{w \mid \varphi(\bar{w} + w, \bar{c} + c) \leq \varphi(\bar{w}, \bar{c})\}$$

  gives the least price at which the agent could sell the option without worsening his financial position.
- The indifference buying price

  $$\pi_b(\bar{w}, \bar{c}; c) = \sup\{w \mid \varphi(\bar{w} - w, \bar{c} - c) \leq \varphi(\bar{w}, \bar{c})\} = -\pi_s(\bar{w}, \bar{c}; -c)$$

  gives the geatest price at which he could buy the option.

Indifference pricing

We denote the super- and subhedging prices by

$$\pi_{\sup}(c) = \inf\{w \mid \exists x \in D: \; S_0(x) \leq w, \; S_T(-x) + c \leq 0\},$$
$$\pi_{\inf}(c) = \sup\{w \mid \exists x \in D: \; S_0(x) \leq -w, \; S_T(-x) - c \leq 0\}.$$

Theorem

The function $\pi_s(\bar{w}, \bar{c}; \cdot)$ is convex and nondecreasing on L^0. If S_t are sublinear, D is a cone and $\pi_s(\bar{w}, \bar{c}; 0) \geq 0$, then

$$\pi_{\inf}(c) \leq \pi_b(\bar{w}, \bar{c}; c) \leq \pi_s(\bar{w}, \bar{c}; c) \leq \pi_{\sup}(c).$$

with equalities throughout when c is replicable in the sense that there exists an $x \in D \cap (-D)$ such that

$$S_0(x) \leq -S_0(-x) \quad \text{and} \quad S_T(x) \leq c \leq -S_T(-x).$$

Pricing of S&P500 options

Assume now that the set J of tradeable assets consists of a cash account, S&P500 index futures and put and call options on the index all with the same maturity. We model the prices as:

$$S_t(x) = \sum_{j \in J} S_t^j(x^j),$$

where

$$S_0^j(x^j) = \begin{cases} s_+^j x^j & \text{if } x^j \geq 0 \\ s_-^j x^j & \text{if } x^j \leq 0 \end{cases} \quad \text{and} \quad S_T^j(x^j) = s_T^j x^j$$

and s_+^j and s_-^j denote the bid- and ask-prices of asset j. $s_-^j \leq s_+^j$, so S_0 is convex. The final prices are

$$s_T^j = \begin{cases} \exp(rT) & \text{if } j \text{ is cash,} \\ Z_T & \text{if } j \text{ is a future,} \\ \max\{Z_T - K_j, 0\} & \text{if } j \text{ is a call with strike } K_j, \\ \max\{K_j - Z_T, 0\} & \text{if } j \text{ is a put with strike } K_j. \end{cases}$$

Subjective factors

- Assume that the claim c also only depends on Z_T. We will consider the case of no claim and the case of an option claim.
- Assume that the disutility function v is:

$$v(P) = \frac{e^{-\lambda P} - 1}{\lambda}$$

  where $\lambda > 0$ is a risk aversion parameter.
- The only random variable we need to model is Z_T, the S&P500 value at maturity. The choice of model is also subjective. Two possiblities we will consider are:
  - $\log(Z_T)$ is normally distributed with mean and variance calibrated using exponentially weighted historic data.
  - $\log(Z_T)$ is follows a student t-distribution calibrated similarly.
- We have now completely specified a finite dimensional convex optimization problem.

Explicit computation

- Our objective function is

$$Ev(c + S_T(x)) = \int v\left(c + \sum_j s^j(z)x^j\right) p(z)\,\mathrm{d}(z)$$

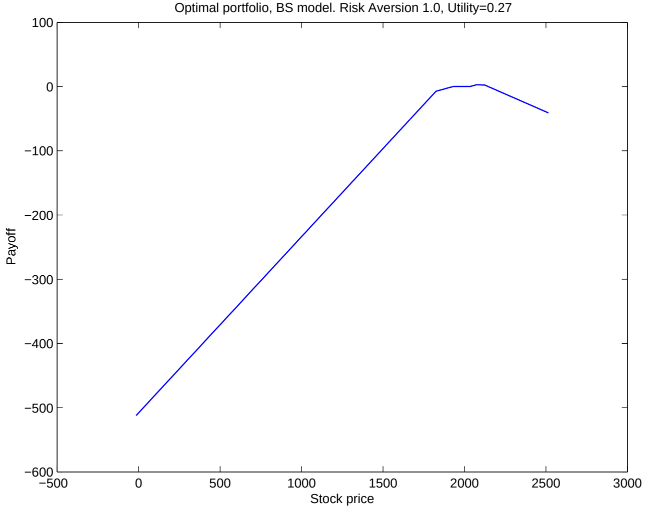
- We can approximate integrals using a quadrature rule:

$$\int f(z)\mathrm{d}(z) \approx \sum_{i=1}^{N} w_i f(z_i)$$

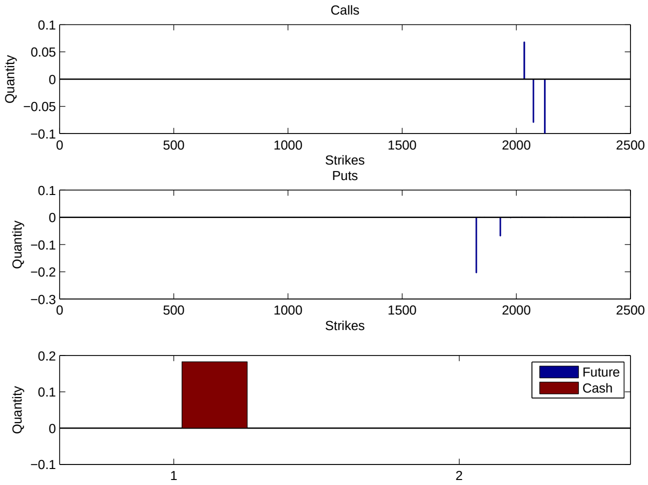
  for some weights w_i and evaluation points z_i.
- Examples: Monte Carlo, quasi-Monte Carlo, mid-point rule, Gaussian quadrature.
- In summary:

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^{N} w_i v\left(c(z_i) + \sum_j s^j(z_i)x^j\right) p(z_i) \\ &\text{subject to} && \sum s_0^j(x^j) \leq w. \end{aligned}$$

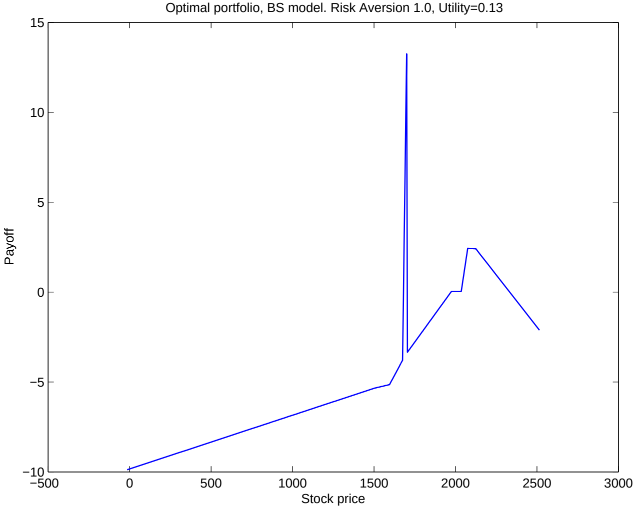
The optimal portfolio



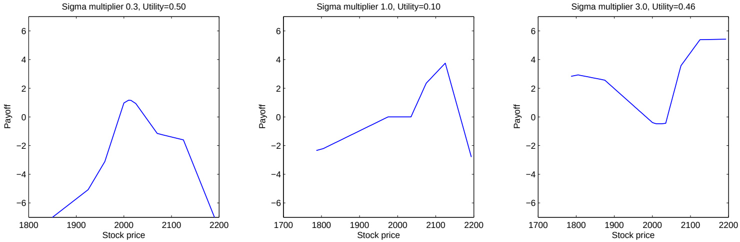
The optimal portfolio



The optimal portfolio - Student-t Model

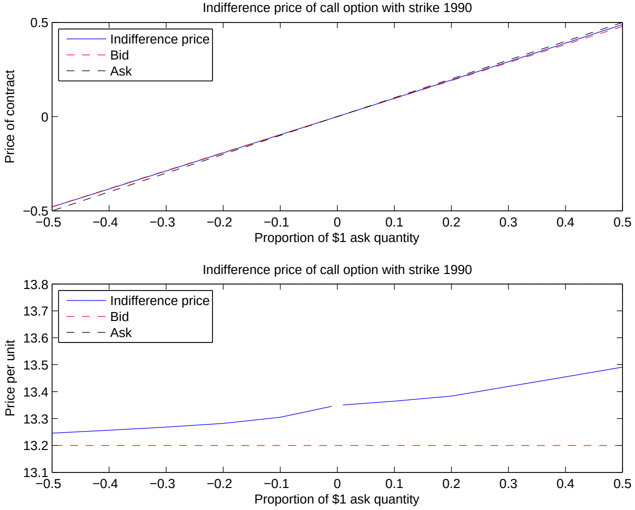


The optimal portfolio - varying beliefs about volatility

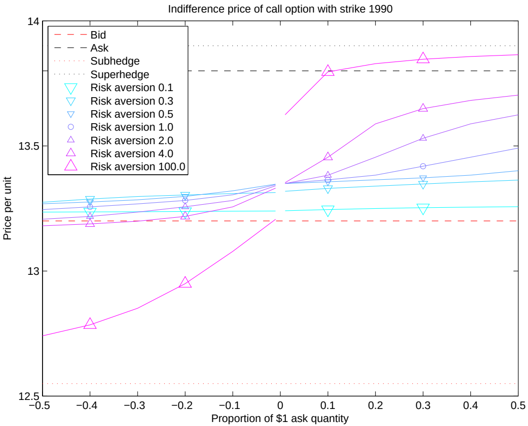


- A belief that the volatility will be lower than the historic trend (LHS) leads to a short straddle
- A belief that the volatility will be higher than the historic trend (RHS) leads to a long straddle

The indifference price - Two Pictures

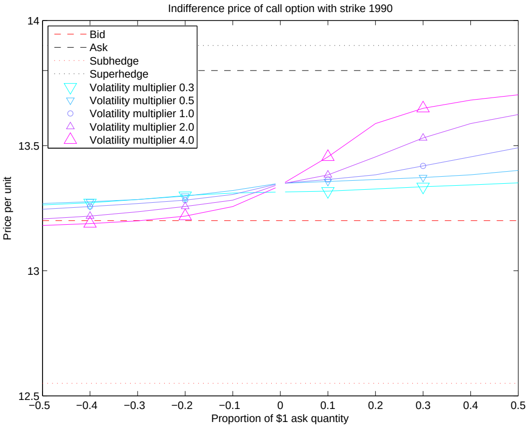


The indifference price - Sensitivity to Risk Preferences



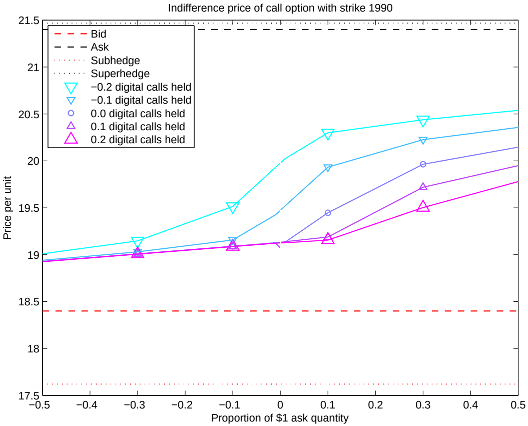
- For high risk aversion, the indifference price is close to a step function
- For low risk aversion, the indifference price is close to a constant function

The indifference price - Sensitivity to Beliefs



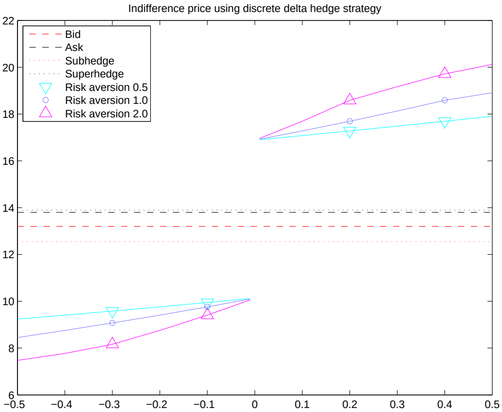
- As volatility increases, the sellers indifference price increases, the buyers price decreases
- The sensitivity to beliefs is less than in classical models
- We can calibrate to the market without changing our beliefs

The indifference price - Sensitivity to financial position



- We hold λ units of a digital call with strike 2000.
- The lower λ, the more we value the call as a hedge for our position

The indifference price - Delta Hedging



- Calibrate Black Scholes model using the mid price
- Assume the bid price is a fixed proportion of the ask price
- Delta hedge at evenly spaced time points. Number of steps chosen to give the best price.

Summary

- Prices offered in practice are subjective (views, risk preferences, financial position).
- Much of classical asset pricing theory can be extended to convex models of illiquid markets.
- Abritrage and martingale measures have little to do with hedging-based pricing.
- Hedging-based pricing allows you to calibrate to market data without discarding your beliefs.