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Rough Paths and Gamma Hedging

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Joint work with: Claudio Bellani (Imperial); Damiano Brigo (Imperial); Tom Cass (Imperial); Andrei Ionescu (King's College London)

What happens if we use rough-path calculus to try and understand replication rather than Itô calculus?

- Motivation: why apply rough-path calculus given Itô calculus works so well?
- A brief introduction to rough paths
- Replication of European options using rough paths calculus
- Replication of exotic derivatives using rough path calculus
- Take home messages: you need to gamma hedge; gamma hedging is very robust.

Motivation

Problem:

Reality is messy

- Transaction costs
- Market impact
- Front-running
- Discrete-time trading
- Model uncertainty
- ...

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Solution:

All applied mathematics needs to cope with messiness.

- Use stylised models
- Check that the model is robust under perturbation.

The Fundamental Theorem of Derivative Trading

For simplicity assume a 1-d market, risk-free rate 0 and assume sufficient regularity throughout.

Theorem

A trader believes that a stock price S_t will follow a diffusion

$$dS_t = S_t(\mu_h(t, S_t)\,dt + \sigma_h(t, S_t)\,dW_t).$$

They purchase a European option with payoff $g(X_T)$ for a price V_0^i and hedge it using the Delta–hedging strategy until time T. In reality the stock follows the diffusion

$$dS_t = S_t(\mu_r(t, S_t)\,dt + \sigma_r(t, S_t)\,dW_t).$$

where the subscript r stands for real as opposed to h for hedge.
Their profit and loss is:

$$\mathrm{P\&L} = V_0^h - V_0^i + \frac{1}{2}\int_0^T (\sigma_r(t, S_t))^2 - \sigma_h(t, S_t)^2) S_t^2 \frac{\partial^2 V^h}{\partial S^2}\,dt$$

(El Karoui, Jeanblanc-Piqué and Shreve, 1998)

Remarks

What's Good

- If the payoff g is convex or concave, we know the sign of Gamma so can work out the sign of the integral if $\sigma_r > \sigma_h$ or $\sigma_h < \sigma_r$.
- We can bound the integral in terms of $\sigma_r^2 - \sigma_h^2$.
- Beyond diffusions: Backhoff-Veraguas, Bartl, Beiglbock, Eder *Adapted Wasserstein distances and stability* 2020.

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But...

- What if S_t doesn't come from any probability model at all?
- The discrete-time delta hedging strategy does not converge almost-surely as the gap between trades tend to zero (unless you are allowed to choose the hedging times adaptively)

Mathematical finance without probability

- Föllmer *Calcul d'Itô sans probabilites* 1981
- Bick and Willinger *Dynamic spanning without probabilities* 1994
- Dupire. *Functional Itô Calculus* 2009
- Cont and co-authors e.g.
  - Riga 2015 — analyses continuous time trading strategies
  - Ananova 2020 — draws connections with rough path theory
- Perkowski and Prömel — Vovhk measure and rough path theory 2016
- Allan, Liu, Prömel — includes jumps 2021
- ...

Our contribution will be to apply rough path theory to classical replication arguments to obtain results giving almost-sure (indeed sure) convergence for gamma hedging strategies with robustness.

Rough Path Theory

We will give a short overview of Rough Path theory, see Friz-Hairer.

A non-continuity result

Lemma

There exists no separable Banach space $\mathcal{B} \subseteq C[0,1]$ where both

- *Sample paths of Brownian motion lie in $\mathcal{B}$ almost surely*
- *The map $(f,g) \to \int_0^t f_u \, dg_u$ defined on smooth functions extends to a continuous map $\mathcal{B} \times \mathcal{B} \to C[0,1]$.*

So we cannot define an integral driven by Brownian motion that will be robust.

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Example:

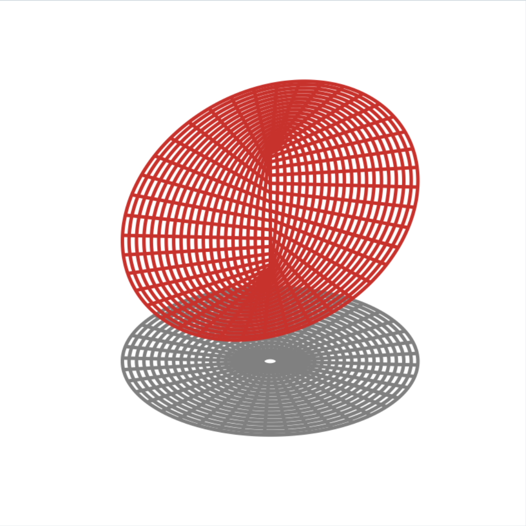
There is no continuous map $f : \mathbb{R}^2 \to [-1,1]$ extending the map

$$f : \mathbb{R}^2 \setminus \{0\} \to [-1,1]$$

given by

$$f : (r\cos\theta, r\sin\theta) \to \cos\theta$$

Solution: define a lift



Reduced Rough Paths

If F takes values in a linear space, write $F_{s,t} := F_t - F_s$.

Definition

For $p \in (2, 3)$ a *reduced rough path* is a pair $\boldsymbol{X_t} := (X_t, \mathbb{X}_t)$ of a path $X : [0, T]$ called the *trace* and a map $\mathbb{X} : [0, T]^2 \to \mathbb{R}^d \odot \mathbb{R}^d$ where the *lift* satisfies the reduced Chen relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \odot X_{u,t} \quad \forall s, u, t \in [0, T].$$

and where X has finite p-variation and $\mathbb{X}$ has finite $\frac{p}{2}$-variation.

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Definition

The rough bracket of a rough path is

$$[\boldsymbol{X}]_{s,t} := X_{s,t} \otimes X_{s,t} - 2\mathbb{X}_{s,t} \in \mathbb{R}^d \odot \mathbb{R}^d$$

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- A reduced rough path is determined by its trace X_t and its rough bracket $[\boldsymbol{X}]_{s,t}$.
- The rough bracket plays an identical role in Ito's Lemma for rough paths to the role played by the quadratic variation in the classical Ito's Lemma.

Rough Integral

We want to define a rough path integral to ensure

$$\int_s^t X_u \, d\mathbf{X}_u := \mathbb{X}_{s,t}$$

Definition

Let X_t be a path of finite p-variation then an *X-controlled path of (p,q)-variation regularity* ($p > q > 1$) is a pair $\boldsymbol{Y} = (Y, Y')$ of p-variation paths such that the remainder

$$R^Y_{s,t} = Y_{s,t} - Y'_s X_{s,t}$$

is of finite q-variation. Y' is a called a *Gubinelli derivative* for Y.

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Theorem

If (p, q) *satisfies* $p \in (2, 3)$, $q \geq \frac{p}{2}$ *and* $p^{-1} + q^{-1} > 1$ *then*

$$\int_0^T \mathbf{Y} \, d\mathbf{X} = \lim_{\|\pi\| \to 0} \sum_{(s,t) \in \pi} (Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t})$$

exists, where π *is a partition of* $[0, T]$ *and* $\|\pi\|$ *denotes the mesh size. (Gubinelli)*

Relationship with Itô Calculus

Let W_t be a Wiener process. Define

$$\mathbb{W}_t^{\mathrm{Ito}} = \int_s^t W_{s,u}\, dW_u$$

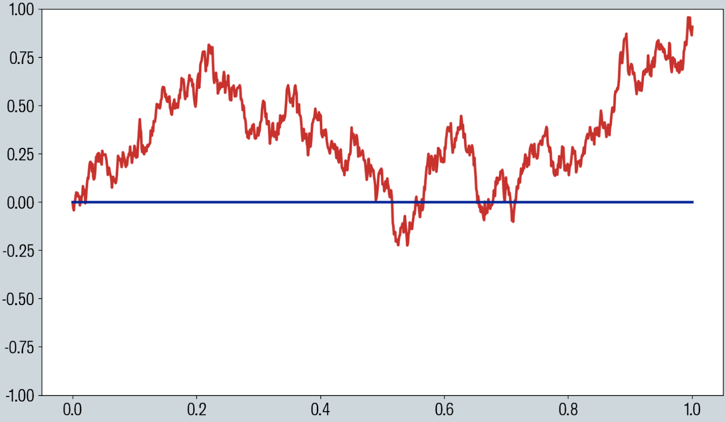
then $\boldsymbol{W} = (W, \mathbb{W}^{\mathrm{Ito}})$ is almost-surely a p-rough path.

Suppose that $\boldsymbol{Y} = (Y, Y')$ are adapted and almost-surely W-controlled paths then

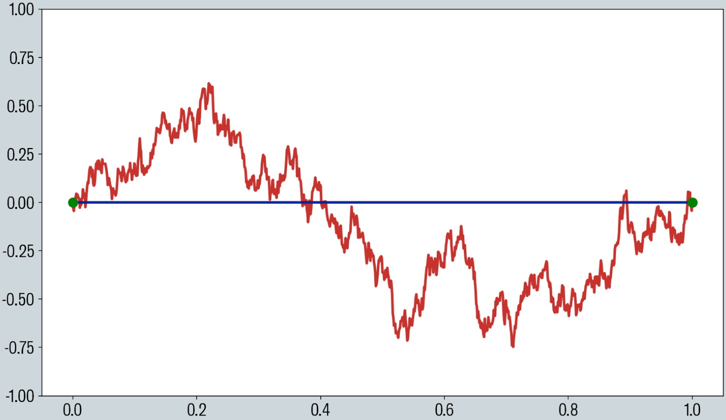
$$\int_0^t Y_u\, dW_u = \int_0^t \boldsymbol{Y}_u\, d\boldsymbol{W}_u$$

almost surely.

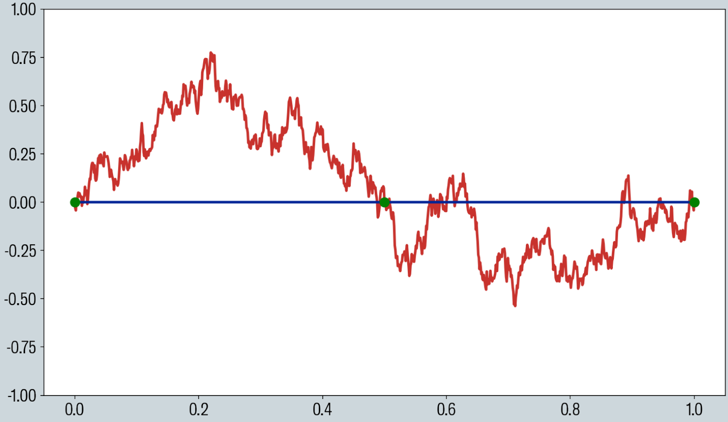
Example – a Brownian motion



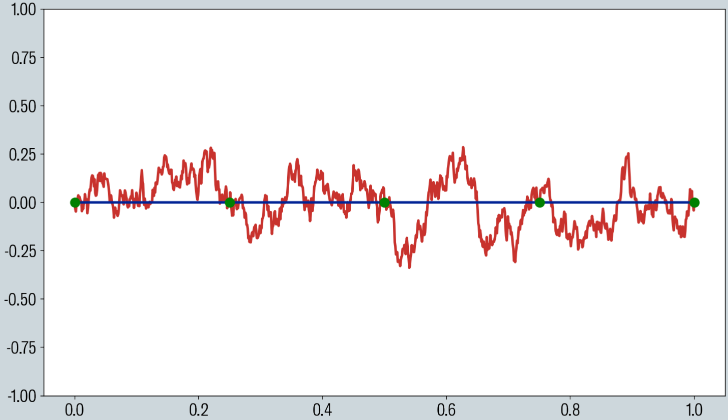
A Brownian Bridge



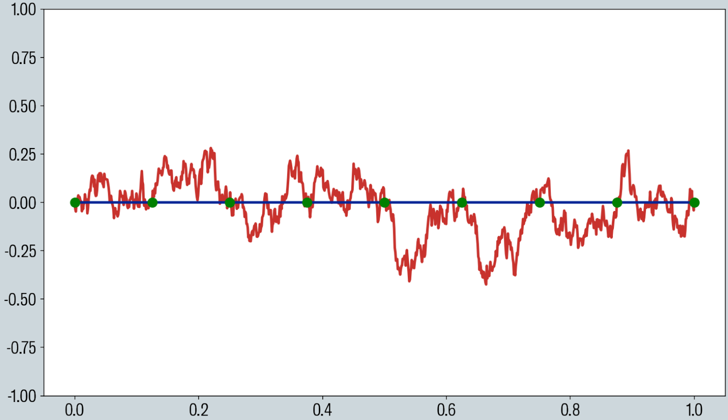
Two Brownian bridges in a row



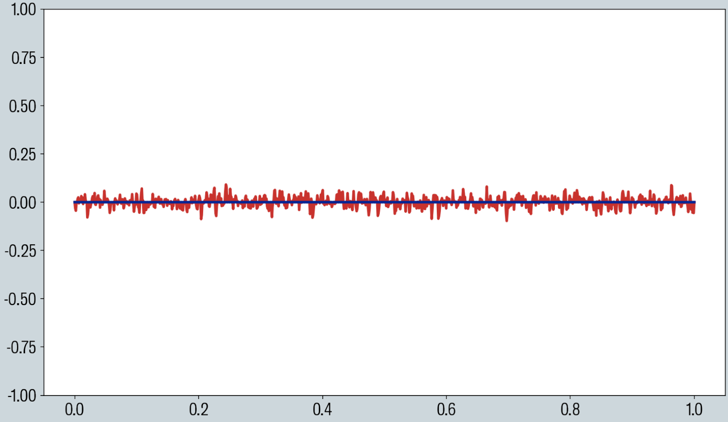
Four Brownian Bridges



Eight Brownian Bridges



256 Brownian Bridges



A non-trivial rough path with a trival trace

This sequence of rough paths converges to a limit $\boldsymbol{X} = (0, \mathbb{X})$ which has a trace of zero, but has the rough bracket

$$[\boldsymbol{X}]_{s,t} = t - s$$

equivalently

$$\mathbb{X}_{s,t} = \frac{1}{2} X_{s,t} \otimes X_{s,t} - \frac{1}{2}(t-s) = -\frac{1}{2}(t-s)$$

Checking the Chen relation when $s < u < t$:

$$\begin{aligned} \mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} &= -\frac{1}{2}(s-t) + \frac{1}{2}(s-u) + \frac{1}{2}(u-t) \\ &= 0 \\ &= X_{s,u} \odot X_{s,t} \end{aligned}$$

Rough Path Summary

- A rough path is an appropriate generalization of a smooth path thought of as the driver of a differential equation.
- We can compute the integral of processes which are locally modelled by X_t.
- Where rough path calculus and Ito calculus intersect, they are equivalent.
- Using the rough intergral, you can define rough differential equations. The resulting solution map exists and is continuous (assuming regular coefficients and in appropriate topologies).
- Discrete-time approximations to the rough integral converge surely

These properties make it very tempting to ask if we can understand replication of derivatives in terms of rough paths and so obtain strong robustness results.

Rough Path Theory and Replication

Suppose that a European derivative has payoff $f(S_T)$ where f is a smooth bounded function and S_T is the stock price at time t.

We want model the trader's Profit and Loss as a process driven by a rough path signal $(S_t, \mathbb{S}_t)$.

Let us suppose initially that the trader will follow the delta hedging strategy associated with the diffusion model

$$dS_t = S_t(\mu(t, S_t))\, dt + \sigma(t, S_t))\, dW_t$$

Let $V(t, S_t)$ denotes the classical price of the option in this model.

Let $\Delta(t, S_t) = \frac{\partial V}{\partial S}$ denote the Delta of the option.

The trader holds $\Delta(t, S_t)$ units of the stock at time t.

How NOT to model P&L

Classically, assuming $r = 0$, the profit and loss of the delta hedging portfolio over the interval $[0, t]$ is, in effect, *defined* to equal:

$$\int_0^t \Delta(u, S_u)\, dS_u$$

The usual way that this is phrased is by defining what is meant by a *self-financing trading strategy*. Since self-financing portfolios are defined using the Ito integral (or with our rephrasing, the profit and loss is defined in terms of the Ito integral) we are making a leap of faith when we use this as our continuous time model for accounting.

We can justify this decision by writing

$$\int_0^t \Delta(S, u)\, dS_u \approx \sum_{(u,v)\in\pi} \Delta(S_u, u) S_{u,v}$$

But if we are working with the rough path integrals we have an additional term

$$\int_0^t \boldsymbol{\Delta}(S, u)\, d\boldsymbol{S}_u := \lim_{\|\pi\|\to 0} \sum_{(u,v)\in\pi} (\Delta S_{u,v} + (\Delta)' \mathbb{S}_{u,v})$$

so the rough integral is *not* in general a good way to model P&L.

Why this is frustrating

Definition

A rough path $(S, t, \mathbb{S}_t)$ is σ-diffusive if

$$[\boldsymbol{S}]_{s,t} = \int_s^t \sigma(u, S_u)^2 \, du$$

Why this is frustrating

Definition

A rough path $(S, t, \mathbb{S}_t)$ is σ-diffusive if

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It is then easy to prove using Ito's Lemma for rough paths that:

Lemma

$\Delta(t, S_t)$ *has Gubinelli dervative* $\Gamma(t, S_t) =: \frac{\partial^2 V}{\partial S^2}$ *and*

$$f(S_T) - V(t, S_0) = \int_0^T \boldsymbol{\Delta}(S, u) \, d\boldsymbol{S}_u$$

So we have a *sure* result, but unfortunately it doesn't appear to have any financial meaning...

Resolution

(A, Bellani, Brigo, Cass 2021)

Suppose over a time period $[s, t]$ you can buy volatility swaps that pay off

$$(S_{s,t})^2 - [\mathbf{S}]_{s,t}$$

then you can offset the additional Gubinelli term by purchasing Γ units of such swaps.

The Black_Scholes price of such swaps will be vanishingly small, so in the limit purchasing these swaps will not affect P&L.

If one postulates the existence of such swaps and bounds on their price, one can prove sure convergence of delta hedge + volatility swaps strategy.

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Or

(A, Ionescu, (Dec 2022?))

Suppose that you engage in the gamma-hedging strategy. This in effect replicates the volatilty swaps, eliminating the Gubinelli terms.

Gamma Hedging

- A trader buys and sells derivatives in response to customer demands
- The trader periodically purchases either the stock or exchange traded derivatives to ensure that their portfolio has a delta of zero and a gamma of zero.
- They will choose which derivatives to purchase dynamically based on market prices and their current portfollio

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Model of gamma hedging

- The trader wishes to replicate a European option with payoff function f^0 and value function V^0
- They wish to do this by trading in two European options with payoff functions f^i with value functions V^1 and V^2
- At each time they hold quantities q_i units of option i where q_i is chosen such that

$$q_1\Delta^1 + q_2\Delta^2 = \Delta^0$$
$$q_1\Gamma^1 + q_2\Gamma^2 = \Gamma^0$$

We will simply assume that these equations can always be solved. In the event of linear dependence, a trader would simply consider an additional exchange traded derivative.

Sure replication of European Options

We cannot interpret rough integrals in terms of P&L. So to get financially meaningful results we must look at discrete-time trading strategies.

Theorem

Let S_t be a p-regular path. Suppose that the prices of the hedging options with payoff f^i are given by $V^i(t, S_t)$. If a trader sells an option with payoff f^0 for the price $V^0(S_0, t)$ and follows a discrete-time gamma-hedging strategy until maturity using a partition π of $[0, T]$ then their profit and loss will tend to 0 as $\pi \to 0$.

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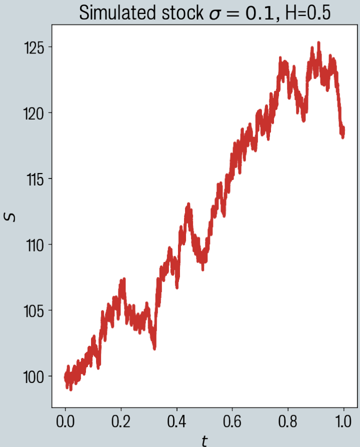
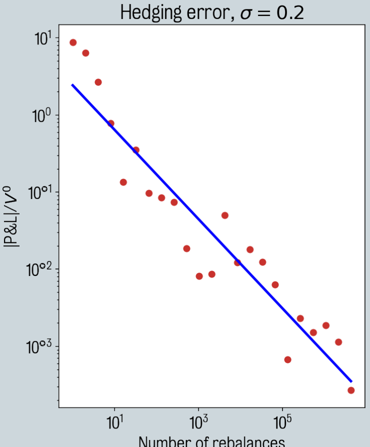
This theorem is inspired by, and proved by rough path theory, but is a straightforward statement about classical stock price paths.

Example: Gamma Hedging in the Black Scholes Model

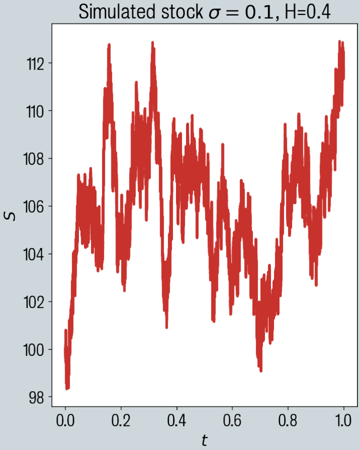
We simulate a stock price path in the Black Scholes Model with $\sigma = 0.1$, but assume that the market prices a hedging option with strike $K = 115$ and maturity $T = 1.2$ using the Black Scholes formula with $\sigma = 0.2$. We sell annother call option with strike $K = 100$ and maturity $T = 2$ and hedge it using the discrete-time gamma-hedging strategy with market prices computed using the Black Scholes formula with $\sigma = 0.2$.

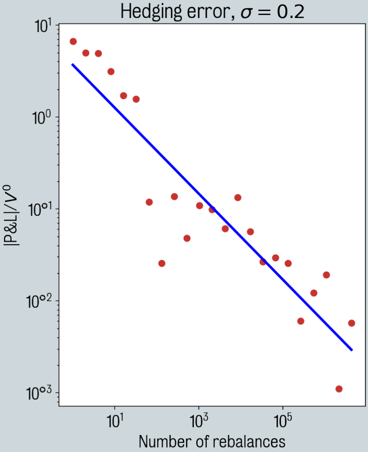




Using a fractional Brownian motion

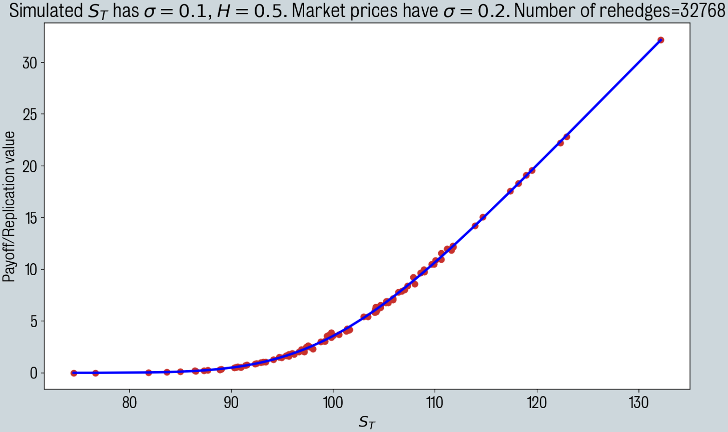
We repeat using a $S_t = S_0 \exp(\sigma W_t^H)$ where W^H is a fractional Brownian motion





Scatter plot of final stock price against replicated value

The replicated values are shown as points in the scatter plot, the blue line shows the payoff (the Black Scholes price at time T=1 using the σ from the market prices)



Proof:

Write Π_t for the bank balance at time t.

$$\Pi_{s,t} = q^1 V^1_{s,t} - q^2 V^2_{s,t}$$

so

$$\begin{aligned}
\Pi_T - V^0_T &= \Pi_{0,T} - (V^0_T - V^0_0) \\
&= \sum_{(s,t)\in\pi} (\Pi_{s,t} - \int_s^t \boldsymbol{\Delta}^0 \, d\boldsymbol{S}_t) \quad \text{by Lemma} \\
&= \sum_{(s,t)\in\pi} (q^1 V^1_{s,t} - q^2 V^2_{s,t} - \int_s^t \boldsymbol{\Delta}^0 \, d\boldsymbol{S}_t) \\
&= \sum_{i=0}^{2} \sum_{(s,t)\in\pi} q^i \int_{s,t} \boldsymbol{\Delta}^i d\boldsymbol{S}_t \quad \text{by Lemma} \\
&\approx \sum_{i=0}^{2} \sum_{(s,t)\in\pi} q^i (\Delta^i_s S_{s,t} + \Gamma^i_s \mathbb{S}_{s,t}) \quad \text{by Gubinelli} \\
&= 0
\end{aligned}$$

Gamma Hedging is VERY robust

- We can gamma-hedge an option successfully even if our stock price model is completely wrong!
- The convergence is sure.
- We need to choose our price and hedging strategy to be compatible with market option prices
- This gives a mathematical explanation for the success of the standard practice of calibrating to market prices

Rough Path Fundamental Theorem of Derivatives Trading

Theorem

Suppose that a trader uses a model for option prices in which

$$V^{i,mdl}_{s,t} = \int_s^t \mathbf{\Delta}^{mdl}_u \, d\mathbf{S}_u$$

and writes and hedges an option according to this model. Suppose that in reality the hedging options satisfy

$$V^{i,mkt}_{s,t} = \int_s^t \mathbf{\Delta}^{mkt}_u \, d\tilde{\mathbf{S}}_u$$

where $\tilde{\mathbf{S}}_t = (S_t, \tilde{\mathbb{S}}_t)$ *then the PnL of the trader is given by*

$$\int_0^T \Gamma^0_u \, d([\mathbf{S}]_u - [\tilde{\mathbf{S}}]_u) + \int_0^T \mathbf{A}_u \, d\tilde{\mathbf{S}}_u + \int_0^T (B - A') \, d\tilde{\mathbb{S}}$$

where

$$A_t = \sum_{i=1}^{2} q^i_t (\Delta^{i,mkt}_t - \Delta^{i,mdl}_t), \quad B_t = \sum_{i=1}^{2} q^i_t (\Gamma^{i,mkt}_t - \Gamma^{i,mdl}_t)$$

Extending the results to exotics

The delta and gamma of a derivative product are usually understood as partial derivatives of the price. This does not give us well-defined notions for path-dependent derivatives.

Definition

In a diffusion model for a stock price S_t, the delta of a contract with payoff $G(\omega)$ where ω is a price path is defined to be the predictable process Δ_t^G satisfying

$$\mathbb{E}_t(G) = \mathbb{E}_t(G) = \int_0^t \Delta_u^G \, dS_u$$

assuming this process exists.

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The results we have described on the convergence of gamma-hedging apply almost-surely to S-controllable derivatives: we can replicate S-controllable derivatives by gamma-hedging with S-controllable derivatives.

Sure results

Since rough path theory is continuous in the driver we obtain sure results for any p-rough paths S for which any p-variation neighbourhood has non-zero measure.

As our example of the zero path as a limit of Brownian motions might suggest, we get sure results for any smooth path and hence for the closure of the space of smooth paths in the p-variation topology.

This is not equal to the space of paths of finite p-variation, so sure results derived from this definition do not apply to all paths. Thus our results for exotics are not quite as strong as for European options.

Examples

If S_t is given by a regular diffusion, S-controllable is equivalent being W-controllable, where W is the driving Brownian motion.

- European options with smooth payoff functions on W_T.
- Continuous-time Asian options on W_T.

Examples

If S_t is given by a regular diffusion, S-controllable is equivalent being W-controllable, where W is the driving Brownian motion.

- European options with smooth payoff functions on W_T.
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Iterated integrals of the form

$$\int_0^T \int_0^{s^1} \int_0^{s^2} \dots \int_0^{s^{n-1}} dX^1_{s_1} dX^2_{s_2} \dots dX^n_{s_n}$$

where each X^i is given by eother W^i_t or t.

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By the Universal Approximation Theorem for signatures we can uniformly approximate any derivative on a compact subset of the the space of paths using a portfolio of derivatives given by iterated integrals.

Polynomial random variables

Definition

Given a smooth function $h : [0, T] \to \mathbb{R}$ define a random variable by

$$W(h) = \int_0^T h(u)\, dW_u$$

this is trivially controllable, wirh $\Delta_t^{W(h)} = h(t)$ and $\Gamma_t^{W(h)} = 0$.
We will call a random variable that can be written in terms of sums and products of such $W(h)$ a {*polynomial random variable*}.

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By the Leibniz rule for Gubinelli derivatives and for the Malliavin derivative one can show that polynomial random variables, G, are all controllable and satisfy

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Since polynomial random variables are dense in L^2, it is possible to define a closable operator (Δ^G, Γ^G) using appropriate topologies, but the rough-path and L^2 topologies don't seem to combine very nicely.

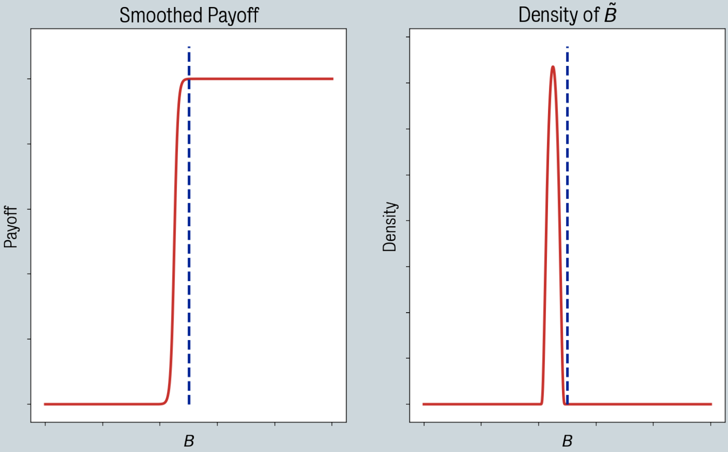
Non-controllable payoffs

- Put and call options have non-differentiable payoffs and so are not controllable. They can easily be smoothed to obtain a super-hedge or sub-hedge with an arbitrarily close risk-neutral price.
- Barrier options have discontinuous delta when the barrier is hit and unbounded gamma.

Smoothing a no-touch

To find a smooth super/sub-hedge a no-touch option (which pays of 1 if a barrier level is never hit and 0 otherwise) we do two things

- Smooth the payoff near the barrier B as shown.
- Use an infinite portfolio of no touch options with smoothed payoffs, with barriers $\tilde{B}$ with the density shown.



Smoothing a continuous payoff

We have ‘dense’ families of controllable payoffs, but the nature of the density is not financially satisfactory.

- Iterated integrals
- Signature payoffs

If a payoff is continuous in the uniform topology and bounded above and below by controllable payoffs we can:

- Approximate the payoff from above/below using a portfolio of one-touch options with smooth, time-varying barriers and the bounding payoff.
- Approximate the one-touch options as shown

We conclude that any continuous payoff which is bounded by controllable payoffs can be super/sub hedged for a price aribtrarily close to the risk-neutral price.

Conclusions

- Rough path theory naturally leads to considering the gamma-hedging strategy
- It may be necessary to smooth a financial derivative before we can replicate it
- The gamma-hedging strategy is robust and has sure convergence properties
- This helps explain why the strategy used by traders of calibrate and gamma-hedge is so effective

Conclusions

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Thank you!