Theorem
(Riemann–Hurwitz formula)
Suppose $X$ and $Y$ are compact Riemann surfaces of genus $g_X$ and $g_Y$ respectively and that $f : X \to Y$ is a branched cover. Then

$$(2 - 2g_Y) = d(2 - 2g_X) - \sum_{x \in X} (k_x - 1)$$

Where $k_x$ is the multiplicity of $f$ at $x$. 
Theorem

(Riemann–Hurwitz formula)

Suppose $X$ and $Y$ are compact Riemann surfaces of genus $g_X$ and $g_Y$ respectively and that $f : X \to Y$ is a branched cover. Then

$$(2 - 2g_Y) = d(2 - 2g_X) - \sum_{x \in X} (k_x - 1)$$

Where $k_x$ is the multiplicity of $f$ at $x$.

Proof is to triangulate $Y$ with a triangulation so that all critical values are vertices. Then lift the triangulation to $Xm$ and compute Euler characteristics.
Theorem
(Riemann–Hurwitz formula)
Suppose $X$ and $Y$ are compact Riemann surfaces of genus $g_X$ and $g_Y$ respectively and that $f : X \to Y$ is a branched cover. Then

$$(2 - 2g_Y) = d(2 - 2g_X) - \sum_{x \in X} (k_x - 1)$$

Where $k_x$ is the multiplicity of $f$ at $x$.
Proof is to triangulate $Y$ with a triangulation so that all critical values are vertices. Then lift the triangulation to $X_m$ and compute Euler characteristics.

Theorem
(Degree genus formula) A smooth plane curve of degree $d$ has genus $\frac{1}{2}(d - 1)(d - 2)$. 

Bezout’s theorem

Definition
Two complex curves in \( \mathbb{C}P^2 \) intersect transversally at a point \( p \) if \( p \) is a non-singular point of each curve and if the tangent space of \( \mathbb{C}P^2 \) at that point is the direct sum of the tangent spaces of the two curves.

Theorem
(Bezout) Two complex curves of degrees \( p \) and \( q \) that have no common component meet in no more than \( pq \) points. If they intersect transversally, they exactly in \( pq \) points.

If the polynomial defining a curve factorizes then each factor defines a component of the curve. Smooth curves have only one component because they would clearly not be smooth at their intersections of the components.
Proof of degree genus formula

- Given a smooth plane curve $C$ of degree $d$ consider the projection from a point $p$ to a line $L$ with $p$ not lying on $C$.
- By the fundamental theorem of algebra, the degree of this projection map will be $d$.
- We can choose coordinates so that the projection of a point $(z, w)$ in affine coordinates is just $z$. If $P(z, w) = 0$ defines the curve then branch points correspond to points where $P_w = 0$. These have ramification index 1 unless $P_{ww} = 0$.
- By Bezout’s theorem we expect there to be $d(d - 1)$ branch points and that so long as $p$ does not lie on a line of inflection (i.e. a tangent to the curve through a point of inflection) there will be exactly $d(d - 1)$ branch points.
- By Bezout’s theorem there are a finite number of lines of inflection (clearly points of inflection will be given by some algebraic condition).
- So for generic $p$ there are exactly $d(d - 1)$ branch points of ramification index 1.
- Apply Riemann–Hurwitz formula.
Complex structures on vector spaces

Definition

A complex structure on a vector space $V$ is a $\mathbb{R}$-linear map $J : V \longrightarrow V$ satisfying $J^2 = -1$. 

Example: rotation of a plane through 90 degrees — equivalently multiplication by $i$.
Complex structures on vector spaces

Definition
A complex structure on a vector space $V$ is a $\mathbb{R}$-linear map $J : V \rightarrow V$ satisfying $J^2 = -1$.

Example: rotation of a plane through 90 degrees — equivalently multiplication by $i$
Complex structures on vector spaces

**Definition**
A complex structure on a vector space $V$ is a $\mathbb{R}$-linear map $J : V \to V$ satisfying $J^2 = -1$.

Example: rotation of a plane through 90 degrees — equivalently multiplication by $i$

**Definition**
A map $T : V \to \mathbb{C}$ is complex linear if $T(Jv) = iTv$ for all $v$. It is complex anti-linear if $T(Jv) = -iTv$.

**Lemma**
Any $\mathbb{R}$-linear map $T$ from $V$ to $\mathbb{C}$ can be written as $T = T' + T''$ where $T'$ is complex linear and $T''$ is complex anti-linear.
Complex structures on vector spaces

**Definition**
A complex structure on a vector space $V$ is a $\mathbb{R}$-linear map $J : V \rightarrow V$ satisfying $J^2 = -1$.

Example: rotation of a plane through 90 degrees — equivalently multiplication by $i$

**Definition**
A map $T : V \rightarrow \mathbb{C}$ is complex linear if $T(Jv) = iTv$ for all $v$. It is complex anti-linear if $T(Jv) = -iTv$.

**Lemma**
Any $\mathbb{R}$-linear map $T$ from $V$ to $\mathbb{C}$ can be written as $T = T' + T''$ where $T'$ is complex linear and $T''$ is complex anti-linear.

$$T' = \frac{1}{2} (T - iTJv)$$
$$T'' = \frac{1}{2} (T + iTJv)$$
Complex structure on a Riemann surface

Definition
Let $T_p^* = \text{Hom}_\mathbb{R}(T_p, \mathbb{C})$ be the complex cotangent space.

Lemma
There is a unique complex structure $J$ on $T_p$ such that $df$ is a complex linear map with respect to $J$ whenever $f$ is holomorphic.

Since the definition only depends on the first order terms of $f$ we only need to check that an $\mathbb{R}$-linear map $T: \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if $T(iv) = i T(v)$. 

Definition
Let $T^*_C = \text{Hom}_\mathbb{R}(T_p, \mathbb{C})$ be the \textit{complex cotangent space}.

Definition
Given a complex valued function $f$ on a $X$ we can define $df \in T^*_C$ using the same formula as before. So $df : T_p \rightarrow \mathbb{C}$. 

Lemma
There is a unique complex structure $J$ on $T^*_p$ such that $df$ is a complex linear map with respect to $J$ whenever $f$ is holomorphic. Since the definition only depends on the first order terms of $f$ we only need to check that an $R$-linear map $T : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic if and only if $T(iv) = iT(v)$. 


Complex structure on a Riemann surface

Definition
Let $T^*_C = \text{Hom}_R(T_p, \mathbb{C})$ be the complex cotangent space.

Definition
Given a complex valued function $f$ on a $X$ we can define $df \in T^*_C$ using the same formula as before. So $df : T_p \to \mathbb{C}$.

Lemma
There is a unique complex structure $J$ on $T_p$ such that $df$ is a complex linear map with respect to $J$ whenever $f$ is holomorphic.
Complex structure on a Riemann surface

**Definition**
Let $T^*_\mathbb{C} = \text{Hom}_\mathbb{R}(T_p, \mathbb{C})$ be the *complex cotangent space*.

**Definition**
Given a complex valued function $f$ on a $X$ we can define $df \in T^*_\mathbb{C}$ using the same formula as before. So $df : T_p \rightarrow \mathbb{C}$.

**Lemma**
*There is a unique complex structure $J$ on $T_p$ such that $df$ is a complex linear map with respect to $J$ whenever $f$ is holomorphic.*

Since the definition only depends on the first order terms of $f$ we only need to check that an $\mathbb{R}$-linear map $T : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic if and only if $T(iv) = iT(v)$. 
Splitting of one forms

With respect to $J$ we can split the cotangent space into complex linear and complex anti-linear parts. We write the splitting of 1-forms as follows:

Definition

$$\Omega^1_C = \Omega^{1,0} \oplus \Omega^{0,1}$$
Splitting of one forms

With respect to $J$ we can split the cotangent space into complex linear and complex anti-linear parts. We write the splitting of 1-forms as follows:

**Definition**

\[ \Omega^1_{\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1} \]

- Correspondingly we can write $d = \partial \oplus \bar{\partial}$ where $\partial$ takes values in $\Omega^{1,0}$ and $\bar{\partial}$ takes values in $\Omega^{0,1}$.

It follows from our definition of $J$ that $f$ is holomorphic if and only if $\partial f = 0$.

Note that complex conjugation of $\mathbb{C}$ allows us to define an $\mathbb{R}$ linear map of $\text{Hom}_{\mathbb{R}}(TX, \mathbb{C})$ to itself. We call this complex conjugation too.

$\Omega^{1,0}$ and $\Omega^{0,1}$ are complex conjugates.
Splitting of one forms

With respect to $J$ we can split the cotangent space into complex linear and complex anti-linear parts. We write the splitting of 1-forms as follows:

Definition

$$\Omega^1_{\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1}$$

- Correspondingly we can write $d = \partial \oplus \overline{\partial}$ where $\partial$ takes values in $\Omega^{1,0}$ and $\overline{\partial}$ takes values in $\Omega^{0,1}$.

- It follows from our definition of $J$ that $f$ is holomorphic if and only if $\overline{\partial}f = 0$. 

Note that complex conjugation of $\mathbb{C}$ allows us to define an $\mathbb{R}$ linear map of $\text{Hom}_\mathbb{R}(TX, \mathbb{C})$ to itself. We call this complex conjugation too. $\Omega^{1,0}$ and $\Omega^{0,1}$ are complex conjugates.
Splitting of one forms

With respect to $J$ we can split the cotangent space into complex linear and complex anti-linear parts. We write the splitting of 1-forms as follows:

Definition

$$\Omega^1_\mathbb{C} = \Omega^{1,0} \oplus \Omega^{0,1}$$

Correspondingly we can write $d = \partial \oplus \overline{\partial}$ where $\partial$ takes values in $\Omega^{1,0}$ and $\overline{\partial}$ takes values in $\Omega^{0,1}$.

It follows from our definition of $J$ that $f$ is holomorphic if and only if $\overline{\partial} f = 0$.

Note that complex conjugation of $\mathbb{C}$ allows us to define an $\mathbb{R}$ linear map of $\text{Hom}_\mathbb{R}(TX, \mathbb{C})$ to itself. We call this complex conjugation too.
Splitting of one forms

With respect to $J$ we can split the cotangent space into complex linear and complex anti-linear parts. We write the splitting of 1-forms as follows:

**Definition**

$$\Omega_\mathbb{C}^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$

- Correspondingly we can write $d = \partial \oplus \overline{\partial}$ where $\partial$ takes values in $\Omega^{1,0}$ and $\overline{\partial}$ takes values in $\Omega^{0,1}$.
- It follows from our definition of $J$ that $f$ is holomorphic if and only if $\overline{\partial} f = 0$.
- Note that complex conjugation of $\mathbb{C}$ allows us to define an $\mathbb{R}$ linear map of $\text{Hom}_\mathbb{R}(TX, \mathbb{C})$ to itself. We call this complex conjugation too.
- $\Omega^{1,0}$ and $\Omega^{0,1}$ are complex conjugates.
Local coordinates

If $z : U \longrightarrow \mathbb{C}$ is a complex coordinate then writing $z = x + iy$, $x$ and $y$ are real coordinates.

\[ dz = dx + idy, \quad d\bar{z} = dx - idy \]
Local coordinates

If \( z : U \to \mathbb{C} \) is a complex coordinate then writing \( z = x + iy \), \( x \) and \( y \) are real coordinates.

\[
\begin{align*}
dz &= dx + i
dy,
\quad d\bar{z} = dx - i
dy
\end{align*}
\]

Equivalently:

\[
\begin{align*}
dx &= \frac{1}{2}(dz + d\bar{z}),
\quad dy = \frac{1}{2i}(dz - d\bar{z})
\end{align*}
\]
df in complex coordinates

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]

\[
= \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \, dz + \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \, d\bar{z}
\]

\[
= \frac{\partial f}{\partial z} \, dz + \frac{\partial f}{\partial \bar{z}} \, d\bar{z}
\]

- This last line should be seen as defining \( \frac{\partial f}{\partial z} \) and \( \frac{\partial f}{\partial \bar{z}} \).
\( df \) in complex coordinates

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]

\[
= \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \, dz + \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \, d\bar{z}
\]

\[
= \frac{\partial f}{\partial z} \, dz + \frac{\partial f}{\partial \bar{z}} \, d\bar{z}
\]

- This last line should be seen as defining \( \frac{\partial f}{\partial z} \) and \( \frac{\partial f}{\partial \bar{z}} \).
- The usual complex analysis definition using limits only makes sense for holomorphic \( f \) in which case the two definitions coincide.
∂ and \overline{\partial} in complex coordinates

\[ \partial f = \frac{\partial f}{\partial z} dz, \quad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z} \]

Where by definition:

\[ \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \]
\[ \frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \]
\( \partial \) and \( \bar{\partial} \) in complex coordinates

\[
\partial f = \frac{\partial f}{\partial z} \, dz, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} \, d\bar{z}
\]

Where by definition:

\[
\begin{align*}
\frac{\partial f}{\partial z} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \\
\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)
\end{align*}
\]

\( f \) is holomorphic is equivalent to the statement \( df = \partial f \) which is equivalent to the statement \( \bar{\partial} f = 0 \).
It is conventional to write $d = \bar{\partial}$ when it acts on $(1, 0)$ forms and as $d = \partial$ when it acts on $(0, 1)$ forms.

**Definition**
A $(1, 0)$ form $\omega \in \Omega^{1,0}$ is **holomorphic** if $\bar{\partial}\omega = 0$. 

**Definition**
A meromorphic $1$-form is one that can be written as $\omega = f dz$ with $f$ a meromorphic function.
Holomorphic and meromorphic 1-forms

It is conventional to write $d = \partial$ when it acts on $(1,0)$ forms and as $d = \partial$ when it acts on $(0,1)$ forms.

**Definition**
A $(1,0)$ form $\omega \in \Omega^{1,0}$ is *holomorphic* if $\bar{\partial} \omega = 0$.
Equivalently it is one that can be written $\omega = f dz$ with $f$ holomorphic local coordinates.
Holomorphic and meromorphic 1-forms

It is conventional to write $d = \partial$ when it acts on $(1, 0)$ forms and as $d = \bar{\partial}$ when it acts on $(0, 1)$ forms.

**Definition**

A $(1, 0)$ form $\omega \in \Omega^{1,0}$ is **holomorphic** if $\bar{\partial} \omega = 0$.

Equivalently it is one that can be written $\omega = f \, dz$ with $f$ holomorphic local coordinates.

**Definition**

A meromorphic 1-form is one that can be written as $\omega = f \, dz$ with $f$ a meromorphic function.
Contour integration

When you have calculated contour integrals, you have been integrating holomorphic one forms.

Theorem
(Cauchy’s theorem) If $S$ is a compact surface with boundary and $\omega$ is a holomorphic one form:

$$\int_{\partial S} \omega = 0$$

Definition
(Residue) If $p$ is a pole of a meromorphic 1-form $\omega$ then the residue of $\omega$ at $p$ is

$$\text{Res}_p(\omega) = \frac{1}{2\pi i} \int_C \omega$$

for a small loop $C$ around $p$. 
The Laplace operator

We define $\Delta$ by:

$$\Delta = 2i \bar{\partial} \partial : \Omega^0 \rightarrow \Omega^2$$

In local coordinates we compute:

$$\Delta f = \left( 2i \frac{1}{4} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f \right) dz \wedge d\bar{z}$$

$$= - \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx \wedge dy$$
Dolbeault Cohomology

We have a splitting of the $d$ operator so it is natural to wonder if the De Rham cohomology splits as well.

Complexified de Rham

Dolbeault
Dolbeault Cohomology

We define Dolbeault cohomology to be the “cohomology” of $\overline{\partial}$. This means we define:

$$H^{0,0} = \ker \overline{\partial} \subseteq \Omega^0 = \Omega^0$$

$$H^{1,0} = \ker \overline{\partial} \subseteq \Omega^{1,0}$$

$$H^{0,1} = \coker \overline{\partial} \subseteq \Omega^{0,1}$$

$$H^{1,1} = \coker \overline{\partial} \subseteq \Omega^{1,1} = \Omega^2$$
Dolbeault Cohomology

We define Dolbeault cohomology to be the “cohomology” of $\bar{\partial}$. This means we define:

\[
\begin{align*}
H^{0,0} & = \ker \bar{\partial} \subseteq \Omega^0 = \Omega^0 \\
H^{1,0} & = \ker \bar{\partial} \subseteq \Omega^{1,0} \\
H^{0,1} & = \text{coker} \bar{\partial} \subseteq \Omega^{0,1} \\
H^{1,1} & = \text{coker} \bar{\partial} \subseteq \Omega^{1,1} = \Omega^2
\end{align*}
\]

- On an $n$-manifold, the wedge product of $p$ $(1,0)$-forms and $q$ $(0,1)$-forms defines the notion of a $(p,q)$-form. So $\Omega^{1,1}$ is just another term for $(\Omega^{1,1})$ and $\Omega^{0,0}$ is just another term for $\Omega^0$. 
Dolbeault Cohomology

We define Dolbeault cohomology to be the “cohomology” of $\bar{\partial}$. This means we define:

\[
H^{0,0} = \ker \bar{\partial} \subseteq \Omega^0 = \Omega^0 \\
H^{1,0} = \ker \bar{\partial} \subseteq \Omega^{1,0} \\
H^{0,1} = \coker \bar{\partial} \subseteq \Omega^{0,1} \\
H^{1,1} = \coker \bar{\partial} \subseteq \Omega^{1,1} = \Omega^2
\]

▶ On an $n$-manifold, the wedge product of $p \ (1, 0)$-forms and $q \ (0, 1)$-forms defines the notion of a $(p, q)$-form. So $\Omega^{1,1}$ is just another term for $(\Omega^{1,1})$ and $\Omega^{0,0}$ is just another term for $\Omega^0$.

▶ An element of the cokernel is an equivalence class. We will refer to this equivalence class as the cohomology class.
Dolbeault Cohomology

One obvious motivation for considering Dolbeault Cohomology is that the dimensions of these cohomology vector spaces will give us invariants of complex manifolds.

\[
\begin{align*}
H^2 & \quad H^{1,1} \\
H^1 & \quad H^{1,0} \quad H^{0,1} \\
H^0 & \quad H^0 \\
\text{Complexified de Rham} & \quad \text{Dolbeault}
\end{align*}
\]
Equivalence of De Rham and Dolbeault Cohomology

On a Riemann surface these invariants are trivial. In particular $H^{1,0}$ is isomorphic to $\overline{H}^{0,1}$ and:

\[
\begin{align*}
H^2 & \cong H^{1,1} \\
H^1 & \cong H^{1,0} \oplus H^{0,1} \\
H^0 & \cong H^0
\end{align*}
\]

Complexified de Rham

Dolbeault
Theorem
("Main Theorem") If $X$ is a compact and connected Riemann surface then there is a solution $f$ to $\Delta f = \rho$ if and only if $\int_X \rho = 0$. The solution is unique up to the addition of a constant.
Theorem

(“Main Theorem”) If $X$ is a compact and connected Riemann surface then there is a solution $f$ to $\Delta f = \rho$ if and only if $\int_X \rho = 0$. The solution is unique up to the addition of a constant. The “only if” is follows from Stoke’s theorem. The uniqueness follows from the maximum principle — by compactness $f$ has a maximum value, but holomorphic functions only have maxima at their boundary.
Theorem
(“Main Theorem”) If $X$ is a compact and connected Riemann surface then there is a solution $f$ to $\Delta f = \rho$ if and only if $\int_X \rho = 0$. The solution is unique up to the addition of a constant. The “only if” is follows from Stoke’s theorem. The uniqueness follows from the maximum principle — by compactness $f$ has a maximum value, but holomorphic functions only have maxima at their boundary.

The if is the deep input. Physical arguments suggest solutions to Laplace’s equation should always exist. Laplace’s equation crops up in gravity, electrostatics, the study of heat etc. It had better have solutions if these theories are going make sense!
Deducing equivalence of homologies

Let \( \omega = \omega^{1,0} \oplus \omega^{0,1} \) satisfy \( d\omega = 0 \).
Deducing equivalence of homologies

Let $\omega = \omega^{1,0} \oplus \omega^{0,1}$ satisfy $d\omega = 0$.
We can represent the cohomology class $[\omega]$ using $\omega + df$ for any $f$.

$$\omega + df = (\omega^{1,0} + \partial f) \oplus (\omega^{0,1} + \overline{\partial f})$$
Deducing equivalence of homologies

Let $\omega = \omega^{1,0} \oplus \omega^{0,1}$ satisfy $d\omega = 0$. We can represent the cohomology class $[\omega]$ using $\omega + df$ for any $f$.

$$\omega + df = (\omega^{1,0} + \partial f) \oplus (\omega^{0,1} + \overline{\partial} f)$$

The condition that the $1,0$ term of $\omega + df$ is holomorphic is equivalent to the requirement $\overline{\partial}\partial f = -\overline{\partial} \omega^{1,0}$ which can easily be seen to be equivalent to the condition that the $0,1$ term lies in the kernel of $\partial$. 

So by the main theorem $\omega$ can be written uniquely as an element of $H^{1,0}$ plus an element of $H^{0,1}$. The results for $H^2$ and $H^0$ and the equivalence of $H^{1,0}$ and $H^{0,1}$ are similarly easy.
Deducing equivalence of homologies

Let $\omega = \omega^{1,0} \oplus \omega^{0,1}$ satisfy $d\omega = 0$. We can represent the cohomology class $[\omega]$ using $\omega + df$ for any $f$.

$$\omega + df = (\omega^{1,0} + \partial f) \oplus (\omega^{0,1} + \overline{\partial}f)$$

The condition that the $1,0$ term of $\omega + df$ is holomorphic is equivalent to the requirement $\overline{\partial}\partial f = -\overline{\partial}\omega^{1,0}$ which can easily seen to be equivalent to the condition that the $0,1$ term lies in the kernel of $\partial$.

So by the main theorem $\omega$ can be written uniquely as an element of $H^{1,0}$ plus an element of $\overline{H}^{1,0}$. 
Deducing equivalence of homologies

Let \( \omega = \omega^{1,0} \oplus \omega^{0,1} \) satisfy \( d\omega = 0 \).
We can represent the cohomology class \([\omega]\) using \( \omega + df \) for any \( f \).

\[
\omega + df = (\omega^{1,0} + \partial f) \oplus (\omega^{0,1} + \bar{\partial} f)
\]

The condition that the 1, 0 term of \( \omega + df \) is holomorphic is equivalent to the requirement \( \bar{\partial}\partial f = -\bar{\partial}\omega^{1,0} \) which can easily seen to be equivalent to the condition that the 0, 1 term lies in the kernel of \( \partial \).
So by the main theorem \( \omega \) can be written uniquely as an element of \( H^{1,0} \) plus an element of \( \overline{H}^{1,0} \).
The results for \( H^2 \) and \( H^0 \) and the equivalence of \( \overline{H}^{1,0} \) and \( H^{0,1} \) are similarly easy.
Corollaries

Corollary

*On a compact Riemann surface of genus* \( g \) \( \dim H^{1,0} = g \), \( \dim H^{0,1} = g \).
Corollaries

Corollary

On a compact Riemann surface of genus $g$ \( \dim H^{1,0} = g \), \( \dim H^{0,1} = g \).

I’ve assumed here that \( \dim H^1 = 2g \). Notice that this result gives a deep explanation for why \( \dim H^1 \) is even dimensional on oriented surfaces. (Poincaré duality gives another deep explanation).
Corollaries

Corollary

On a compact Riemann surface of genus $g$ \( \dim H^{1,0} = g \), \( \dim H^{0,1} = g \).

I’ve assumed here that \( \dim H^1 = 2g \). Notice that this result gives a deep explanation for why \( \dim H^1 \) is even dimensional on oriented surfaces. (Poincaré duality gives another deep explanation).

For the purposes of this course we only really need the result for the sphere and the torus. \( H^1(S^2) \) vanishes since the sphere is simply connected. \( H^1(T^2) = \mathbb{R}^2 \) is a homework exercise.
Interpretation of $H^{0,1}$

Suppose that $p$ is a point in a Riemann surface $X$ and we want to find a meromorphic function $f$ with a simple pole at $p$ at no other poles.
Interpretation of $H^{0,1}$

- Suppose that $p$ is a point in a Riemann surface $X$ and we want to find a meromorphic function $f$ with a simple pole at $p$ at no other poles.
- Define $\rho$ to be a cut-off function equal to 1 in a neighbourhood of $p$ but equal 0 outside of a slightly larger neighbourhood.
- Finding $f$ is equivalent to finding a smooth $g$ on $X$ with $g + \rho \frac{1}{z}$ holomorphic.
Interpretation of $H^{0,1}$

This is equivalent to finding smooth $g$ with:

$$\overline{\partial} g = -(\overline{\partial} \rho) \frac{1}{z} =: A$$
Interpretation of $H^{0,1}$

- This is equivalent to finding smooth $g$ with:

$$\bar{\partial}g = -\left(\bar{\partial}\rho\right)\frac{1}{z} =: A$$

- Since $\rho$ is equal to 1 in a neighbourhood of $p$ we can regard $A$ as a $(0,1)$ form with value 0 at $p$. 
Interpretation of $H^{0,1}$

- This is equivalent to finding smooth $g$ with:

$$\bar{\partial} g = - \left( \bar{\partial} \rho \right) \frac{1}{z} =: A$$

- Since $\rho$ is equal to 1 in a neighbourhood of $p$ we can regard $A$ as a $(0,1)$ form with value 0 at $p$.

- By definition of $H^{0,1}$ a solution will exist if and only if $[A]$ represents a the trivial element of $H^{0,1}$.
Interpretation of $H^{0,1}$

- This is equivalent to finding smooth $g$ with:
  \[
  \bar{\partial} g = - (\bar{\partial} \rho) \frac{1}{z} =: A
  \]

- Since $\rho$ is equal to 1 in a neighbourhood of $p$ we can regard $A$ as a $(0, 1)$ form with value 0 at $p$.

- By definition of $H^{0,1}$ a solution will exist if and only if $[A]$ represents a the trivial element of $H^{0,1}$.

- One can say that $[A] \in H^{0,1}$ is the obstruction to finding a meromorphic function with a simple pole at $p$ and no other poles.
Uniqueness of holomorphic structure on $S^2$

**Corollary**

*The Riemann sphere is the only Riemann surface of genus 0.*
Uniqueness of holomorphic structure on $S^2$

**Corollary**

*The Riemann sphere is the only Riemann surface of genus 0.*

This is because on a surface of genus 0, $H^{0,1}$ is trivial so one can always find such a meromorphic function. But we have already proved that $S^2$ is the only Riemann surface that admits a meromorphic function with a simple pole.
Repeating the same argument, if \( p_1, p_2, \ldots, p_d \) are \( d \)-distinct points then we define \( A_1, A_2, \ldots, A_d \) in the same way. We can find a smooth function \( g \) satisfying

\[
\overline{\partial} g = \lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_d A_d
\]

if and only if the right hand side represents a trivial cohomology class in \( H^{0,1} \).
Repeating the same argument, if \( p_1, p_2, \ldots, p_d \) are \( d \)-distinct points then we define \( A_1, A_2, \ldots, A_d \) in the same way. We can find a smooth function \( g \) satisfying

\[
\overline{\partial} g = \lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_d A_d
\]

if and only if the right hand side represents a trivial cohomology class in \( H^{0,1} \). Since \( H^{0,1} \) is \( g \) dimensional on a compact Riemann surface of genus \( g \) any \( g + 1 \) cohomology classes must be linearly dependent.

**Theorem**

*Given \( g + 1 \) points on a compact Riemann surface \( X \) of genus \( g \) then exists a non-constant meromorphic function having at worst simple poles at the \( p_i \) and no other poles.*
Genus 1 surfaces as branched covers

Corollary

A compact Riemann surface of genus 1 has a meromorphic function with precisely two poles, both of which are simple.
Corollary

A compact Riemann surface of genus 1 has a meromorphic function with precisely two poles, both of which are simple.

Corollary

A compact Riemann surface of genus 1 is a branched cover of the Riemann sphere of degree 2 with four branch points.
Genus 1 surfaces as branched covers

Corollary

A compact Riemann surface of genus 1 has a meromorphic function with precisely two poles, both of which are simple.

Corollary

A compact Riemann surface of genus 1 is a branched cover of the Riemann sphere of degree 2 with four branch points.

The Riemann Hurwitz formula for a branched cover

$2 - 2g_Y = d(2 - 2g_X) - R_f$ allows you to compute that there must be four branch points.
Genus 1 surfaces as branched covers

Corollary

A compact Riemann surface of genus 1 has a meromorphic function with precisely two poles, both of which are simple.

Corollary

A compact Riemann surface of genus 1 is a branched cover of the Riemann sphere of degree 2 with four branch points.

The Riemann Hurwitz formula for a branched cover $2 - 2g_Y = d(2 - 2g_X) - R_f$ allows you to compute that there must be four branch points.

This is a classification theorem for surfaces of genus 1 surfaces because two sheeted branched covers of a sphere are essentially unique.
Uniqueness of branched covers up to monodromy
Uniqueness of branched covers up to monodromy

- Given two branched covers $f_i : X_i \to Y$ of degree $d$ with the same critical values, pick a generic point $y$ and label the pre-images of $y$, $x_i^1, x_i^2, \ldots, x_i^d$. 

- Lift a path $\gamma$ starting at $y$ to paths $\gamma_i$ in $X_i$ based at $x_i^1$. Attempt to define $\phi : X_i \to Y_i$ by sending the end point of $\gamma_1$ to the end point of $\gamma_2$. In other words try to use "parallel transport" to define a homeomorphism.

- This map is well defined if we can choose our labelling such that the "monodromy" around closed loops is the same. The monodromy is defined to be the homomorphism from $\pi_1$ to $S_d$ that sends $\gamma$ to the permutation induced by parallel transport around $\gamma$. 

Uniqueness of branched covers up to monodromy

- Given two branched covers $f_i : X_i \to Y$ of degree $d$ with the same critical values, pick a generic point $y$ and label the pre-images of $y$ $x_i^1, x_i^2, \ldots, x_i^d$.

- Given a path $\gamma$ starting at $y$ lift it to paths $\gamma_i$ in $X_i$ based at $x_i^1$. Attempt to define $\phi : X_i \to Y_i$ by sending the end point of $\gamma_1$ to the end point of $\gamma_2$. In other words try to use “parallel transport” to define a homeomorphism.
Uniqueness of branched covers up to monodromy

- Given two branched covers \( f_i : X_i \to Y \) of degree \( d \) with the same critical values, pick a generic point \( y \) and label the pre-images of \( y \) \( x_i^1, x_i^2, \ldots, x_i^d \).

- Given a path \( \gamma \) starting at \( y \) lift it to paths \( \gamma_i \) in \( X_i \) based at \( x_i^1 \). Attempt to define \( \phi : X_i \to Y_i \) by sending the end point of \( \gamma_1 \) to the end point of \( \gamma_2 \). In other words try to use “parallel transport” to define a homeomorphism.

- This map is well defined if we can choose our labelling such that the “monodromy” around closed loops is the same. The monodromy is defined to be the homomorphism from \( \pi_1 \) to \( S_d \) that sends \( \gamma \) to the permutation induced by parallel transport around \( \gamma \).
Two sheeted branched covers

- $S_2$ only contains two elements so the monodromy of two sheeted covers is very easy to analyse.
Two sheeted branched covers

- $S_2$ only contains two elements so the monodromy of two sheeted covers is very easy to analyse.
- The fundamental group of $S_2$ with $p$ points removed is generated by loops around the points that have been removed.
Two sheeted branched covers

- $S_2$ only contains two elements so the monodromy of two sheeted covers is very easy to analyse.
- The fundamental group of $S_2$ with $p$ points removed is generated by loops around the points that have been removed.
- Thus two sheeted branched covers of $S_2$ are uniquely determined by the critical values.
Genus 1 surfaces are all cubics

Theorem

All genus 1 surfaces are biholomorphic to smooth cubic curves.
Genus 1 surfaces are all cubics

Theorem
All genus 1 surfaces are biholomorphic to smooth cubic curves.

- We know they can be expressed as 2 sheeted covers with 4 branch points. We can assume without loss of generality that one of the branch points is infinity.
Genus 1 surfaces are all cubics

Theorem

All genus 1 surfaces are biholomorphic to smooth cubic curves.

- We know they can be expressed as 2 sheeted covers with 4 branch points. We can assume without loss of generality that one of the branch points is infinity.
- The cubic curve written in inhomogeneous coordinates

\[ y^2 = (x - \alpha)(x - \beta)(x - \gamma) \]

gives a smooth plane curve in \( \mathbb{C}P^2 \) so long as the \( \alpha, \beta \) and \( \gamma \) are distinct.
Genus 1 surfaces are all cubics

**Theorem**

*All genus 1 surfaces are biholomorphic to smooth cubic curves.*

- We know they can be expressed as 2 sheeted covers with 4 branch points. We can assume without loss of generality that one of the branch points is infinity.

- The cubic curve written in inhomogeneous coordinates

\[ y^2 = (x - \alpha)(x - \beta)(x - \gamma) \]

gives a smooth plane curve in \( \mathbb{C}P^2 \) so long as the \( \alpha, \beta \) and \( \gamma \) are distinct.

- This is obviously a 2 sheeted cover of the sphere with at least 3 branch points using the map \((x, y) \rightarrow x\). There must be a branch point at infinity too by the Riemann Hurwitz formula and the fact that a smooth cubic has genus 0. It is easy to check this directly if you prefer.
Existence of non-vanishing holomorphic one form

Lemma

All smooth cubic surfaces admit a non-vanishing holomorphic one form.
Existence of non-vanishing holomorphic one form

Lemma

All smooth cubic surfaces admit a non-vanishing holomorphic one form.

- Let $P(w, z) = 0$ be the defining equation of the cubic in inhomogeneous coordinates.
- We have meromorphic functions $w$ and $z$ defined on the cubic. Hence we have a meromorphic forms $dw$ and $dz$. By the defining equation for the cubic:

$$P_w dw + P_z dz = 0$$
Existence of non-vanishing holomorphic one form

Lemma

All smooth cubic surfaces admit a non-vanishing holomorphic one form.

Let $P(w, z) = 0$ be the defining equation of the cubic in inhomogeneous coordinates.

We have meromorphic functions $w$ and $z$ defined on the cubic. Hence we have a meromorphic forms $dw$ and $dz$. By the defining equation for the cubic:

$$P_w dw + P_z dz = 0$$

So where $P_w$ and $P_z$ are non zero, we have:

$$\frac{dz}{P_w} = -\frac{dw}{P_z} =: \omega$$

Since the cubic is smooth, there are no points where both $P_w$ and $P_z$ vanish. So this defines a single meromorphic 1-form $\omega$ that is holomorphic and non-vanishing on $\mathbb{C}^2 \subset \mathbb{CP}^2$. 
One can choose coords s.t. $\omega$ has no zeros on a cubic

- In general suppose that $P(z, w)$ defines a smooth degree $d$ curve. Let $p(Z_1, Z_2, Z_3)$ be equivalent to $P$ but in homogeneous coordinates, so $p(1, z, w) = P(z, w)$.

- By the fundamental theorem of algebra, there will be $d$ points where the curve intersects the line at infinity (counted with multiplicity). Perturb our coords to ensure that there are exactly $d$ points.

- Suppose that $x = [0, 1, 0]$ is an intersection point with multiplicity $m$. Take inhomogeneous coords $[u, 1, v] \rightarrow (u, v)$. So $u = \frac{1}{z}$, $v = \frac{w}{z}$.

- Define $q$ to be the homogeneous polynomial of degree $d - 1$ corresponding to $P_w$. Since there are exactly $d$ points on the line at infinity, $q$ is non-zero at $x$ and $u$ is a local coordinate.

- $dz = -u^{-2} du$. $P_w(z, w) = q(1, z, w) = z^{d-1} q(u, 1, v)$ by homogeneity of $q$.

- One form is given by: $\frac{u^{d-3}}{q(u, 1, v)} du$. 
Non-vanishing 1-forms implies torus

Theorem

Any compact Riemann surface \( X \) with a non-vanishing holomorphic 1-form \( \omega \) is biholomorphic to \( \mathbb{C}/\Lambda \) for some lattice \( \Lambda \).
Non-vanishing 1-forms implies torus

Theorem

Any compact Riemann surface $X$ with a non-vanishing holomorphic 1-form $\omega$ is biholomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda$.

- Define $f : \tilde{X} \rightarrow \mathbb{C}$ by integrating $\omega$ along paths. This is well defined on $\tilde{X}$ the universal cover.
- Show that $f$ is a covering map.
- $X$ is quotient of $\mathbb{C}$ hence equivalent to either $\mathbb{C}/\Lambda$ or a cylinder. Since $X$ is compact, cylinders are ruled out.
Non-vanishing 1-forms implies torus - proof

Definition
A continuous map $F : X \rightarrow Y$ is a covering map if around each point $y \in Y$ there exists an open neighbourhood $V$ such that $F^{-1}(V)$ is a disjoint union of open sets $U_\alpha$ and $F$ restricted to each $U_\alpha$ is a homeomorphism onto its image.
Non-vanishing 1-forms implies torus - proof

Definition
A continuous map $F : X \rightarrow Y$ is a covering map if around each point $y \in Y$ there exists an open neighbourhood $V$ such that $F^{-1}(V)$ is a disjoint union of open sets $U_\alpha$ and $F$ restricted to each $U_\alpha$ is a homeomorphism onto its image.

Notice that so long as $Y$ is connected this implies that $X$ is onto.
Non-vanishing 1-forms implies torus - proof

Definition
A continuous map $F : X \rightarrow Y$ is a covering map if around each point $y \in Y$ there exists an open neighbourhood $V$ such that $F^{-1}(V)$ is a disjoint union of open sets $U_\alpha$ and $F$ restricted to each $U_\alpha$ is a homeomorphism onto its image.

▶ Notice that so long as $Y$ is connected this implies that $X$ is onto.
▶ In our case $f : \tilde{X} \rightarrow Y$ is defined by integrating $\omega$. Since $\omega$ is non-vanishing, $f'$ is non-vanishing and so $f$ is a local homeomorphism.
Non-vanishing 1-forms implies torus - proof

Definition
A continuous map $F : X \rightarrow Y$ is a covering map if around each point $y \in Y$ there exists an open neighbourhood $V$ such that $F^{-1}(V)$ is a disjoint union of open sets $U_\alpha$ and $F$ restricted to each $U_\alpha$ is a homeomorphism onto its image.

► Notice that so long as $Y$ is connected this implies that $X$ is onto.
► In our case $f : \tilde{X} \rightarrow Y$ is defined by integrating $\omega$. Since $\omega$ is non-vanishing, $f'$ is non-vanishing and so $f$ is a local homeomorphism.
► Given $x \in \tilde{X}$ define $f_x$ mapping a neighbourhood of $x$ to some disc $D(f(x), r)$ to be this local homeomorphism — so $f_x$ has a well defined inverse on $D(f(x), r)$. 

Compactness of $X$ means that we can choose a single value for $r$ that will work for the whole manifold.
Non-vanishing 1-forms implies torus - proof

Definition
A continuous map $F : X \rightarrow Y$ is a covering map if around each point $y \in Y$ there exists an open neighbourhood $V$ such that $F^{-1}(V)$ is a disjoint union of open sets $U_\alpha$ and $F$ restricted to each $U_\alpha$ is a homeomorphism onto its image.

- Notice that so long as $Y$ is connected this implies that $X$ is onto.
- In our case $f : \tilde{X} \rightarrow Y$ is defined by integrating $\omega$. Since $\omega$ is non-vanishing, $f'$ is non-vanishing and so $f$ is a local homeomorphism.
- Given $x \in \tilde{X}$ define $f_x$ mapping a neighbourhood of $x$ to some disc $D(f(x), r)$ to be this local homeomorphism — so $f_x$ has a well defined inverse on $D(f(x), r)$.
- Compactness of $X$ means that we can choose a single value for $r$ that will work for the whole manifold.
Two points less than $r$ apart
We say that \( x \) and \( y \) are *less than \( r \) apart* if 
\[ y \in f_x^{-1}(D(f(x), r)). \]
This relationship is symmetric.
Non-vanishing one form implies torus — proof (cont.)

- We say that $x$ and $y$ are *less than* $r$ *apart* if $y \in f_x^{-1}(D(f(x), r))$. This relationship is symmetric.
- If $f(x_1) = f(x_2)$ and $x_1$ and $x_2$ are less than $r$ apart then $x_1 = x_2$. This is because $f_x^{-1}$ is one to one.
We say that $x$ and $y$ are less than $r$ apart if $y \in f_x^{-1}(D(f(x), r))$. This relationship is symmetric.

If $f(x_1) = f(x_2)$ and $x_1$ and $x_2$ are less than $r$ apart then $x_1 = x_2$. This is because $f_x^{-1}$ is one to one.

If $x$ and $y$ are less than $\frac{r}{2}$ apart and $y$ and $z$ are less than $\frac{r}{2}$ apart then $x$ and $z$ are less than $r$ apart. This is the triangle law on $\mathbb{C}$ pulled back onto $f_x^{-1}(D(f(x), r))$. 
We say that \( x \) and \( y \) are *less than \( r \) apart* if 
\( y \in f_x^{-1}(D(f(x), r)) \). This relationship is symmetric.

If \( f(x_1) = f(x_2) \) and \( x_1 \) and \( x_2 \) are less than \( r \) apart then 
\( x_1 = x_2 \). This is because \( f_x^{-1} \) is one to one.

If \( x \) and \( y \) are less than \( \frac{r}{2} \) apart and \( y \) and \( z \) are less than \( \frac{r}{2} \) apart then \( x \) and \( z \) are less than \( r \) apart. This is the triangle law on \( \mathbb{C} \) pulled back onto \( f_x^{-1}(D(f(x), r)) \).

Write \( \Delta_x \) for the set of points less than \( \frac{r}{2} \) apart from \( x \).
We say that $x$ and $y$ are less than $r$ apart if $y \in f_x^{-1}(D(f(x), r))$. This relationship is symmetric.

If $f(x_1) = f(x_2)$ and $x_1$ and $x_2$ are less than $r$ apart then $x_1 = x_2$. This is because $f_x^{-1}$ is one to one.

If $x$ and $y$ are less than $\frac{r}{2}$ apart and $y$ and $z$ are less than $\frac{r}{2}$ apart then $x$ and $z$ are less than $r$ apart. This is the triangle law on $\mathbb{C}$ pulled back onto $f_x^{-1}(D(f(x), r))$.

Write $\Delta_x$ for the set of points less than $\frac{r}{2}$ apart from $x$.

If $f(x_1) = f(x_2)$ and $y \in \Delta_{x_1}$ and $y \in \Delta_{x_2}$ then $x_1$ and $x_2$ must be less than $r$ apart. So $x_1 = x_2$. 
We say that \( x \) and \( y \) are \textit{less than} \( r \) apart if \( y \in f_x^{-1}(D(f(x), r)) \). This relationship is symmetric.

If \( f(x_1) = f(x_2) \) and \( x_1 \) and \( x_2 \) are less than \( r \) apart then \( x_1 = x_2 \). This is because \( f_x^{-1} \) is one to one.

If \( x \) and \( y \) are less than \( \frac{r}{2} \) apart and \( y \) and \( z \) are less than \( \frac{r}{2} \) apart then \( x \) and \( z \) are less than \( r \) apart. This is the triangle law on \( \mathbb{C} \) pulled back onto \( f_x^{-1}(D(f(x), r)) \).

Write \( \Delta_x \) for the set of points less than \( \frac{r}{2} \) apart from \( x \).

If \( f(x_1) = f(x_2) \) and \( y \in \Delta_{x_1} \) and \( y \in \Delta_{x_2} \) then \( x_1 \) and \( x_2 \) must be less than \( r \) apart. So \( x_1 = x_2 \).

So \( f^{-1}D(y, \frac{r}{2}) \) is the disjoint union of \( \Delta_x \) where \( x \in f^{-1}y \).
Summary

- Equivalence of de Rham and Dolbeault cohomology shows that any genus 1 surface is a two sheeted cover with four branch points.
- Monodromy of two sheeted cover of sphere shows that this is a classification result.
- Any two sheeted cover with four branch points can be realised by a non-singular cubic.
- Any non-singular cubic is equivalent to $\mathbb{C}/\Lambda$ because they all have non-vanishing holomorphic one forms.
What have we learned?

- There is only one genus 0 Riemann surface.
- All genus 1 Riemann surfaces are $\mathbb{C}/\Lambda$.
- All genus 1 Riemann surfaces are smooth cubics.
- (Homework) the moduli space of genus 1 Riemann surfaces is $\mathbb{C}$.
- Learning how to prove that Laplace’s equation has a unique solution will be a very rewarding pursuit. (Chapter 9 of Donaldson)
Where did we cheat?

- The classification of surfaces assumed lots of Morse theory.
- We have only discussed the fundamental group informally.
- We motivated but didn’t prove Bezout’s theorem. See Kirwan for details — and perhaps read about “complex quantifier elimination” to understand the handwaving motivation in more detail.
- We didn’t prove the existence and uniqueness of solutions to Laplace’s equation.