Introduction to Riemann Surfaces — Lecture 4

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KCL

26 November 2012
Overview of Course

1. Definition and examples of Riemann Surfaces
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2. Understand statement: $S^2$ is unique genus 0 Riemann surface.
3. Understand statement: All genus 1 surfaces are given as $\mathbb{C}/\Lambda$. The moduli space is biholomorphic to $\mathbb{C}$. 
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4. $S^2$ is unique surface with a meromorphic function with exactly 1 pole of degree 1.

TODO:
5. $\mathbb{C}/\Lambda$ are the only compact surfaces with a non-vanishing holomorphic 1 form.
6. Definition and examples of De Rham cohomology.
7. Definition of Dolbeault cohomology.
8. Understand statement: The existence and uniqueness of meromorphic functions and forms is encoded by Dolbeault cohomology.
9. Equivalence of De Rham and Dolbeault cohomology on surfaces.
10. 2 and 3 follow from 4 and 5 given 7 and 8.
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Except...

1. We may only get as far as the results — i.e. may not prove equivalence of Dolbeault and De Rham cohomology.
2. We won't prove equivalence of Dolbeault and De Rham cohomology.
3. We will show that it is equivalent to the existence and uniqueness of solutions to a certain partial differential equation.
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Easier reading

1. Our description of the fundamental group has been ultra brief. Any algebraic topology book can fill in the gaps. I learned this from M Armstrong, Basic Topology.

2. Our description of differential forms and calculus on surfaces will proceed at a break-neck pace. Spivak’s “Comprehensive introduction to differential geometry” is much much slower.

3. Kirwan’s “Complex Algebraic Curves” covers similar ground to this course at a slower pace.
Integration on one manifolds

Suppose $x : U \rightarrow \mathbb{R}$ and $y : U \rightarrow \mathbb{R}$ and $X$ are two coordinates on a 1 manifold. Let $\psi = x \circ y^{-1}$ be the transition function. If $f$ is a real valued on $\mathbb{R}$ then:

$$
\int_{x(U)} f(x) \, dx = \int_{y(U)} f(\psi(y)) \frac{dx}{dy} \, dy
$$

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Densities on one manifolds

Definition

A *density at a point* \( p \) on a 1-manifold is an equivalence class of a pair \((f, x)\) where \( f \) is a number and \( x \) is a chart \( x \rightarrow \mathbb{R} \) centered at \( p \). The equivalence relation is given by:

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(f, x) \sim (g, y) \iff g = f \frac{dx}{dy}
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We denote the equivalence class $[f, x]$ by $f \, dx$. 
Densities on $n$-manifolds

If $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism we have:

$$\int_U f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n = \int_U f(x(y)) \partial(x, y) \, dy_1 \ldots dy_n$$

$$= \int_{\psi(U)} f(\psi^{-1}(y)) \partial(\psi, x)^{-1} \, dy_1 \ldots dy_n$$

Where $\partial(x, y)$ is shorthand for the determinant of the Jacobian matrix.

**Definition**

A *density* on an $n$-manifold is an equivalence class $(f, \phi)$ where:

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We can now define the integral of a density over a manifold. Use a “partition of unity” to define the integral over the entire atlas.
Tangent vectors on 1-manifolds

Definition
A *tangent vector* at a point \( p \) on a 1-manifold is an equivalence class of a number \( v \) and a chart \( x \) with:

\[
(v, x) \sim (v \frac{dy}{dx}, y)
\]

 Whereas for a density we had:

\[
(f, x) \sim (v \frac{dx}{dy}, y)
\]
Transformation of densities and vectors on a 1-manifold

If we change coordinates using $y = 2x$ then, in local coordinates, vectors double in length but densities halve. On a 1-manifold, densities are dual to vectors. Given a density $(p, x)$ and a vector $(v, x)$ the quantity $pv$ is independent of $x$. So a density defines an invariant map from the tangent space of $p$ to $\mathbb{R}$. A density is an element of the dual vector space of the tangent space.
Tangent vectors on $n$-manifolds

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A tangent vector $p$ on an $n$-manifold is an equivalence class of an element $v = (v^i) \in \mathbb{R}^n$ and a chart $x = (x^1, \ldots, x^n)$ centered at $p$ with:

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The upper indices are simply labels not powers. So $x^2$ is a completely different coordinate from $x^1$. It isn’t its square. Surprisingly this convention doesn’t end up causing too much confusion!
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A vector field is a smoothly varying choice of vector at each point. The tangent space $T_p M$ at a point $p$ on a manifold $M$ is the set of all tangent vectors at $p$. It has an obvious vector space structure.
Cotangent vectors on $n$-manifolds

Definition
A cotangent vector $p$ on an $n$-manifold is an equivalence class of an element $\omega = (\omega_i) \in \mathbb{R}^n$ and a chart $x = (x^1, \ldots, x^n)$ centered at $p$ with:

$$(\omega_i, x) \sim (\sum_j \frac{\partial x^j}{\partial y^i} \omega_j, y)$$
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- It is a standard convention to use upper-indices for components of vectors and coordinates and lower-indices for components of forms.
- Equivalently a cotangent vector is an element of $(T_p M)^*$ the dual space of the tangent space. To see this, given a cotangent vector $(\omega_i)$ we define a map from the tangent space to $\mathbb{R}$ by $(v^i) \mapsto \sum_i \omega_i v^i$. This map does not depend on the choice of coordinates.
The exterior derivative of a function

Given a function $f$ on a manifold and coordinates $\mathbf{x}$ define

$$d_{\mathbf{x}} f = \left( \frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n} \right)$$
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This looks like the definition of the gradient of a function. What happens if we change coordinates?

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We conclude that \( (d_x f, x) \) and \( (d_y f, y) \) are equivalent cotangent vectors. Hence we have a well defined cotangent vector \( df \) given independently of our choice of coordinates.
Transformation of covectors and vectors

A good way to draw $df$ is to draw its contours. If we rescale by a factor of 2, the terrain becomes shallower by a factor of two as vectors become longer by a factor of 2. The total distance travelled up or down remains constant.
Summary so far:

▶ A vector is a collection of $n$-numbers in local coordinates that transform like a vector.
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On 1-manifolds covectors and densities are the same thing — but they’re completely different concepts in higher dimensions.
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- We can associated a smooth covector field $df$ to a smooth function $f$. It is somewhat analagous to the gradient of a function, but it is defined independent of coordinates. The standard gradient is only defined up to isometries of $\mathbb{R}^n$ — it depends on the metric.
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▶ On 1-manifolds covectors and densities are the same thing — but they’re completely different concepts in higher dimensions.
Pushing vectors forward

Given a smooth map $F : X \rightarrow Y$ between smooth manifolds if sending a point $p \in X$ to $q \in Y$ we can define a mapping $F_* : T_pX \rightarrow T_qY$. 
Formal definition of $F_*$

Given charts $\mathbf{x}$ for $X$ and $\mathbf{y}$ for $Y$. If $v^i$ are the components of a vector $V$ define $F_*(V)$ to have components:

$$(F_*(V))^i = \sum_a \frac{\partial y^i}{\partial x^a} v^a$$

It is easy to check that this definition is independent of the choice of chart.
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(Notice that our sums always combine a lower index and an upper index — so long as we think of $\frac{d}{dx^i}$ as having a lower index on account of being the denominator of a fraction. In the *Einstein summation convention*, one drops the $\sum$ symbols and always sums over repeated indices.)
Pulling back

By standard linear algebra we have can define a dual map $F^*: T^*_q Y \rightarrow T^*_p X$. We can “pull back” covectors using $F^*$. Notice that if we have a function $g: Y \rightarrow \mathbb{R}$ we can define $F^*(g) = g \circ f$ so functions on a manifold “pull back” too. Notice that $d(F^*g) = F^*(dg)$. You can prove this by a direct calculation, or you can think in terms of contours and say that it is obvious. Both are worth doing!
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- Notice that $d(F^*g) = F^*(dg)$. You can prove this by a direct calculation, or you can think in terms of contours and say that it is obvious. Both are worth doing!
Areas and volumes in vector spaces

Given a vector space \( V \) a good definition of an area \( A \) for \( V \) would be a function that associates an area \( A(v_1, v_2) \) to any two vectors \( v_1 \) and \( v_2 \) that also satisfies:

- **Linearity**: \( A(v_1 + \lambda v_2, v_3) = A(v_1, v_3) + \lambda A(v_2, v_3) \)
- **Anti-symmetry**: \( A(v_1, v_2) = -A(v_2, v_1) \).
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In other words we want something that behaves rather like the cross product on 2-vectors. The anti-symmetry condition means that our concept of area detects orientation just as the cross product does.
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Similarly if we wanted to define a concept of a 3-volume on a vector space we could define it as an antisymmetric multi-linear map from $V \times V \times V \rightarrow \mathbb{R}$. Antisymmetric means that the value changes sign if you swap any two vectors.
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With these ideas in mind we define $\Lambda^p V^*$ of a vector space to be the vector space of antisymmetric multi-linear maps from $V$ to $\mathbb{R}$. 
Integration on submanifolds

- A smooth $p$-form $\omega$ on an $n$-dimensional manifold $M$ is a smoothly varying choice from $\Lambda^p T^* M$. This is usually called a section of $\Lambda^p T^* M$. 

- Locally a $p$-dimensional submanifold $V$ of $M$ is given by a smooth map $F : \mathbb{R}^p \to M$.

- Divide $\mathbb{R}^p$ into cubes of length $\epsilon$. The edges of each cube correspond to vectors so we can push them forward into $M$ using $F$. We can then use $\omega$ to measure the volume of the cube we have pushed forward.

- Define the integral of $\omega$ over $V$ by:

$$
\int_V \omega = \lim_{\epsilon \to 0} \sum \text{cubes} \left( p\text{-volume given by } \omega \right)
$$
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$$\int_V \omega = \lim_{\epsilon \to 0} \sum \text{(p-volume given by } \omega)$$

over cubes
Integrating $\text{df}$ on a 1-dimensional submanifold
The fundamental theorem of calculus is obvious. Given a 1-form $\omega$ we write $\omega(X)$ for the length that $\omega$ associated to a vector $X$.

$$
\int_V df = \lim_{\epsilon \to 0} \sum_i ((df)X_i)
$$

$$
\approx \lim_{\epsilon \to 0} \sum_i \text{change in } f \text{ over interval}
$$

$$
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$$\approx \lim_{\epsilon \to 0} \sum_i \text{change in } f \text{ over interval}$$

$$= \text{Total change in } f$$

The geometry of the situation is clear. To make the argument rigorous one just needs to use Taylor’s theorem to get a bound on the error in the approximation.
Geometric definition of the exterior derivative

Definition
(Non standard) Given a $p$ form $\omega$ on a manifold $M$ and vectors $X_1, X_2, \ldots, X_{p+1}$ at a point in $M$ choose a smooth map $F$ from $\mathbb{R}^{p+1}$ to $M$ such that $F_*$ sends the coordinate axes to the $X_i$. Let $\Delta_\epsilon$ be the the tetrahedron:

$$\Delta_\epsilon = \{ (x_1, x_2, \ldots, x_p) : x_i \geq 0, \sum_{i} x_i \leq \epsilon \}$$
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$$\Delta_\epsilon = \{(x_1, x_2, \ldots, x_p) : x_i \geq 0, \sum_i (x_i) \leq \epsilon\}$$

Define $d\omega$ by:

$$d\omega(X_1, X_2, \ldots X_{p+1}) = \lim_{\epsilon \to 0} \frac{(p + 1)!}{\epsilon^{p+1}} \int_{F(\partial \Delta_\epsilon)} (\omega)$$
d on 0-forms.

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Let \( x \) be a chart centered at a point \( p \) on the manifold. Let \( V \) be a tangent vector at \( p \) and assume that the path \( \gamma : \mathbb{R} \to \mathbb{R} \) has tangent vector \( V \) at 0.
\textbf{d on 0-forms.}

- A 0-form is just a function, $f$, on a manifold. The integral of 0-form over a 0-dimensional submanifold is just the sum of $f$ over the points in the 0-dimensional submanifold.

- Let $x$ be a chart centered at a point $p$ on the manifold. Let $V$ be a tangent vector at $p$ and assume that the path $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ has tangent vector $V$ at 0.

- Use $t$ to denote the coordinate on $\mathbb{R}$.

\[ df(X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\gamma(\partial[0,\epsilon])} f \]

\[ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (f(\gamma(\epsilon)) - f(\gamma(0))) \]
\( \text{d on 0-forms.} \)

- A 0-form is just a function, \( f \), on a manifold. The integral of 0-form over a 0-dimensional submanifold is just the sum of \( f \) over the points in the 0-dimensional submanifold.

- Let \( x \) be a chart centered at a point \( p \) on the manifold. Let \( V \) be a tangent vector at \( p \) and assume that the path \( \gamma : \mathbb{R} \to \mathbb{R} \) has tangent vector \( V \) at 0.

- Use \( t \) to denote the coordinate on \( \mathbb{R} \)

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= \lim_{\epsilon \to 0} \frac{1}{\epsilon} (f(\gamma(\epsilon)) - f(\gamma(0)))
\]

- It is now clear from the chain rule that the two definitions we have given for \( \text{d} \) on 0-forms are equivalent.
Properties of $d$

- It is well defined because it only depends on first order term of $F$.

- It generalizes the notion of the derivative of a function.

- It measures the rate at which the notion of length/area/volume changes over an infinitesimal tetrahedron.

- $d\omega$ is alternating in the $X_i$.

- (Less obvious) it is linear in the $X_i$ so is a $(p+1)$-form.

- It satisfies Stokes' theorem $\int_V d\omega = \int_{\partial V} \omega$.

- It satisfies $dd\omega = 0$. This follows from Stoke's theorem because $\partial \partial \Delta \epsilon$ is empty.
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- It satisfies $dd\omega = 0$. This follows from Stoke’s theorem because $\partial\partial\Delta_\epsilon$ is empty.
Proof of Stokes’ theorem

The definition of $d$ ensures that Stokes’ theorem is infinitessimally true.
The wedge product

Given two 1-forms \( \omega \) and \( \nu \) we define \( \omega \wedge \nu \) as follows:

\[
(\omega \wedge \nu)(X_1, X_2) = \omega(X_1)\nu(X_2) - \omega(X_2)\nu(X_1)
\]

Where \( X_1 \) and \( X_2 \) are vectors.

\( \omega \wedge \nu \) is clearly a two form. This definition is pure linear algebra on the tangent space.

In general if \( \omega \) and \( \nu \) are \( p \) and \( q \) forms we can define:

\[
(\omega \wedge \nu)(X_1, X_2, ..., X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \omega(X_{\sigma(1)}, X_{\sigma(2)}, ..., X_{\sigma(p)}) \times \nu(X_{\sigma(p+1)}, X_{\sigma(p+2)}, ..., X_{\sigma(p+q)})
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Note that \( \omega \wedge \nu = (-1)^{pq} \nu \wedge \omega \). So \( \wedge \) is anti-commuting on 1-forms.
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Formal definition of $\mathbf{d}$

Definition

$\Omega^p(M)$ is defined to be the space of smooth forms on $M$.

$\mathbf{d} : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ is defined to be the unique $\mathbb{R}$-linear map satisfying:

1. $\mathbf{d} f$ is the differential of $f$ for smooth functions $f$ as defined earlier.
2. $\mathbf{d}(\mathbf{d} f) = 0$ for any smooth function $f$.
3. $\mathbf{d}(\alpha \wedge \beta) = \mathbf{d} \alpha \wedge \beta + (-1)^p \alpha \wedge \mathbf{d} \beta$ when $\alpha$ is a $p$-form.

For surfaces, this last item simplifies to the special case $\mathbf{d}(f \alpha) = \mathbf{d} f \wedge \alpha + f \mathbf{d} \alpha$ if $f$ is a function.
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Calculating $d$ on a surface

- If $(x^1, x^2)$ are coordinates for $S$ centered at $p$ then $\{dx^1, dx^2\}$ gives a basis for $T^*_pM$. 

\[ d \text{ determined by the axioms (on a surface).} \]
Calculating $\text{d}$ on a surface

- If $(\chi^1, \chi^2)$ are coordinates for $S$ centered at $p$ then $\{\text{d}\chi^1, \text{d}\chi^2\}$ gives a basis for $T_p^*M$.
- $\text{d}\chi^1 \wedge \text{d}\chi^2$ gives a basis for $\Lambda^2 T_p^*M$.

Notice that this proves that $\text{d}$ is determined by the axioms (on a surface).
Calculating $d$ on a surface

- If $(x^1, x^2)$ are coordinates for $S$ centered at $p$ then $\{dx^1, dx^2\}$ gives a basis for $T_p^*M$.
- $dx^1 \wedge dx^2$ gives a basis for $\Lambda^2 T_p^*M$.
- We can write any 1-form as $\alpha_1 dx^1 + \alpha_2 dx^2$. Using the axioms we compute:

$$
d(\alpha_1 dx^1 + \alpha_2 dx^2) = (d\alpha_1) \wedge dx^1 + (d\alpha_2) \wedge dx^2$$

$$= \left( \frac{\partial \alpha_1}{\partial x^1} \right) dx^1 \wedge dx^1 + \left( \frac{\partial \alpha_1}{\partial x^2} \right) dx^2 \wedge dx^1$$

$$+ \left( \frac{\partial \alpha_2}{\partial x^1} \right) dx^1 \wedge dx^2 + \left( \frac{\partial \alpha_2}{\partial x^2} \right) dx^2 \wedge dx^2$$

$$= \left( \frac{\partial \alpha_2}{\partial x^1} - \frac{\partial \alpha_1}{\partial x^2} \right) dx^1 \wedge dx^2$$
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&= \left( \frac{\partial \alpha_2}{\partial x^1} - \frac{\partial \alpha_1}{\partial x^2} \right) dx^1 \wedge dx^2
\end{align*}
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- Notice that this proves that $\mathbf{d}$ is determined by the axioms (on a surface).
Remarks

- We could have used the formula from the previous slide to define $d$ on a surface.
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- The condition \( ddf = 0 \) is equivalent to \( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0 \).
- To check that my non-standard definition is correct, simply check that it satisfies the axioms.
- The standard definition is the more practical choice for most computations.
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The standard definition is the more practical choice for most computations.
The Poincaré Lemma

**Theorem**

On $\mathbb{R}^2$ if $\omega$ is a 1-form and $d\omega = 0$ there exists a function $f$ with $df = \omega$. 

**Definition**

A closed $p$-form $\omega$ is one that satisfies $d\omega = 0$.

**Definition**

An exact $p$-form $\omega$ is one that can be written $\omega = d\nu$ for some $(p - 1)$-form. Exact forms are always closed.
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An exact \( p \)-form \( \omega \) is one that can be written \( \omega = d\nu \) for some \( (p-1) \)-form. exact forms are always closed.
Proof of the Poincaré lemma

Theorem

On \( \mathbb{R}^2 \) a closed 1-form \( \omega \) is always exact.
Proof of the Poincaré lemma

Theorem

On $\mathbb{R}^2$ a closed 1-form $\omega$ is always exact.

Define $f(x) = \int_{\gamma_1} \omega$. Since $\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_R d\omega = 0$ we see that $f$ is well defined. By the fundamental theorem of calculus $df = \omega$.

(Result follows because $\mathbb{R}^2$ is simply connected.)
A closed form $\omega$ on $\mathbb{R}^2 \setminus \{0\}$ which is not exact
De Rham cohomology

- For clarity write \( d_i = d : \Omega^{i-1}(M) \to \Omega^i(M) \) on an \( n \)-manifold \( M \). We have the **exact sequence**:

\[
\begin{align*}
0 & \xrightarrow{d_0} \Omega^0(M) \xrightarrow{d_1} \Omega^1(M) \xrightarrow{d_2} \Omega^2(M) \xrightarrow{d_3} 0
\end{align*}
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- Define $H^i(M)$ to be the cohomology of the sequence:

$$H^i(M) = \ker d_i / (\text{Im } d_{i-1})$$
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- The dimension of $H^i(M)$ is a topological invariant of $M$ called the $i$-th betti number. (It is not obvious whether or not the betti numbers are finite.)
De Rham cohomology

For clarity write $d_i = d : \Omega^{i-1}(M) \to \Omega^i(M)$ on an $n$-manifold $M$. We have the exact sequence:

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We have shown that the 1-st betti number is zero for simply connected spaces, but non-zero for $\mathbb{R}^2$. 
Bezout’s theorem

Definition
Two complex curves in $\mathbb{C}P^2$ intersect transversally at a point $p$ if $p$ is a non-singular point of each curve and if the tangent space of $\mathbb{C}P^2$ at that point is the direct sum of the tangent spaces of the two curves.

Theorem
(Bezout) Two complex curves of degrees $p$ and $q$ that have no common component meet in no more than $pq$ points. If they intersect transversally, they exactly in $pq$ points.

If the polynomial defining a curve factorizes then each factor defines a component of the curve. Smooth curves have only one component because they would clearly not be smooth at their intersections of the components.
Proof of degree genus formula

Given a smooth plane curve \( C \) of degree \( d \) consider the projection from a point \( p \) to a line \( L \) with \( p \) not lying on \( C \).

By the fundamental theorem of algebra, the degree of this projection map will be \( d \).

We can choose coordinates so that the projection of a point \((z, w)\) in affine coordinates is just \( z \). If \( P(z, w) = 0 \) defines the curve then branch points correspond to points where \( P_w = 0 \). These have ramification index 1 unless \( P_{ww} = 0 \).

By Bezout’s theorem we expect there to be \( d(d - 1) \) branch points and that so long as \( p \) does not lie on a line of inflection (i.e. a tangent to the curve through a point of inflection) there will be exactly \( d(d - 1) \) branch points.

By Bezout’s theorem there are a finite number of lines of inflection (clearly points of inflection will be given by some algebraic condition).

So for generic \( p \) there are exactly \( d(d - 1) \) branch points of ramification index 1.

The degree genus formula now follows from the Riemann-Hurwitz formula.