MOCK EXAM "Introduction to Riemann Surfaces"

August 22, 2020

1. Prove the inverse function theorem for holomorphic functions:

   Theorem 1 *Let f be a holomorphic function on an open neighbourhoood U of 0 in $\mathbb{C}$. Suppose that $f(0) = 0$, but $f'(0) \neq 0$. Then there exists a neighbourhood U' of 0 such that f is a homeomorphism of U' onto $f(U')$ and such that f^{-1} is holomorphic.*

2. (i) State Liouville's theorem.

   (ii) Give an example of two non-compact Riemann surfaces which are diffeomorphic but not biholomorphic. Justify your answer.

   (iii) Use Liouville's theorem to show that the holomorphic automorphisms of $\mathbb{C}$ are the maps $z \to az + b$ for $a \neq 0$.

   (iv) Give an example of two compact Riemann surfaces which are diffeomorphic but not biholomorphic. Justify your answer.

3. Prove that the for the torus T^2, $H^1(T^2) \cong \mathbb{R}^2$. You may assume that $H^1(\mathbb{R}^2) \cong \{0\}$.

4. Give a summary of the key steps of the proof given in the lectures that all compact genus 1 Riemann surfaces are biholomorphic to a torus $\mathbb{C}/\Lambda$ for some lattice Λ. You may assume without proof that $\dim H^{1,0} = \dim H^{0,1} = \frac{1}{2} \dim H^1$.