

# King's College London

UNIVERSITY OF LONDON

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**PLACE THIS PAPER AND ANY ANSWER BOOKLETS in the EXAM ENVELOPE provided**

**Candidate No:** ..... **Desk No:** .....

MSC EXAMINATION

7CCMF06 NUMERICAL AND COMPUTATIONAL METHODS IN  
FINANCE

MAY 2018

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS.

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO ALL FOUR QUESTIONS.

YOU ARE PERMITTED TO USE A CALCULATOR.

ONLY CALCULATORS FROM THE CASIO FX83 AND FX85 RANGE ARE ALLOWED.

**DO NOT REMOVE THIS PAPER  
FROM THE EXAMINATION ROOM**

**TURN OVER WHEN INSTRUCTED**

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1. (i) Describe mathematically how you could simulate  $M$  stock price paths in the Black-Scholes model at the discrete time points  $(0, \delta t, 2\delta t, \dots, N\delta t = T)$ . Justify your answer [30%].

**Answer:** (Bookwork) The stock price  $S_t$  obeys

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

hence by Itô's Lemma the log of the stock price  $z_t$  satisfies

$$dz_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

This is simply Brownian motion with drift, so we may simulate  $z$  at time point  $i$  by

$$z_{i\delta t} = z_{(i-1)\delta t} + (\mu - \frac{1}{2}\sigma^2)\delta t + \epsilon_i\sigma\sqrt{\delta t}$$

where the  $\epsilon_i$  are i.i.d. normally distributed random variables with mean 0 and standard deviation 1.

- (ii) A trader sells a call option with strike  $K$  and maturity  $T$  for the Black-Scholes price  $P$ . They then delta hedge the option at the discrete time points defined above. Derive finite difference equations for the trader's bank balance  $b_i$  at each time point  $i\delta t$ . [30%]

**Answer:** (Bookwork) Let  $\Delta_i$  denote the delta. At the initial time they receive the principal  $P$  and purchase  $\Delta_i$  units of stock. Hence

$$b_0 = P - \Delta_i S_i$$

At intermediate times the bank balance gains interest. They purchase  $\Delta_i - \Delta_{i-1}$  units of stock to ensure that they own exactly  $\Delta_i$  units.

$$b_i = e^{r\delta t} b_{i-1} - (\Delta_i - \Delta_{i-1}) S_i$$

At the final time they gain interest, liquidate their stock portfolio and pay off any liability.

$$b_n = e^{r(t_n - t_{n-1})} b_{n-1} + \Delta_{n-1} S_i - (S_n - K)^+$$

- (iii) A second trader chooses not to trade in options at all. They have an initial principal  $P$  which they initially invest entirely in stock. At each subsequent time point they:
- invest all their wealth in the stock if the stock price increased over the last time interval

- otherwise they place all their wealth in a risk-free bank account.

Let  $b_i$  denote their bank balance,  $q_i$  denote the quantity of stock they hold and  $W_o$  denote their total wealth (all taken at time  $i \delta t$ ). Derive finite difference equations for these quantities which allow their wealth at each time to be computed. [20%]

**Answer:** (Unseen)

$$b_0 = 0$$

$$q_0 = \frac{P}{S_0}$$

and for  $i \geq 0$

$$w_i = e^{r\delta t} b_{i-1} + q_{i-1} S_i$$

$$b_i = \begin{cases} w_i & \text{if } S_i \leq S_{i-1} \\ 0 & \text{otherwise} \end{cases}$$

$$q_i = \frac{(w_i - b_i)}{S_i}.$$

- (iv) In the Black-Scholes model, how would you expect the expected return of each trader to depend upon the drift of the stock  $\mu$  assuming that the time interval  $\delta t$  is small? Justify your answer. [20%]

**Answer:** (Unseen) For the first trader, the expected return should be approximately zero irrespective of the drift by Black-Scholes theory. For the second trader they are investing approximately half the time in assets that grow at a rate  $r$  and in assets that grow at a rate  $\mu$  the rest of the time. So their profit would be  $(e^{\frac{r}{2} + \frac{\mu}{2}} - 1)P$  which increases as  $\mu$  increases.

2. (i) Write a MATLAB function that approximates the integral

$$\int_{-1}^1 e^{\cos(x)} dx \quad (1)$$

using the Monte Carlo method. [30%]

**Answer:** (Similar to bookwork) In anticipation of the final part of the question we divide our code into two functions.

```
function res = integrateByMonteCarlo(f,a,b,N)
x = rand(N,1)*(b-a)+a;
res = 1/N * sum( f( x));
end

function integral = answerQuestion(N)
function r = integrand(x)
r = exp(cos(x));
end
integral = integrateByMonteCarlo(@integrand,-1,1,N);
end
```

- (ii) If  $X$  and  $Y$  are random variables with finite mean and variance, compute the variance of  $X + \lambda Y$ , where  $\lambda$  is a real number, in terms of the variance and covariance of  $X$  and  $Y$ . [10%]

**Answer:** (Bookwork)

$$\text{Var}(X + \lambda Y) = \text{Var}(X) + 2\lambda \text{Cov}(X, Y) + \lambda^2 \text{Var}(Y)$$

- (iii) For what value of  $\lambda$  is this covariance minimized? [10%]

**Answer:** (Bookwork) Solve

$$2 \text{Cov}(X, Y) + 2\lambda \text{Var}(Y) = 0$$

to get  $\lambda = -\frac{\text{Cov}(X, Y)}{\text{Var}(Y)}$ .

- (iv) Use your answer to the questions above to describe the *control-variate method* for improving the accuracy of Monte Carlo integration. [10%]

**Answer:** (Bookwork) Let  $X$  be a random variable whose expectation we wish to compute and let  $Y$  be a random variable whose expectation is known which is correlated with  $X$ , then rather than attempt to estimate the expectation

of  $X$  directly by the Monte Carlo method we try to estimate the value of  $X + \lambda Y$  instead as this will have a lower variance. We may then estimate  $E(X)$  using  $E(X) = E(X + \lambda Y) - \lambda E(Y)$ . This can be applied to Monte Carlo integration since the Monte Carlo integration method is simply a particular case of computing an expectation.

- (v) Suggest an appropriate control variate to improve the calculation of the integral (1). Justify your answer. [20%]

**Answer:** (Unseen) I would use  $\cos(x)$  as the control variate since it would clearly be highly correlated with the integrand in the equation and its integral is known analytically.

- (vi) Could antithetic sampling be used to improve the calculation of the integral (1). Justify your answer. [10%]

**Answer:** (Unseen) No. Antithetic sampling only works when one is calculating expectations of the form  $f(x)$  where  $f(x)$  and  $f(-x)$  are anti-correlated. For the integrand in the question  $f(x)$  and  $f(-x)$  are perfectly correlated so antithetic sampling will be useless.

- (vii) How would you test your MATLAB function? (You must not use the MATLAB function `integral` in your answer to this part.) [10%]

**Answer:** (Unseen) Divide the code into two functions (as we have done) so the Monte Carlo integration routine can be tested on integrals whose value is known analytically. To get a reliable test, one should seed the random number generator before running the test.

3. (i) Let  $f(x)$  be a smooth function. What is meant by the forward, backward and central estimates for the derivative? [20%]

**Answer:** (Bookwork)

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

respectively.

- (ii) Write a MATLAB function to compute the central estimate for the derivative of a function  $f$ .

**Answer:** (Unseen)

```
function [ fDash ] = centralEstimate(f,x,h)
fDash = (f(x+h)-f(x-h))./(2*h);
end
```

Note the brackets. [20%]

- (iii) Write a unit test for this function. You may use the function `assertApproxEqual` that was defined in the lectures if you wish. [20%]

**Answer:** (Unseen)

```
function testCentralEstimate()
x = 1.0;
h = 0.0000001;
actual = centralEstimate(@sin,x,h);
expected = cos(x);
assert( abs(actual-expected) < 0.001 );
end
```

- (iv) Write the MATLAB code to generate a log-log plot illustrating the error in the estimate. You should use built-in function `loglog(x,y)` to draw a log-log plot of the points  $x$  against the points  $y$ . [20%]

**Answer:** (Unseen) Students may use a for loop instead of vectorizing the calculation.

```

function errorPlot()
x = 1.0;
logH = -30:0;
h = exp(logH);
error = abs(centralEstimate(@sin,x,h) - cos(x));
loglog(h, error);
xlabel('h');
ylabel('error');
end

```

- (v) What would you expect the plot to look like? Justify your answer. [20%]  
**Answer:** (Unseen) On the right the gradient is 2 indicating convergence of

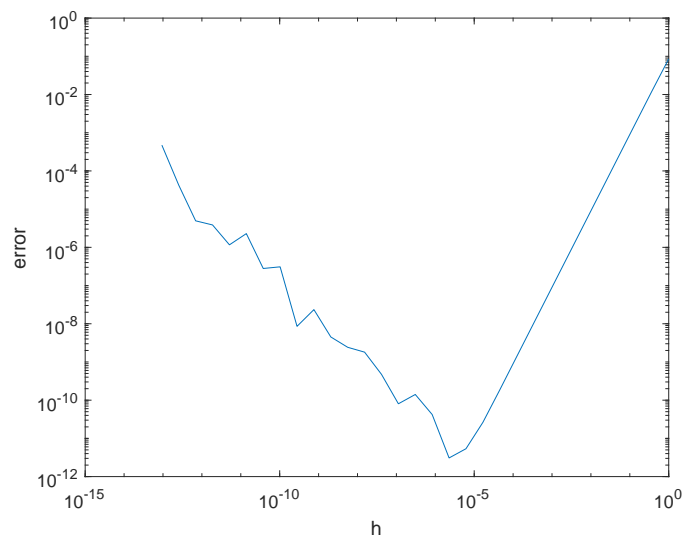


Figure 1: Log-Log plot of the error of the central estimate against  $h$

order  $O(h^2)$ . However, for excessively small  $h$  rounding errors will dominate the calculation resulting in increasing errors as shown on the left of the chart.

4. (i) What is meant by Cholesky decomposition? [20%]

**Answer:** (Bookwork) The Cholesky decomposition of a positive definite matrix,  $A$ , is the unique lower triangular matrix  $L$  satisfying  $A = LL^T$ .

- (ii) Explain how Cholesky decomposition can be used to simulate a multivariate normal distribution with mean vector  $\mu$  and covariance matrix  $\Sigma$ .

[20%]

**Answer:** (Bookwork) Let  $L$  be the Cholesky decomposition of  $\Sigma$  and  $\epsilon$  a vector whose components are i.i.d. normally distributed with mean 0 and standard deviation 1. Then  $L\epsilon + \mu$  will be normally distributed with the desired mean and covariance.

- (iii) Suppose that a financial market consists of  $n$  assets whose value at time  $T$  follow such a multivariate normal distribution. Suppose also the initial prices of the assets are given by the components of a vector  $c$ . An investor has an amount  $P_1$  to invest at time 0 and they wish to purchase a portfolio of assets with an expected payout of  $P_2$ . They wish to choose a portfolio that minimizes the risk of their position. Short selling is allowed. Describe mathematically what optimization problem they should solve, being careful to justify how you measure risk. [20%]

**Answer:** (Bookwork) Let  $q \in \mathbb{R}^n$  be the vector of quantities of each asset. The total cost is  $q \cdot c$ . The expected payout is  $q \cdot \mu$ . The variance is  $q^T \Sigma q$ . The final payoff must be normally distributed with mean  $P_2$ . This means that the final distribution is determined completely by its variance. Hence the variance of the final distribution completely captures the risk of the position. Thus the optimization problem to solve is:

$$\begin{array}{ll} \underset{q \in \mathbb{R}^n}{\text{minimize}} & q^T \Sigma q \\ \text{subject to} & q \cdot c = P_1 \\ \text{and} & q \cdot \mu = P_2 \end{array}$$

- (iv) How could you solve this optimization problem in MATLAB? [10%]

**Answer:** (Similar to bookwork) Let  $A$  be the matrix with rows  $c$  and  $\mu$  and  $b$  be the vector

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

then the constraints may be written  $Aq = b$ . We have now written our optimization problem explicitly as a quadratic programming problem with linear constraints which can be solved using `quadprog` in MATLAB.



- (v) In practice, the covariance matrix  $\Sigma$  and mean  $\mu$  would need to be estimated from historic data. Describe briefly how you might use computer simulations to estimate the magnitude of this model risk. [30%]

**Answer:** (Unseen) You might choose a covariance matrix and return vector representing the “true” model and then simulate (as described in (ii)) the daily changes in the asset price over a one year period. One could then calibrate a new model by calculating the sample mean and covariance of this simulation. One can then solve the optimization problem for this new model to find the optimal portfolio  $q'$  with respect to this sample mean and covariance. One could then use the “true” model again to simulate the outcome of investing in this way. By repeating the process one could find the distribution of profits and loss for the entire process and compare it with the theoretical distribution obtained by solving the optimization problem for the true model.