King's College London

UNIVERSITY OF LONDON

PLACE THIS PAPER AND ANY ANSWER BOOKLETS **in the EXAM ENVELOPE provided**
Candidate No: .... **Desk No:** ....

MSc EXAMINATION

7CCMFM06 NUMERICAL AND COMPUTATIONAL METHODS IN FINANCE

SUMMER 2017

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS.

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO FOUR QUESTIONS.

IF MORE THAN FOUR QUESTIONS ARE ATTEMPTED, THEN ONLY THE BEST FOUR WILL COUNT.

NO CALCULATORS ARE PERMITTED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. (i) What does it mean to say that "a trader sells a call option on a stock and then hedges their exposure using the discrete-time delta-hedging trading strategy"? [**20%**]

(ii) Suppose that a trader does pursue this investment strategy and rehedges at a finite number of time points t_i. Derive the difference equations you would use to simulate the outcome of this strategy. [**30%**]

(iii) Explain what is meant by a utility function and describe how you would estimate the trader's expected utility. [**20%**]

(iv) How would you estimate the accuracy of your calculation of the expected utility? [**10%**]

(v) Suppose that the stock follows the Black-Scholes model and that the trader re-hedges at N evenly spaced time points. Is it always true that the trader's expected utility will increase as $n \to \infty$? Justify your answer. [**20%**]

2. (i) A stock price process S_t follows geometric Brownian motion

$$\mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t)$$

write a MATLAB function to simulate M price paths at the $N+1$ evenly spaced points $S_0, S_{\delta t}, S_{2\delta t}, \ldots S_T$ where $\delta t = \frac{T}{N}$. [**30%**]

(ii) Describe how you would use this code to approximate the risk-neutral price of a discrete-time Knock-out call option with strike K, maturity T and barrier $B > S_0$ in the Black–Scholes model by the Monte Carlo method. (Recall that by definition of this option, if at any time $i\delta t$ where $i \in \{0, 1, 2, \ldots, N\}$ the price is above the barrier B the option will have a payoff of zero. Otherwise its payoff is given by $\max\{S_T - K, 0\}$.) [**20%**]

(iii) How would you estimate the error in your answer? [**10%**]

(iv) How would you apply the control variate method to decrease the error in your answer? [**30%**]

(v) Suppose that simple Monte Carlo with $100,000$ samples is accurate to within 2 cents and that using the control variate method with the same number of samples the answer is accurate to within 1 cent. Estimate how many samples would be needed to make the simple Monte Carlo method as accurate as the control variate method. Explain your answer. [**10%**]

3. (i) Define the term *pseudo square root.* [**10%**]

(ii) Define the term *Cholesky decomposition.* [**10%**]

(iii) Let N_1, N_2 and N_3 be independent Gaussian random variables of mean 0 and standard deviation 1. Suppose that X_1, X_2 and X_3 are random variables defined by:

$$\begin{array}{lcccccc} X_1 & = & 2N_1 & & & + & N_3 \\ X_2 & = & & & 3N_2 & & \\ X_3 & = & N_1 & + & 4N_2 & + & N_3 \end{array} .$$

What is the covariance matrix of (X_1, X_2, X_3)? [**20%**]

(iv) Write down two distinct pseudo square roots of this covariance matrix. [**20%**]

(v) In the Markowitz model, assets returns over a time period T are assumed to be normally distributed with covariance matrix Σ and mean vector μ. Explain how the Cholesky decomposition could be used to simulate asset returns in this model. [**20%**]

(vi) Prove that there is no real valued square matrix L such that

$$LL^T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

[**20%**]

4. (i) Let X_t be a stochastic process which solves the SDE

$$\mathrm{d}X_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)\,\mathrm{d}t + \left(\sqrt{1+X_t^2}\right)\,\mathrm{d}W_t, \qquad X_0 = 0$$

where W_t is Brownian motion. Write down the Euler scheme for X_t. [**20%**]

(ii) Show that $X_t = \sinh(t + W_t)$. [**20%**]

(iii) How would you simulate X_t in practice? Write a MATLAB function that produces a matrix of M simulations of X_t over a time interval T with N time steps. [**30%**]

(iv) Describe a graph you could plot to test how rapidly the Euler scheme for X_t converges to the true solution of the stochastic differential equation. Briefly describe how you could produce this graph in MATLAB. What result would you expect? [**30%**]

5. (i) The backwards heat equation is

$$\frac{\partial u}{\partial t} = -\sigma^2 \frac{\partial^2 u}{\partial x^2}.$$

You are given the condition $u(x,T) = f(x)$ where f is a piecewise smooth bounded real function, and wish to solve this equation numerically using the explicit finite difference method. Derive the difference equations you would use to find the solution to the backward heat equation at time 0. [**30%**]

(ii) Explain briefly how solving the heat equation can be used to price derivatives in the Black–Scholes model with an interest rate of 0. [**20%**]

(iii) When is the explicit finite difference method stable? [**10%**]

(iv) Give a probabilistic interpretation of the difference equations you derived in the first part of the question. [**20%**]

(v) Let δt be the time step used for finite difference method and δx the space step. Suppose we wish to compute $u(0,T)$ for some fixed time T. Let $u^N(0,0)$ denote the value computed by using the finite difference method with $\delta T = \delta X = \frac{T}{N}$. Show that in this case we can find a function f such that $u^N(0,0)$ does not converge to the correct answer as $N \to \infty$. [**20%**]

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