1. **Monte Carlo.** Denote with X a real-valued random variable having a probability density function $f_X(x)$, and consider a function $h : R \to R$.

   (a) Explain how the expectation $Eh(X)$ can be estimated using Monte-Carlo simulation. What is the convergence behaviour as the number of scenarios increases? What is a measure of the accuracy of the estimator that you propose. **[30%]**

   (b) What is a 'variance reduction technique'? Give three different variance reduction techniques, and give an intuitive interpretation of what they are trying to achieve. **[40%]**

   (c) Produce pseudo-code that implements the estimation of $Eh(X)$ that you provided in (a). Also give pseudo-code that implements one of the variance reduction techniques that you proposed in (b). **[30%]**

2. **Hedging.** For this question make the assumptions of the Black-Scholes framework.

   (a) Explain the concept of Delta hedging, and how it relates to the assumptions and derivation of the Black-Scholes formula. In particular, explain how failure of these assumptions might impact Delta hedging in practice. **[40%]**

   (b) Consider a European put option with maturity T and strike X. Assume that the pricing function $P(t, S)$ and Delta $\Delta(t, S)$ are known. Also assume that a bank account is available, which offers a constant borrowing and lending rate r (compounded over a trading period). Write down the equations that govern the evolution of the bank balance of a desk which writes a put option and then Delta-hedges it. **[40%]**

   (c) Say that you observe a path of the underlying stock $\{S_t\}_{t=0}^{t=T}$. Produce pseudo-code that implements your answer to (b). What are the typical paths of the Delta and the bank balance for different price paths, as the option expires in- or out-of-the-money? **[20%]**

3. Risk.

(a) Say that an investor exhibits a utility function of wealth $U(W)$. Explain why the concavity of the utility function can serve as a measure of her risk aversion, when faced with a gamble where there is a 50% chance of winning or losing an amount ΔW. Explain why an investor with a concave utility function would be averse to volatility. Show why gambles with higher volatility must be accompanied by higher expected returns, for the investor to remain indifferent (between taking and not taking the gamble). **[40%]**

(b) Describe the concept of Value-at-Risk (VaR) and explain which shortcomings of volatility (as a risk measure) it tries to address. Describe the concept of Expected Shortfall (ES) and explain which shortcomings of VaR it tries to address. **[30%]**

(c) Describe the Exponentially Weighted Moving Average (EWMA) methodology for filtering volatility. Given a time series of historical returns $\{r_t\}_{t=1}^{t=T}$ produce pseudo-code that calculates the stock return volatility of the next period σ_{T+1} using this method. Explain how one can produce estimates of VaR (Value-at-Risk) and ES (Expected Shortfall) using the estimate of σ_{T+1}. **[30%]**

4. **Finite differences.** The Black-Scholes PDE is $f_t + (\mu - \sigma^2/2) f_s + \frac{1}{2}\sigma^2 f_{ss} = rf$, for a pricing function $f = f(s,t)$, where s is the log-price of the underlying and t denotes time. Subscripts denote partial derivatives.

   (a) Write down the finite difference approximations for the first and second order derivatives involved, and produce the Crank-Nicolson (CN) approximation scheme. Say that you are pricing a European put option which is at-the-money and has a maturity of 90 days. Use the known value of the option Delta to derive and apply the boundary conditions to the CN scheme. Show how the Black-Scholes PDE can be solved recursively to obtain the current value of the option. **[40%]**

   (b) Say that the option in (a) now has a down-and-out barrier feature, where the barrier is set at 85% of the strike price, monitored on day 10, 20, etc. Explain which parts of your answer to (a) need to be modified to account for this feature. **[30%]**

   (c) Produce pseudo-code that implements your answer to (a) and (b). Show graphically how you expect the pricing function to look like (as a function of the spot price) in those two cases, at time $t = 0$. **[30%]**

5. **Empirical findings.**

   (a) What are the main stylised features of the yield curve? Explain briefly the stylised behaviour across time and across tenors. Explain how parametric models attempt to replicate these features. **[30%]**

   (b) What are the main stylised features of equity volatility? How does historical (realised) volatility compare to implied volatility? **[30%]**

   (c) Describe the concept of Principal Component Analysis (PCA) in the context of interest rate modelling. Produce pseudo-code that implements PCA on a cross-sectional data set of yields $\{y_{t,\tau}\}$, where t is the recorded time and τ is the tenor. What are your expected results, if the three first factors are kept. **[40%]**

6. Credit.

(a) Define the Gaussian copula, and describe how it can be used to simulate dependent random variables that have different marginal distributions. Explain what we mean when we say that 'the Gaussian copula has zero tail-dependence'. Why is that undesirable? **[40%]**

(b) Consider a portfolio of $N = 500$ credits. 250 of these credits have default probability $p_1 = 1\%$ each, and the other 250 has default probability $p = 3\%$ each. The recovery rate distribution is the same for all credits, and is Gaussian with mean 30% and standard error 5%. Produce pseudo-code that simulates the loss distribution, when default events are correlated via a Gaussian copula, while recoveries are independent. How do you expect the loss distribution to change as the correlation coefficient increases from zero to 0.70? **[60%]**

Final Page