King's College London

UNIVERSITY OF LONDON

PLACE THIS PAPER AND ANY ANSWER BOOKLETS **in the EXAM ENVELOPE provided**

Candidate No: .... **Desk No:** ....

MSc EXAMINATION

7CCMFM06 NUMERICAL AND COMPUTATIONAL METHODS IN FINANCE

AUGUST 2019

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS. FULL MARKS ARE AWARDED FOR CORRECT ANSWERS TO FOUR QUESTIONS. IF MORE THAN FOUR ARE ATTEMPTED THEN ONLY THE BEST FOUR WILL COUNT.

YOU ARE PERMITTED TO USE A CALCULATOR.
ONLY CALCULATORS FROM THE CASIO FX83 AND FX85 RANGE ARE ALLOWED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. (i) What is meant by the terms $\mathbb{P}$-measure and $\mathbb{Q}$-measure? [**20%**]

(ii) In the Black–Scholes model one can simulate a stock price using either the $\mathbb{P}$- or the $\mathbb{Q}$-measure. State which should you use for the following tasks:

(a) Testing the effectiveness of discrete time delta hedging by simulating the use of this strategy in practice.

(b) Computing Value at Risk by the Monte Carlo method.

(c) Computing the Black–Scholes price of an option by the Monte Carlo method.

[**10%**]

(iii) What stochastic differential equation does a stock price satisfy in the Black–Scholes model? Give the answer for both the $\mathbb{P}$- and $\mathbb{Q}$-measures. [**10%**]

(iv) Describe mathematically how you would simulate the stock price in the $\mathbb{P}$-measure model at a final time T given the stock price at time 0, justifying your answer. [**30%**]

(v) Write a MATLAB function to simulate n samples of the stock price at time T. [**20%**]

(vi) Use your answer to write the MATLAB code to price a digital call option of strike K and maturity T. [**10%**]

2. (i) Explain what is meant by a utility function. Give an example and sketch its graph. [**20%**]

(ii) Trader A writes a call option for a price P price with strike K and maturity T. They hedge the option using the delta hedging trading strategy at the discrete time points $0, \delta t, 2\delta t, \ldots, n\delta t = T$. Derive difference equations that allow their bank balance to be computed at each time point given the stock prices and a risk free interest rate r. [**30%**]

(iii) Trader B decides not to trade in options at all. Instead they follow a trading strategy involving only investing in the stock and a risk free bank account. At each discrete time point they rebalance their portfolio to ensure they have invested a fixed proportion, α, of their total wealth in stock. The remainder is placed in the risk free account. Their initial wealth is W_0. Describe how you would numerically compute the expected utility of this trader assuming you have been given a function `simulate-StockPrices` which can simulate the stock price in discrete time in either the $\mathbb{P}$ or $\mathbb{Q}$ measure. [**20%**]

(iv) Suppose that the ask price of the stock at each time point, $i\delta t$, is given by a random variable S_i and the bid price is $(1-\epsilon)S_i$ for some fixed $\epsilon > 0$. How would you change the difference equations for Trader A to account for this (you need only describe the change you would make at intermediate time points)? How would you compute their total wealth W_i at each time point $i\delta t$?

[**20%**]

(v) Assume $\epsilon > 0$. If the stock prices are simulated in accordance with the Black–Scholes model and Trader A has utility given by an exponential utility function, what will happen to the expected utility of Trader A as δt tends to zero? [**10%**]

3. (i) Suppose that a stock price follows geometric Brownian motion

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t).$$

Compute an SDE for the log of the stock price, $z_t = \log S_t$. Hence write down the probability density function for the log of the stock price z_T at time T. **[30%]**

(ii) Suppose that you wish to calculate the price of a call option with strike K and maturity T in the Black–Scholes model. Write down a 1-dimensional integral in terms of z_T which can be used to compute the price of the option. **[20%]**

(iii) Name three integration methods which can be used to evaluate an integral of the form

$$\int_a^b f(x)\,\mathrm{d}x.$$

State their rate of convergence (assuming a sufficiently well-behaved function f). **[10%]**

(iv) How would you transform the integral found in part (ii) so that these integration rules can be applied? **[20%]**

(v) Is the integral you have computed sufficiently well-behaved for you to be able to obtain the maximum rate of convergence? Justify your answer and, if necessary, explain what changes you would need to make to achieve the desired rate of convergence. **[20%]**

4. (i) (a) What is meant by a pseudo square root of a positive definite symmetric matrix A? **[10%]**

(b) What is meant by the Cholesky decomposition of a positive definite symmetric matrix A? **[10%]**

(c) Compute the Cholesky decomposition of the matrix:

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

[30%]

(ii) The Box–Muller algorithm is an algorithm to generate independent normally distributed random numbers with mean 0 and standard deviation 1. One first generates uniformly distributed random numbers U_1 and U_2 between 0 and 1. One then defines $Z_1 = R\cos(\theta)$, $Z_2 = R\sin(\theta)$ where $R^2 = -2\log U_1$ and $\theta = 2\pi U_2$.

(a) Write a function `boxMuller` which takes a parameter n and returns an $2 \times n$ sample of independent normally distributed random numbers generated by the Box–Muller algorithm. **[20%]**

(b) Using your answer to parts (i)(c) and (ii)(a), show how you would write a function which takes a parameter n and returns a $2 \times n$ sample from a two-dimensional multivariate normal distribution with mean 0 and covariance matrix

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

[20%]

(c) How would you test this function? **[10%]**

5. (i) When the risk-free interest rate, r, is zero, the Black–Scholes partial differential equation simplifies to:

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} = 0.$$

Show that this can be transformed by a change of variables into a constant coefficient equation:

$$C_t = -\frac{1}{2}\sigma^2 C_{xx}.$$

[**20%**]

(ii) Derive the difference equations for the implicit finite difference scheme for solving this equation. [**30%**]

(iii) What boundary conditions would you use to price a put option in the Black-Scholes model when using this implicit method? Justify your answer. [**30%**]

(iv) Explain why it is essential to provide top and bottom boundary conditions to solve the heat equation by the implicit method but only a final condition is required for the explicit method. [**20%**]