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Flexibility of Steklov eigenvalues via boundary homogenisation

Mikhail Karpukhin[1] · **Jean Lagacé**[2]

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Abstract
Recently, D. Bucur and M. Nahon used boundary homogenisation to show the remarkable flexibility of Steklov eigenvalues of planar domains. In the present paper we extend their result to higher dimensions and to arbitrary manifolds with boundary, even though in those cases the boundary does not generally exhibit any periodic structure. Our arguments use a framework of variational eigenvalues and provide a different proof of the original results. Furthermore, we present an application of this flexibility to the optimisation of Steklov eigenvalues under perimeter constraint. It is proved that the best upper bound for normalised Steklov eigenvalues of surfaces of genus zero and any fixed number of boundary components can always be saturated by planar domains. This is the case even though any actual maximisers (except for simply connected surfaces) are always far from being planar themselves. In particular, it yields sharp upper bound for the first Steklov eigenvalue of doubly connected planar domains.

Résumé
D. Bucur et M. Nahon ont récemment démontré une flexibilité remarquable pour le spectre de Steklov de domaines planaires grâce à une homogénéisation périodique de la frontière. Dans cet article, nous généralisons leur résultat aux dimensions plus grandes ainsi qu'à des variétés à bord arbitraire, même lorsque le bord n'est en général pas muni d'une structure périodique. Nos arguments sont fondés dans le cadre des valeurs propres variationnelles et donnent une preuve différente des résultats originaux. De plus, nous présentons une application de cette flexibilité à l'optimisation des valeurs propres de Steklov sous contrainte de périmètre. Nous démontrons que la meilleure borne supérieure pour les valeurs propres de Steklov normalisées pour des surfaces de genre nul et n'importe quel nombre de composantes connexes du bord est saturé par des domaines planaires. Ceci est le cas même si tous les maximiseurs (sauf pour les surfaces simplement connexes), sont très loin d'être planaires eux-mêmes. En particulier, nous avons une borne supérieure optimale explicite pour la première valeur propre de domaines planaires doublement connexes.

✉ Jean Lagacé
jean.lagace@kcl.ac.uk

Mikhail Karpukhin
m.karpukhin@ucl.ac.uk

1 Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

2 Department of Mathematics, King's College London, The Strand, London WC2R 2LS, UK

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1 Introduction, main results and setting

1.1 Optimisation of Steklov eigenvalues

Let $(\mathcal{M}, g)$ be a complete smooth Riemannian manifold, $\Omega \subset \mathcal{M}$ be a domain with non-empty Lipschitz boundary and $0 \not\equiv \beta : \partial\Omega \to [0, \infty)$ be a non-negative function. We refer to such an Ω as a *manifold with Lipschitz boundary*; any abstract manifold with smooth boundary can be realised in this way. Consider the eigenvalue problem

$$\begin{cases} \Delta_g u = 0 & \text{in } \Omega, \\ \partial_\nu u = \sigma \beta u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Under some integrability conditions on β to be made explicit later (see Theorem 1.5), the eigenvalues are discrete and form a sequence

$$0 = \sigma_0(\Omega, g, \beta) < \sigma_1(\Omega, g, \beta) \le \sigma_2(\Omega, g, \beta) \le \ldots \nearrow \infty.$$

For every k, the *naturally normalised eigenvalue* is

$$\overline{\sigma}_k(\Omega, g, \beta) = \sigma_k(\Omega, g, \beta) \frac{\int_{\partial\Omega} \beta \, \mathrm{d}A_g}{\mathrm{Vol}_g(\Omega)^{1-\frac{2}{d}}},$$

see [10, 14] for a discussion around the naturality of that normalisation. The case $\beta \equiv 1$ is of particular interest and is referred to as the Steklov problem. The corresponding Steklov eigenvalues $\sigma_k(\Omega, g, 1)$ are denoted simply as $\sigma_k(\Omega, g)$. For many known results and open questions about the Steklov problem, the reader can refer to the survey [12] and the references therein. In the present paper we are mainly concerned with the optimisation problem for normalised Steklov eigenvalues.

The first result of this type was obtained by Weinstock [20] who proved that the round disk maximises the first normalised Steklov eigenvalue in the class of all bounded simply connected smooth planar domains. The optimisation problem for other topologies of domains in $\mathbb{R}^2$ remains unsolved. At the same time, if one does not impose any assumptions on the topology of the planar domain, then the optimal upper bound for all normalised Steklov eigenvalues is

$$\text{for all } k \in \mathbb{N} \qquad \overline{\sigma}_k(\Omega, g) \le 8\pi k,$$

was found in [10]. The main goal of the present paper is to apply the ideas of [4] to the optimisation problem for planar domains of fixed topology. Among other things, this allows us to determine the optimal upper bound for the first normalized Steklov eigenvalue in the class of planar domains with exactly 2 boundary components.

As a starting point, let us note that an examination of Weinstock's proof yields that the round disk continues to be the maximiser in the much larger class of all simply connected Riemannian surfaces. The main observation of the present paper is that the same holds for other topologies as well, namely, the optimal upper bound for normalized Steklov eigenvalues for planar domains of fixed topology does not increase after including arbitrary Riemannian surfaces of the same topological type. To give a precise statement, for any $\gamma \ge 0$ and $b \ge 1$, we let $\Omega_{\gamma,b}$ be the compact connected surface with boundary of genus γ with b boundary components, and we define

$$\Sigma_k(\gamma, b) = \sup_g \overline{\sigma}_k(\Omega_{\gamma,b}, g).$$

Theorem 1.1 *For every $b \geq 1$ and $k \geq 0$ one has*

$$\sup_{\Omega\subset\mathbb{R}^2} \overline{\sigma}_k(\Omega) = \Sigma_k(0, b),$$

where the supremum is taken over the set of all bounded Lipschitz domains in $\mathbb{R}^2$ with b boundary components.

Remark 1.2 It follows from our proof that in fact, $\Sigma_k(\gamma, b)$ is saturated by domains in a surface of constant curvature for every $\gamma \geq 0$.

The quantities $\Sigma_k(\gamma, b)$ have received a lot of attention following the influential work of Fraser and Schoen [8], who established the connection between $\Sigma_k(\gamma, b)$ and free boundary minimal immersions of $\Omega_{g,b}$ into a Euclidean ball. In particular, they showed that for the annulus $\Omega_{0,2} = \mathbb{A}$, $\Sigma_1(0, 2)$ is achieved by a metric g_{cc} on the so-called *critical catenoid* in $\mathbb{B}^3$. Combining this result with Theorem 1.1 one obtains the following.

Corollary 1.3 *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain with 2 boundary components. Then one has*

$$\overline{\sigma}_1(\Omega) < \overline{\sigma}_1(\mathbb{A}, g_{cc}) \approx 4\pi/1.2. \tag{1.2}$$

The inequality is sharp, i.e. there exists a sequence of domains Ω_n such that $\overline{\sigma}_1(\Omega_n) \to \overline{\sigma}_1(\mathbb{A}, g_{cc})$.

Remark 1.4 Theorem 1.1 and Corollary 1.3 can be extended to Lipschitz rather than smooth domains. In such a case, however, inequality (1.2) would stop being strict. In order to rule out the equality case one would need to show a regularity theorem for $\overline{\sigma}_1$-maximisers in the spirit of [17, Theorem 1.4].

Many sequences of planar domains saturate bound (1.2). For any bounded $\Omega \subset \mathbb{R}^2$ conformal to $(\mathbb{A}, g_{cc})$ one can find a maximizing sequence Ω_n such that $\Omega_n \to \Omega$ in Hausdorff distance. Here is a concrete example of one of those maximising sequences, which follows from the proof of Theorem 1.1 and the geometry of g_{cc}. Let t_1 be the unique solution of $\coth t = t$. Set $\Omega_0 = \{z \in \mathbb{R}^2,\ r < |z| < R\}$, where $\log \frac{R}{r} = 2t_1$. Then define $\Omega_n \subset \mathbb{R}^2$ to be the (topological) annulus whose outer boundary is the same as Ω_0, but whose inner boundary oscillates uniformly with period $2\pi/n$, where the amplitude of the oscillations is chosen so that the length of the inner boundary component coincides with the length of the outer boundary. As $n \to \infty$, the amplitude in this construction is of order $O\left(n^{-1}\right)$ and the domains Ω_n converge in the Hausdorff metric to Ω_0 while $\overline{\sigma}_1(\Omega_n) \to \overline{\sigma}_1(\mathbb{A}, g_{cc})$ as $n \to \infty$.

1.2 Flexibility of the Steklov spectrum

Theorem 1.1 can be proved by using as a main tool the material already contained in [4]. Despite that, we take this opportunity to give an alternative proof using the framework of measure eigenvalues developed in [10]. This allows and to extend the results of [4] to higher dimension and in a geometric context.

We first make the observation that every compact connected smooth manifold with boundary can be realised as a bounded smooth domain in a complete Riemannian manifold $(\mathcal{M}, g)$. Through this equivalence, we define manifolds with Lipschitz boundary as bounded Lipschitz domains in a complete Riemannian manifold. The weighted Steklov problem (1.1) can be defined for those manifolds as well, the normal derivative being only well-defined almost everywhere.

We prove the following flexibility result for Steklov eigenvalues, which was first observed in [4] for planar domains.

Theorem 1.5 *Let Ω be a compact connected Riemannian manifold with Lipschitz boundary and let $0 \not\equiv \beta : \partial\Omega \to [0, \infty)$. Suppose that $\beta \in \mathrm{L}^{d-1}(\partial\Omega)$ (if $d \geq 3$) or $\beta \in \mathrm{L}\log\mathrm{L}(\partial\Omega)$ (if $d = 2$). Then, there exists a family of domains $\Omega^\varepsilon \subset \Omega$ with Lipschitz boundary such that*

1. *As $\varepsilon \to 0$, $\partial\Omega^\varepsilon \to \partial\Omega$ in the Hausdorff distance.*
2. *For every $k \in \mathbb{N}$ the normalised eigenvalues $\overline{\sigma}_k(\Omega^\varepsilon, g) \to \overline{\sigma}_k(\Omega, g, \beta)$ as $\varepsilon \to 0$.*
3. *For every $\varepsilon > 0$, Ω and Ω^ε have the same topological type.*

As with [4], the proof is based on homogenisation of the boundary. However, when $d \geq 3$ the boundary may no longer carry a periodic structure which means that classical homogenisation constructions do not work in that setting. Instead, we adapt the geometric homogenisation ideas from [11], which do not require any periodic structure. Furthermore, we interpret the statement (2) of Theorem 1.5 in the formalism of variational eigenvalues, which in turn allows us to apply the general convergence results presented in [10]. In particular, this approach results in a more streamlined proof compared to [4]. Let us note that boundary homogenisation of the Steklov problem in dimension $d \geq 3$ was studied by Ferrero–Lamberti in [7], however as with most boundary homogenisation results it required domains in Euclidean space to be of product type; we make no such geometric assumptions.

Theorem 1.1 is a consequence of Theorem 1.5, Koebe uniformization theorem and conformal invariance of Steklov eigenvalues for $d = 2$. For $d \geq 3$, Steklov eigenvalues are no longer conformally invariant, but one still has the following corollary of Theorem 1.5.

Corollary 1.6 *Let (M, g) be a closed Riemannian manifold. Then for any $k \geq 0$ one has*

$$\sup_{\Omega} \overline{\sigma}_k(\Omega, g) = \sup_{\Omega,\, \beta \in \mathrm{C}_+(\partial\Omega)} \overline{\sigma}_k(\Omega, g, \beta),$$

where Ω varies over all smooth domains $\Omega \subset M$.

Informally, this corollary states that the introduction of density does not change the optimal upper bound for the normalized Steklov eigenvalues. At the same time, the problem with density is more natural from the geometric viewpoint [14].

1.3 Plan of the paper

In Sect. 2, we prove Theorem 1.1 and its Corollary 1.3 using conformal changes of variable and assuming Theorem 1.5. Then, in Sect. 3 we prove Theorem 1.5. This is done by first assuming that Ω and β are smooth, using a geometric homogenisation procedure on the boundary. Then, we relax the smoothness assumption and in turn approximate eigenvalues for singular densities, then domains with Lipschitz boundary, in the end extracting a diagonal subsequence from these procedures.

1.4 Notation

We make extensive use throughout the paper of Landau's asymptotic notation. We write

- indiscriminately, $f_1 = O(f_2)$ or $f_1 \ll f_2$ to mean that there exists $C > 0$ such that $|f_1| \le C f_2$;
- $f_1 \asymp f_2$ to mean that $f_1 \ll f_2$ and $f_2 \ll f_1$;
- $f_1 = o(f_2)$ to mean that $f_1/f_2 \to 0$.

The limit in that last bullet point will be either as a parameter tends to 0 or ∞ and will be clear from context. The use of a subscript, for instance $f_1 \ll_\Omega f_2$ means that the constant C or the quantities involved in the definition of the limit may depend on the subscript.

We make use of a generalisation of L^p spaces, called Orlicz spaces. Given Φ an increasing, nonegative convex function on $[0,\infty)$, $\Phi(L)(\Omega)$ is the space

$$\Phi(L)(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \exists \eta > 0 \text{ s.t. } \int_\Omega \Phi(|f/\eta|)\, \mathrm{d}v_g < \infty \right\}.$$

In addition to $\Phi(x) = x^p$ (which corresponds to L^p spaces), we also will refer to the case $\Phi(x) = e^x$, denoted $\exp \mathrm{L}$, $\Phi(x) = x\log(1+x)$, denoted $\mathrm{L}\log\mathrm{L}$ which is dual to $\exp\mathrm{L}$, and $\Phi(x) = x^2\log(1+x)^{-1/2}$ denoted $\mathrm{L}^2(\log\mathrm{L})^{-1/2}$. For a reference on Orlicz space, see [5].

2 Conformal changes of the metric

In this section we prove Theorem 1.1 and its corollary assuming Theorem 1.5. We start by introducing the notion of variational eigenvalues and look at how they behave under a conformal change of variables.

2.1 Function spaces and variational eigenvalues

We study the weighted Steklov problem 1.1 through the formalism developed in [10], see also [13, 16]. For any domain with Lipschitz boundary $\Omega \subset \mathcal{M}$ and any Radon measure μ supported on $\overline{\Omega}$, we define the Sobolev spaces $\mathrm{W}^{1,p}(\Omega,\mu)$ as the closure of $\mathrm{C}^\infty(\overline{\Omega})$ under the norm

$$\|f\|^p_{\mathrm{W}^{1,p}(\Omega,\mu)} = \int_\Omega |\nabla f|^p \, \mathrm{d}v_g + \int_\Omega |f|^p \, \mathrm{d}\mu;$$

we write $\mathrm{W}^{1,p}(\Omega) := \mathrm{W}^{1,p}(\Omega, \mathrm{d}v_g)$ for the usual Sobolev space.

We say that a measure μ is admissible if the trace operator $T_\mu : \mathrm{W}^{1,2}(\Omega) \to \mathrm{L}^2(\Omega,\mu)$ is compact. We note that under such conditions $\mathrm{W}^{1,2}(\Omega,\mu)$ is isomorphic to $\mathrm{W}^{1,2}(\Omega)$, see [10, Theorems 3.4 and 3.5]. For an admissible measure μ and $f \in \mathrm{C}^\infty(\overline{\Omega})$ we define the Rayleigh quotients

$$R_{g,\mu}(f) := \frac{\int_\Omega |\nabla f|^2 \, \mathrm{d}v_g}{\int_\Omega |f|^2 \, \mathrm{d}\mu}.$$

From this Rayleigh quotient we define the variational eigenvalues

$$\lambda_k(\Omega, g, \mu) = \inf_{F_{k+1}} \sup_{f \in F_{k+1}\setminus\{0\}} R_{g,\mu}(f)$$

where the infimum is taken over all $(k+1)$-dimensional subspaces $F_{k+1} \subset C^\infty(\overline{\Omega})$ that remain $(k+1)$-dimensional in $L^2(\Omega, \mu)$. Admissibility of μ ensures that the variational eigenvalues are discrete and form a sequence (see [10, Proposition 4.1])

$$0 = \lambda_0(\Omega, g, \mu) < \lambda_1(\Omega, g, \mu) \le \lambda_2(\Omega, g, \mu) \le \dots \nearrow \infty.$$

The main example of variational eigenvalues employed in the present paper is the following. Let $0 \not\equiv \beta \in L^{d-1}(\partial\Omega; [0,\infty))$ (if $d \ge 3$) or $\beta \in L\log L(\partial\Omega; [0,\infty))$ (if $d = 2$) and $\mathcal{H}^{d-1}\lfloor_{\partial\Omega}$ be the restriction of the Hausdorff measure to $\partial\Omega$. Then $\lambda_k(\Omega, g, \beta\mathcal{H}^{d-1}\lfloor_{\partial\Omega}) = \sigma_k(\Omega, g, \beta)$ as defined in (1.1).

2.2 Conformal optimisation

We are now ready to prove the optimisation theorems for $d = 2$ under the assumption of Theorem 1.5.

Proof of Theorem 1.1 Let $(\Omega_{0,b}, g)$ be a surface with Lipschitz boundary of genus 0 with b boundary components. To prove our claim, it is sufficient to find a family of domains $\Omega^\varepsilon \subset \mathbb{R}^2$ with b boundary components so that $\overline{\sigma}_k(\Omega^\varepsilon, g_0) \to \overline{\sigma}_k(\Omega_{0,b}, g)$.

By Koebe's uniformisation theorem [15], there exists a circle domain $\Omega \subset \mathbb{R}^2$ (i.e. a domain whose boundary is disjoint union of circles) and a conformal diffeomorphism $\varphi : \Omega \to \Omega_{0,b}$ such that $g_0 = \varphi^* g$. It follows from [13, Theorem 1.6] that $\overline{\sigma}_k(\Omega_{0,b}, g) = \overline{\sigma}_k(\Omega, g_0, |d\varphi|)$ for every $k \in \mathbb{N}$. Furthermore, it follows from the proof of [2, Lemma 5.1] that there is $p > 1$ so that $|d\varphi| \in L^p(\partial\Omega)$. Therefore, by Theorem 1.5 there exists a sequence of domains $\Omega^\varepsilon \subset \Omega$ with the same topological type so that $\overline{\sigma}_k(\Omega^\varepsilon, g_0) \to \overline{\sigma}_k(\Omega, g_0, |d\varphi|)$ as $\varepsilon \to 0$, concluding the proof. □

Proof of Corollary 1.3 The inequality (1.2) and its sharpness follows immediately from Theorem 1.1 and [8, Theorem 1.3]. It remains to show that the equality can not be achieved by a smooth domain Ω. Suppose that it does, then by [8, Theorem 1.3] there exists $\omega \in C^\infty(\mathbb{A})$ such that $\omega = 0$ on $\partial\mathbb{A}$ and (Ω, g_0) is isometric to $(\mathbb{A}, e^{-2\omega} g_{cc})$, where g_{cc} is a metric on a free boundary minimal annulus in $\mathbb{B}^3$. Then the formula for Gauss curvature in a conformal metric implies that ω is solution to the following problem

$$\begin{cases} \Delta_{g_{cc}}\omega = -K_{g_{cc}} & \text{on } \mathbb{A}; \\ \omega = 0 & \text{on } \partial\mathbb{A}. \end{cases} \tag{2.1}$$

Let κ and κ_{cc} be the geodesic curvature of Ω and critical catenoid respectively. Recall that the isometry group of the critical catenoid acts transitively on its boundary. Thus, κ_{cc} is constant. Similarly, since the solution to (2.1) is unique, the function $\partial_\nu\omega$ is also constant on $\partial\mathbb{A}$. Then one has $\kappa = \kappa_{cc} - \partial_\nu\omega$ is also constant. The only curves of constant geodesic curvature κ on $\mathbb{R}^2$ are circles of radius κ^{-1}. Hence $\partial\Omega$ consists of two circles of the same radius, which is impossible. □

3 Flexibility of the spectrum

In this section we prove Theorem 1.5, first under the assumptions that $\partial\Omega$ is smooth and $\beta > 0$ is a smooth density, then under the weaker assumption that $\partial\Omega$ is Lipschitz and β is in an appropriate integrability class. We first describe the boundary homogenisation construction yielding the appropriate domains Ω^ε. Then, we briefly recall abstract tools defined in [10] to study eigenvalue continuity results, and we use them in order to obtain continuity of the Steklov eigenvalues of Ω^ε to weighted Steklov eigenvalue on Ω. Finally, we extend the results to the rough case.

3.1 Boundary homogenisation

This construction combines elements found in [11, Section 2] (for the geometric distribution of the perturbations) and in [4] (for the type of perturbation). A distinction from the construction in [4] is that the approximation is done "from the inside", allowing us to perform the construction intrinsically in the geometric setting.

In this subsection we assume that Ω has smooth boundary and $0 < \beta \in C^\infty(\partial\Omega)$. This assumption will be relaxed later in Sect. 3.3. Invariance of normalised eigenvalues under scaling of the density allows us to furthermore assume that $\beta > 1$. Let h be the induced metric on $\partial\Omega$, and assume that $\varepsilon > 0$ is small enough that h is uniformly almost Euclidean in balls of radius 3ε. In other words assume that in geodesic polar coordinates around any $z \in \partial\Omega$, h reads

$$h(\rho,\theta) = \mathrm{d}\rho^2 + \rho^2 g_{\mathbb{S}^{d-2}} + r(\rho,\theta)$$

where $g_{\mathbb{S}^{d-2}}$ is the round metric on the $d-2$-dimensional sphere, and r is a symmetric 2-tensor such that

$$\|r\|_{C^1(B_\varepsilon(z))} = O_\Omega(\varepsilon)\,.$$

For every $\varepsilon > 0$, let $\mathbf{S}^\varepsilon$ be a maximal ε-separated subset of $\partial\Omega$ and let $\mathbf{V}^\varepsilon$ be the Voronoĭ tesselation associated with $\mathbf{S}^\varepsilon$, i.e. $\mathbf{V}^\varepsilon := \left\{V_z^\varepsilon : z \in \mathbf{S}^\varepsilon\right\}$, where

$$V_z^\varepsilon := \left\{x \in \partial\Omega : \mathrm{dist}(x,z) \le \mathrm{dist}(x,y) \text{ for all } y \in \mathbf{S}^\varepsilon\right\}$$

and the distance is computed with respect to the metric h. We construct a sequence of domains $\Omega^\varepsilon \subset \Omega$ in the following way. For every $z \in \mathbf{S}^\varepsilon$ and $\theta \in \mathbb{S}^{d-2}$, let $\rho_{\theta,z}$ be the distance from z to ∂V_z^ε along the geodesic starting with direction θ. Then, define $w_z^\varepsilon : V_z^\varepsilon \to \mathbb{R}$ as

$$w_z^\varepsilon(\rho,\theta) = \varepsilon\left(1 - \frac{\rho}{\rho_{\theta,z}}\right).$$

Then, w_z^ε is piecewise smooth, vanishes on ∂V_z^ε and satisfies the estimates

$$\left\|w_z^\varepsilon\right\|_\infty = \varepsilon \qquad \text{and} \qquad \left\|\nabla w_z^\varepsilon\right\|_\infty \asymp 1.$$

For any smooth nonnegative function $\alpha : \partial\Omega \to \mathbb{R}$, we have that

$$\nabla(\alpha w_z^\varepsilon) = \alpha \nabla w_z^\varepsilon + O(\varepsilon)\,.$$

In a neighbourhood of size $2\varepsilon\|\alpha\|_\infty$ of the boundary $\partial\Omega$, write Fermi coordinates as $x = (y,t)$, where t is the distance along the unit speed geodesic normal to the boundary at y. Define

$$Q_z^\varepsilon := \left\{(y,t) : y \in V_z^\varepsilon \text{ and } t < \alpha(y) w_z^\varepsilon(y)\right\}$$

and

$$Z_z^\varepsilon := \left\{(y,t) : y \in V_z^\varepsilon \text{ and } t = \alpha(y) w_z^\varepsilon(y)\right\}.$$

Finally, we define Ω^ε as

$$\Omega^\varepsilon := \Omega \setminus \bigcup_{z\in\mathbf{S}^\varepsilon} Q_z^\varepsilon,$$

which has boundary

$$\partial\Omega^\varepsilon = \bigcup_{z\in\mathbf{S}^\varepsilon} Z_z^\varepsilon.$$

We note that the family Ω^ε has equi-Lipschitz boundary, with the constant depending only on g, $\partial\Omega$, and α. Furthermore,

$$\mathrm{Vol}_g(\Omega \setminus \Omega^\varepsilon) \ll \varepsilon.$$

Finally, for almost every $y \in V_z^\varepsilon$, if $x = (y, t) \in Z_z^\varepsilon$ then the area element of $\partial\Omega^\varepsilon$ at x is given by

$$\mathrm{d}A_{\partial\Omega^\varepsilon}\big|_x = \left(\sqrt{1 + \alpha^2 \left|\nabla w_z^\varepsilon\right|^2} + O(\varepsilon)\right) \mathrm{d}A_{\partial\Omega}\big|_y. \tag{3.1}$$

We choose

$$\alpha = \left(\frac{\beta^2 - 1}{\left|\nabla w_z^\varepsilon\right|}\right)^{1/2}. \tag{3.2}$$

Since $\beta > 1$ and w_z^ε is piecewise smooth, this implies that the measures in (3.1) are mutually absolutely continuous with a piecewise smooth weight.

3.2 Continuity of eigenvalues—the smooth setting

We start by introducing conditions under which which variational eigenvalues are continuous with respect to the measures used to define them. For $n \in \mathbb{N}$, let $\Omega_n \subset \Omega$. Let μ_n, μ be Radon measures supported respectively on Ω_n, Ω, we introduce the following three conditions:

(**M1**) $\mu_n \overset{*}{\rightharpoonup} \mu$ as measures on Ω and $\mathrm{Vol}_g(\Omega \setminus \Omega_n) \to 0$;
(**M2**) the measures μ, μ_n are admissible for all n;
(**M3**) there is an equibounded family of extension maps $J_n : \mathcal{W}^{1,2}(\Omega_n, \mu_n) \to \mathcal{W}^{1,2}(\Omega, \mu_n)$.

The following proposition appears as [10, Proposition 4.11].

Proposition 3.1 *Suppose that $\Omega_n \subset \Omega$ is a sequence of domains and μ, μ_n are Radon measures on respectively Ω, Ω_n satisfying (**M1**)–(**M3**). If $d \geq 3$, assume that $\mu_n \to \mu$ in $\mathrm{W}^{1,\frac{d}{d-1}}(\Omega)^*$. If $d = 2$, assume that $\mu_n \to \mu$ in $\mathrm{W}^{1,2,-1/2}(\Omega)^*$. Then, for all $k \in \mathbb{N}$*

$$\lim_{n\to\infty} \lambda_k(\Omega_n, \mu_n) = \lambda_k(\Omega, \mu).$$

Remark 3.2 The space $\mathrm{W}^{1,2,-1/2}(\Omega)$ is the space of all functions in $\mathrm{L}^2(\log \mathrm{L})^{-1/2}$ such that their distributional gradient also belongs in that space. It is a space which is contained $\mathrm{W}^{1,p}(\Omega)$ for all $1 \leq p < 2$ so that the convergence in the previous theorem can be verified in the dual of any of those spaces.

Proof of Theorem 1.5 under smoothness assumptions Our goal is to apply Proposition 3.1 with

$$\mu_\varepsilon = \mathcal{H}^{d-1}\Big|_{\partial\Omega^\varepsilon} \text{ and } \mu = \beta\mathcal{H}^{d-1}\Big|_{\partial\Omega}.$$

It is a simple observation to see that $\mathrm{Vol}(\Omega^\varepsilon) \to \mathrm{Vol}(\Omega)$, and (3.1) and (3.2) tell us that $\mu^\varepsilon(\Omega) \overset{*}{\rightharpoonup} \mu(\Omega)$, so that Condition (**M1**) is verified.

Condition (**M2**) follows from the trace inequality and the fact that $\beta \geq 1$. Condition (**M3**) follows from the fact that for all ε, Ω^ε are Lipschitz domains whose Lipschitz constant is controlled by $C_\Omega \sup_{x\in\Omega} \beta(x)$, and C_Ω depends on Ω through the metric.

It only remains to show that $\mu_\varepsilon \to \mu$ in $\mathrm{W}^{1,p}(\Omega)^*$ for all $p > 1$. Let N^ε be a $2\varepsilon \|\alpha\|_\infty$-tubular neighbourhood of $\partial\Omega$, so that $\partial\Omega^\varepsilon \subset N^\varepsilon$. It is sufficient to show that for every $f \in \mathrm{W}^{1,1}(\Omega)$,

$$\langle \mu_\varepsilon - \mu, f\rangle_{\mathrm{W}^{1,1}(\Omega)} \leq c \|f\|_{\mathrm{W}^{1,1}(N_\varepsilon)} \tag{3.3}$$

for some $c > 0$. Indeed, it follows from (3.3) that for $f \in \mathrm{W}^{1,p}(\Omega)$, $p > 1$,

$$\langle \mu_\varepsilon - \mu, f\rangle_{\mathrm{W}^{1,p}(\Omega)} \leq c \|f\|_{\mathrm{W}^{1,1}(N_\varepsilon)} \leq c\varepsilon^{\frac{p-1}{p}} \|f\|_{\mathrm{W}^{1,p}(\Omega)}.$$

By density, it is sufficient to prove (3.3) assuming that f is of class C^1. Write

$$\begin{aligned}\langle \mu_\varepsilon - \mu, f\rangle_{\mathrm{W}^{1,1}(\Omega)} &= \int_{\partial\Omega^\varepsilon} f\,\mathrm{d}A_{\partial\Omega^\varepsilon} - \int_{\partial\Omega} f\beta\,\mathrm{d}A_{\partial\Omega} \\ &= \sum_{z\in\mathbf{S}^\varepsilon}\left[\int_{Z_z^\varepsilon} f\,\mathrm{d}A_{\partial\Omega^\varepsilon} - \int_{V_z^\varepsilon} f\beta\,\mathrm{d}A_\Omega\right]. \end{aligned} \tag{3.4}$$

For any $t \in [0, 2\varepsilon\|\alpha\|_\infty)$ and $y \in \partial\Omega$ write

$$f(y,t) = f(y,0) + \int_0^t \partial_s f(y,s)\,\mathrm{d}s. \tag{3.5}$$

It follows from (3.1) and (3.2) that for all $z \in \mathbf{S}^\varepsilon$,

$$\begin{aligned}\int_{Z_z^\varepsilon} f\,\mathrm{d}A_{\partial\Omega^\varepsilon} &= \int_{V_z^\varepsilon} f(y,\alpha(y)w(y))(\beta(y) + O(\varepsilon))\,\mathrm{d}A_{\partial\Omega} \\ &= \int_{V_z^\varepsilon} (\beta + O(\varepsilon)) f\,\mathrm{d}A_{\partial\Omega} + \int_{V_z^\varepsilon}(\beta + O(\varepsilon))\int_0^{t(y)} \partial_s f(y,s)\,\mathrm{d}s\,\mathrm{d}A_{\partial\Omega},\end{aligned}$$

where $t(y) = \alpha(y)w(y)$. This means that we can rewrite (3.4) as

$$\langle \mu_\varepsilon - \mu, f\rangle_{\mathrm{W}^{1,1}(\Omega)} = \sum_{z\in\mathbf{S}^\varepsilon} O(\varepsilon)\int_{V_z^\varepsilon} f\beta\,\mathrm{d}A_{\partial\Omega} + O(1)\int_{V_z^\varepsilon}\int_0^{t(y)} \partial_s f(y,s)\,\mathrm{d}s\,\mathrm{d}A_{\partial\Omega}.$$

We claim that the operator $T^\varepsilon : \mathrm{W}^{1,1}(N^\varepsilon) \to \mathrm{L}^1(\partial\Omega)$ has norm $\|T^\varepsilon\| \ll \varepsilon^{-1}$. Indeed, integrating (3.5) over t yields

$$2\varepsilon\|\alpha\|_\infty f(y,0) = \int_0^{2\varepsilon\|\alpha\|_\infty} f(t)\,\mathrm{d}t - \int_0^{2\varepsilon\|\alpha\|_\infty}\int_0^t \partial_s f(y,s)\,\mathrm{d}s\,\mathrm{d}t.$$

Changing the order of integration and integrating over y completes the proof of the claim.

Thus, one has

$$\left|\sum_{z\in\mathbf{S}^\varepsilon} O(\varepsilon)\int_{V_z^\varepsilon} f\beta\,\mathrm{d}A_{\partial\Omega}\right| \ll \|\beta\|_\infty \|f\|_{\mathrm{W}^{1,1}(N^\varepsilon)}.$$

By monotonicity, we have that

$$\sum_{z\in\mathbf{S}^\varepsilon} O(1)\left|\int_{V_z^\varepsilon}\int_0^t \partial_s f(y,s)\,\mathrm{d}s\,\mathrm{d}A_{\partial\Omega}\right| \ll \int_{N^\varepsilon} |\nabla f|\,\mathrm{d}v_g \leq \|f\|_{\mathrm{W}^{1,1}(N^\varepsilon)}.$$

This completes the proof that (3.3) holds, which was enough for our purposes, and the proof of Theorem 1.5 under smoothness assumptions is complete. □

3.3 Continuity of eigenvalues—the singular setting

3.3.1 Singular densities

We first give a condition on β so that $\beta\mathcal{H}^{d-1}\lfloor_{\partial\Omega}$ is an admissible measure.

Lemma 3.3 *Suppose that $d \geq 3$ (respectively $d = 2$) and that $0 \not\equiv \beta \in \mathrm{L}^{d-1}(\partial\Omega; [0,\infty))$ (respectively in $\mathrm{L}\log\mathrm{L}(\partial\Omega; [0,\infty))$) is a nonnegative function. Then, the trace $T_\beta : \mathrm{W}^{1,2}(\Omega) \to \mathrm{L}^2(\partial\Omega, \beta\mathcal{H}^{d-1}\lfloor_{\partial\Omega})$ is compact; in other words $\mu_\beta = \beta\mathcal{H}^{d-1}\lfloor_{\partial\Omega}$ is an admissible measure.*

Proof The case $d = 2$ is proven in [13, Proposition 2.2], by factoring T_β through the bounded trace $\mathrm{W}^{1,2}(\Omega) \to \exp\mathrm{L}^2(\partial\Omega)$ and appropriate multiplication operators, so that T_β is seen to be a norm limit of compact operators. The case $d \geq 3$ is dealt with in the same way, using instead the bounded trace $\mathrm{W}^{1,2}(\Omega) \to \mathrm{L}^{\frac{2(d-1)}{d-2}}(\partial\Omega)$ given by Gagliardo's trace theorem [9]. □

Proposition 3.4 *Let $d \geq 3$ (respectively $d = 2$) and let β_n be a sequence of non-negative densities converging in $\mathrm{L}^{d-1}(\partial\Omega)$ (respectively $\mathrm{L}\log\mathrm{L}(\partial\Omega)$) to a non-negative density β. Then, as $n \to \infty$ we have $\lambda_k(M, g, \beta_n\,\mathrm{d}A_g) \to \lambda_k(M, g, \beta\,\mathrm{d}A_g)$.*

Proof Conditions (**M1**)–(**M3**) are respected, the only non-trivial one being (**M2**) which follows from Lemma 3.3. Let $u \in \mathrm{W}^{1,\frac{d}{d-1}}(\Omega)$, for $d \geq 3$. Then, the embedding $\mathrm{W}^{1,\frac{d}{d-1}}(\Omega) \to \mathrm{L}^{\frac{d-1}{d-2}}(\partial\Omega)$ given by Gagliardo's trace theorem [9] and Hölder's inequality with exponents $d-1$ and $\frac{d-1}{d-2}$ yield

$$\left|\int_{\partial\Omega} u(\beta_n - \beta)\,\mathrm{d}A_g\right| \ll_{d,\Omega} \|u\|_{\mathrm{W}^{1,\frac{d}{d-1}}(\Omega)} \|\beta_n - \beta\|_{\mathrm{L}^{d-1}(\partial\Omega)}\,.$$

This precisely means that $\beta_n\,\mathrm{d}A_g \to \beta\,\mathrm{d}A_g$ in $\mathrm{W}^{1,\frac{d}{d-1}}(\Omega)^*$, so that the eigenvalues converge. For $d = 2$, the same proof holds replacing Gagliardo's trace theorem with the trace operator $\mathrm{W}^{1,2,-1/2}(\Omega) \to \exp\mathrm{L}(\partial\Omega)$, see [6, Theorem 5.3], and Hölder's inequality on $\exp\mathrm{L}(\partial\Omega)$ and $\mathrm{L}\log\mathrm{L}(\partial\Omega)$. □

3.3.2 Lipschitz boundary

In order to study convergence of eigenvalues of domains with Lipschitz boundary, we need [3, Theorem 4.1]. Note that this result is proven in the Euclidean setting, but its proof extends to the Riemannian setting directly, see [11, Lemma 3.1] for an adaptation to the Riemannian setting of the only part of the proof which is not completely local.

Proposition 3.5 *Let Ω be a manifold with Lipschitz boundary and for all $n \in \mathbb{N}$, let $\Omega_n \subset \Omega$ be Lipschitz domains such that $\mathbf{1}_{\Omega_n} \to \mathbf{1}_\Omega$, strongly in $\mathrm{L}^1(\Omega)$, $\mathcal{H}^{d-1}(\partial\Omega_n) \to \mathcal{H}^{d-1}(\partial\Omega)$ and*

$$\sup_n \|T_n\|_{\mathrm{BV}(\Omega_n)\to\mathrm{L}^1(\partial\Omega_n)} < \infty$$

where T_n is the trace operator. Then, for every $k \in \mathbb{N}$, $\sigma_k(\Omega_n, g) \to \sigma_k(\Omega, g)$.

Proof of Theorem 1.5 for manifolds with Lipschitz boundary We first prove that we can exhaust any compact manifold with Lipschitz boundary with a sequence of domains with smooth boundary in such a way that the Steklov eigenvalues are stable.

Let Ω be a compact manifold with Lipschitz boundary. Following [19, Theorem A.1], (see [18, Appendix A] for a discussion of the adaptation to the Riemannian case), there exists a sequence of smooth domains $\Omega_n \subset \Omega$ converging to Ω such that the boundaries of Ω, Ω_n may be respectively parametrised by a finite number of equi-Lipschitz maps γ_j, $\gamma_{j,n}$ such that $\gamma_{j,n} \to \gamma_j$ uniformly. This implies in particular that if $T^n : \mathrm{BV}(\Omega_n) \to \mathrm{L}^1(\partial\Omega_n)$ is the trace operator, their norms remains uniformly bounded since it can be estimated in terms of the Lipschitz constants of $\partial\Omega_n$ and the volume of Ω [1]. In particular, it follows from Proposition 3.5 that for all $k \in \mathbb{N}$, $\sigma_k(\Omega_n, g) \to \sigma_k(\Omega, g)$.

It also follows from [19, Theorem A.1] that there are bi-Lipschitz homeomorphisms Φ_n from $\partial\Omega_n$ to $\partial\Omega$, whose bi-Lipschitz character is preserved uniformly in n. In particular, Φ_n induces an isomorphism $\Phi_n^* : \mathrm{L}^p(\partial\Omega) \to \mathrm{L}^p(\partial\Omega)$ for every $p \in [1, \infty]$, whose norms are uniformly bounded in n. Therefore, extracting a diagonal subsequence from

- first finding a sequence of domains Ω_n with smooth boundary converging to Ω;
- then approximating the weight $\beta \circ \Phi_n \in \mathrm{L}^{d-1}(\partial\Omega_n)$ by smooth weights $\beta_{m,n}$;
- finally finding a sequence of domains with Lipschitz boundary $\Omega^\varepsilon_{m,n}$ so that $\mathcal{H}^{d-1}\lfloor_{\partial\Omega^\varepsilon_{m,n}}$ converges to $\beta_{m,n}\mathcal{H}^{d-1}\lfloor_{\partial\Omega_n}$;

provides us with the required sequence of domains proving our claim. □

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