Tranversality

- definitions / examples
- parametric transversality
- generic simplicity
- 2 nodal domains

① $V :=$ vector space

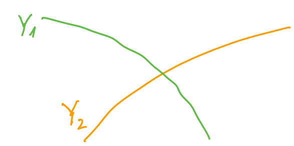
$W_1, W_2 \subset V$ "transverse" $\iff$ $W_1 + W_2 = V$

② $f : V' \longrightarrow V$ linear $\quad W \subset V$

f "transverse" to W $\iff$ $f(V'), W$ transverse $\iff$ $f(V') + W = V$

①' $X =$ diff manifold

$Y_1, Y_2 \subset X$ transverse $\iff$ $T_x Y_1 + T_x Y_2 = T_x X \quad \forall x \in Y_1, Y_2$

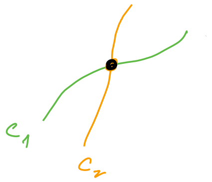


②' $f : X' \longrightarrow X$ C^1 map, $Z \subset X$

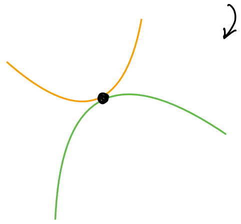
f is "transverse" to Z $\iff$ $df_x (T_x X') + T_{f(x)} Z = T_x X$

E.g.

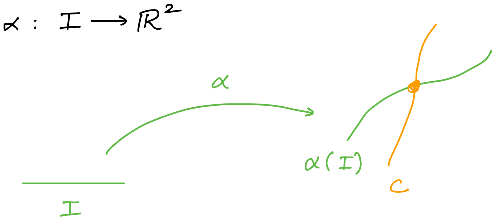
$c_1, c_2 \subset \mathbb{R}^2$ curves

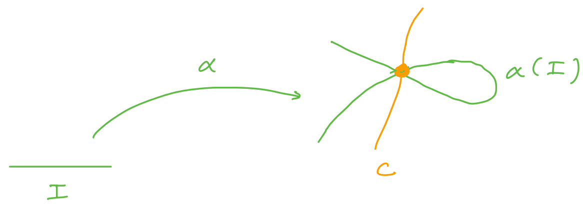


not transverse



$\alpha : I \longrightarrow \mathbb{R}^2$





<u>Remark</u>: Transversality is "stable" under perturbation

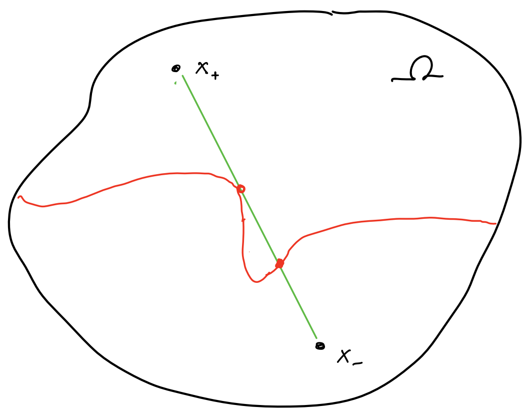
<u>Problem</u>: $\Omega \subset \mathbb{R}^2$ open, simply connected

$c :=$ C^1 curve s.t. $\Omega \setminus c = \Omega_+ \amalg \Omega_-$ open

$x_+ \in \Omega_+$ and $x_- \in \Omega_-$

Does $\exists$ C^1 path $\alpha : [-1,1] \longrightarrow \Omega$ s.t.

- $\alpha(\pm 1) = x_\pm$
- α transverse to c



Parametric transversality theorem

(Thom, Abraham)
1960's

$$\begin{tikzcd} X \times B \arrow[r, "F"] \arrow[d, "\pi"'] & Y \\ B & Z \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

F transverse to Z $\Longrightarrow$ $\exists$ residual subset $R \subset B$ s.t.

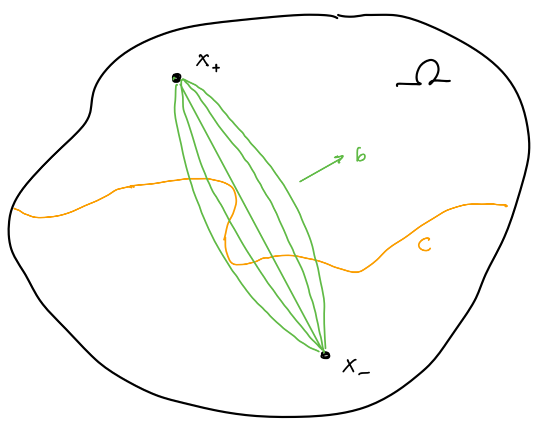
$b \in R \Rightarrow F\big|_{\pi^{-1}(b)}$ transverse to Z

$B =$ "space of parameters"

e.g. $X = [-1,1]$, $Y = \Omega$, $Z = c$ curve

$F : [-1,1] \times [-\varepsilon, \varepsilon] \longrightarrow \Omega$

$F(t,b) = \alpha_b(t)$



$dF_{(t,b)}$ surjective if $t \neq \pm 1$

$\Rightarrow$ F transverse to c

P.T. $\Rightarrow$ $\exists$ residual $R \subset B$

$b \in R \Rightarrow$ b transverse to c

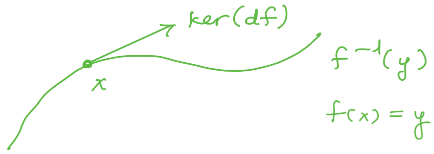
Regular values

$f : X \to Y$

$y = f(x)$ "regular value" $\iff$ df_x surjective $\forall x \in f^{-1}(y)$ $\iff$ f transverse to $\{y\}$

Thm: y regular $\implies$ $f^{-1}(y)$ submanifold
tangent space $= \ker(df_x)$

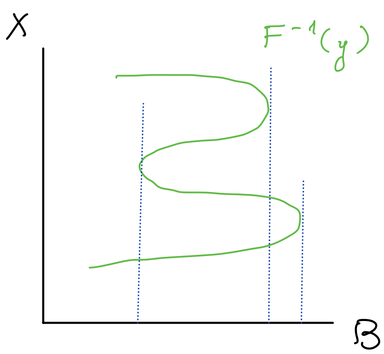
Pf: implicit fun thm



Coro of parametric transversality thm

$$\begin{tikzcd} X \times B \arrow[r, "F"] \arrow[d, "\pi"'] & Y \\ B & \{y\} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

y regular value of F $\implies$ $\left\{ \begin{array}{l} \exists \text{ residual } R \subset B \text{ s.t. if } b \in R \\ y \quad \text{regular value of } F\big|_{\pi^{-1}(b)} \end{array} \right.$



V = vector space

$A : V \longrightarrow V$ linear and symmetric

$f : V \setminus \{0\} \times \mathbb{R} \longrightarrow V$

$f(u,\lambda) := Au - \lambda u$

$f(u,\lambda) = 0 \iff u$ is eigenvector with eigenvalue λ

<u>Proposition</u> (Uhlenbeck '76)

0 is regular value of $f \Rightarrow \dim(\ker(A-\lambda)) \leq 1 \quad \forall \lambda \in \mathbb{R}$

<u>Pf:</u> Suppose $f(u,\lambda) = 0$

i.e. $u \in \ker(A - \lambda I) \setminus \{0\}$

Product rule $\rightsquigarrow df_{(u,\lambda)}(v,\mu) = (A - \lambda I)v - \mu u$

$$V \overset{hyp}{=} \operatorname{im}(df_{(u,\lambda)}) = \operatorname{im}(A - \lambda I) + \langle u \rangle$$

A symmetric $\Rightarrow \operatorname{im}(A - \lambda I) \cap \ker(A - \lambda I) = \{0\}$

$$\Rightarrow V = \operatorname{im}(A - \lambda I) \oplus \langle u \rangle$$

i.e. $\ker(A - \lambda I) = \langle u \rangle$ $\square$

<u>Theorem:</u> (Uhlenbeck)

$$\begin{tikzcd} V' \times \mathbb{R} \times B \arrow[r, "F"] \arrow[d] & V \\ B & \end{tikzcd}$$

$F(v,\lambda,b) := A_b - \lambda I \qquad A_b$ symmetric

0 regular value of $F \Rightarrow \exists$ residual set $R \subset B$ s.t.

$b \in R \Rightarrow A_b$ has simple spectrum.

In the ∞-dimensional setting

V' V' Hilbert space

A_b self-adjoint

F Fredholm

<u>Example</u> $M=$ compact mfld $\partial M \neq \emptyset$

$\mathcal{B} = \{ C^\ell$ Riemannian metrics on $M \}$

$V = H^k(M)$

$F(u,\lambda,b) = \Delta_b u - \lambda u$

<u>Theorem</u> (Uhlenbeck) 0 is a regular value of F

<u>Pf idea</u>: Suppose $F(u,\lambda,b)=0$

$$dF_{(u,\lambda,b)}(v,\mu,c) = \underbrace{(d(\Delta_b)_c)}_{\dot{\Delta}_b} u + (\Delta_b - \lambda I)v - \mu u$$

$$\operatorname{im}(dF_{(u,\lambda,b)}) \supset \underbrace{\{ \dot{\Delta}_c u \mid c \in T_b \mathcal{B} \}}_{\text{ISTS } = V \text{ (or dense in } V)}$$

<u>Application:</u> (Jung–Zelditch '20)

$M = UTX$ $\quad$ $X =$ compact surface

"unit"

$S^1 \curvearrowright UTX$ free action

$\mathcal{B} = \{ S^1$ invariant metrics on $UTX \}$ "Kaluza–Klein"

$\dim(\ker(\Delta_b - \lambda I)) \leq 2$ for generic b

$$\boxed{\text{For generic } b: \quad \left.\begin{array}{l} u \in \ker(\Delta_b - \lambda I) \\ u \text{ not invt under } S^1 \end{array}\right\} \Rightarrow M \setminus u^{-1}(0) \text{ has exactly 2 components}}$$

$u(x,\theta) = u_1(x)\cos(\theta) + u_2(x)\sin(\theta) \qquad (x,\theta) \in UTM$