Variations of geometry and spectrum

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Continuity

Analyticity

Transversality

Reference: Tosio Kato

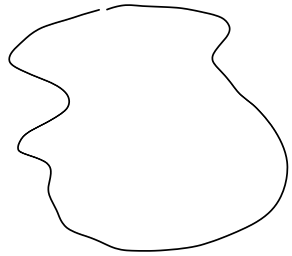
"Perturbation thy for Linear Ops"

Franz Rellich

"Störungstheorie der Spektralzerung"

"Perturburbation thy of spectral decomp"

Ω bounded open



Ω

$\partial\Omega$ = Lipschitz

$d\nu$ = Lebesgue measure

$X \cdot \nabla u = Xu \qquad \forall$ v.f. X

$L^2 = L^2(\Omega, d\nu) \qquad \langle u, v\rangle = \int_\Omega u \cdot v \, d\nu$

$\|u\|^2 = \langle u, u\rangle \qquad \|\nabla u\|^2 = \int_\Omega |\nabla u|^2 d\nu$

$F_\Omega(u) := \dfrac{\|\nabla u\|^2}{\|u\|^2} \qquad F : H^1 \setminus \{0\} \longrightarrow \mathbb{R}$

$F(c \cdot u) = F(u)$ homogeneous!

<u>CLAIM</u>: critical values/pts of F_Ω $\longleftrightarrow$ eigenvalues/functions of Δ_Ω Neumann conditions

u critical pt $\iff \dfrac{d}{dt}\Big|_{t=0} F(u + th) = 0 \qquad \forall h \in H^1$

$\iff \langle \nabla u, \nabla h\rangle = F(u)\langle u, h\rangle \qquad \forall h \in H^1$

$\iff \Delta u = F(u) \cdot u \qquad \Delta : H^1 \longrightarrow H^{-1}$

& $\dfrac{\partial}{\partial n} u \equiv 0$ "natural ∂ condition"

<u>Question</u>: Do eigenspaces vary continuously with Ω?

Is $\Omega \longmapsto \underbrace{\mathrm{crit}(F_\Omega)}$ continuous ?

Hausdorff topology:

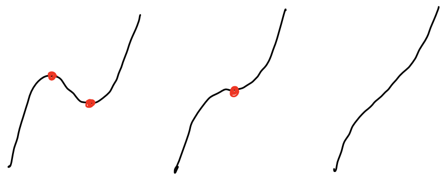
$A_n \longrightarrow A$ means

(1) $a_n \longrightarrow b \Rightarrow b \in A$

(2) $a \in A \Rightarrow \exists\, a_n \in A_n$ w/ $a_n \longrightarrow a$ "stability"

In general, does $F_n \longrightarrow F$ imply $\operatorname{crit}(F_n) \longrightarrow \operatorname{crit}(F)$?

No! $f_t(x) = x^3 + tx$



$t=-1$ $t=0$ $t=1$

Set $F_n = f_{1/n} \longrightarrow f_0$

$\operatorname{crit}(f_{1/n}) = \emptyset \underset{\text{Hausdorff}}{\not\longrightarrow} \{0\}$

stability is violated

On the other hand, F $\underbrace{\text{Morse}}_{\substack{u \text{ critical} \\ \Rightarrow \det(\operatorname{Hess}(u)) \neq 0}}$ $\Longrightarrow$ $\operatorname{crit}(F_n) \underset{\text{Hausdorff}}{\longrightarrow} \operatorname{crit}(F)$

$$F_\Omega\big|_{\{|u|=1\}} \text{ Morse} \iff \underbrace{\dim\big(\ker(\Delta - \mu I)\big) \equiv 1}_{\text{simplicity}} \quad \forall \mu \in \operatorname{spec}(\Delta)$$

Remark:

$$\left.\begin{array}{l}\Omega \text{ bdd} \\ \partial\Omega \text{ Lipschitz}\end{array}\right\} \Longrightarrow H^{s+\varepsilon} \underset{\text{compact}}{\hookrightarrow} H^s \Longrightarrow (\Delta + zI)^{-1} \text{ compact when it exists}$$

$$\text{Riesz-Schauder} \Longrightarrow \begin{cases} \operatorname{spec}(\Delta) \text{ discrete } \nearrow \infty \\ \dim\big(\ker(\Delta - \mu I)\big) < \infty \\ L^2 = \bigoplus_{\mu \in \operatorname{spec}(\Delta)} \ker(\Delta - \mu I) \end{cases}$$

Moral: Bounded energy $\Rightarrow$ finite dimensional space

$\Rightarrow$ compact spheres

$V =$ (finite dim) vector space w/ inner product $\langle \cdot , \cdot \rangle$

$B : V \times V \longrightarrow \mathbb{R}$ symmetric bilinear form

$A : V \longrightarrow V \quad A^* = A \quad$ s.t. $\quad B(v,w) = \langle Av, w \rangle$

$F(u) := \dfrac{B(u,u)}{\langle u,u \rangle} \qquad F : V \setminus \{0\} \longrightarrow \mathbb{R}$

$F(cu) = F(u)$

$u =$ crit pt of $F \iff Au = \mu \cdot u \qquad \mu = F(u)$ eigenvalue

<u>Minimax principle</u>

Fix $M > 0$

$E = E^M = \langle u$ critical $: |F(u)| < M \rangle = \bigoplus_{|\mu| < M} \ker(A - \mu I)$

$W \subset V$ subspace

$W \longmapsto G(W) := \sup |F(W)|$

$$\boxed{E = \text{unique minimizer of } G \text{ over } W \text{ w/ } \dim(W) = \dim(E)}$$

$Gr_k(V) := \{ W \subset V : \dim(W) = k \}$ Grassmannian

topology : $W_n \longrightarrow W \iff$ o.n.b. bases converge

$\dim(V) < \infty \implies Gr_k(V)$ is compact

$V = H^1(\Omega) \Rightarrow \dim(V) = \infty$

But eigenspaces lie in $H^k(\Omega)$ $\forall k$ by "elliptic regularity"

$\Rightarrow E^M \subset V^M := \left\{ u \in H^2(\Omega) : \dfrac{\|u\|_{H^2}}{\|u\|_{L^2}} \leq M + 1 \right\}$

$\Rightarrow$ May restrict G to $Gr_k(V^M) \subset\subset Gr_k(H^1(\Omega))$

Lemma: X compact

$f_n \longrightarrow f$ in $C(X)$

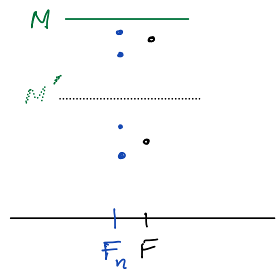
$x_n \longrightarrow x$

$x_n \underbrace{\text{minimizer}}_{\text{"mzr"}}$ of $f_n \implies x$ mzr of f

Proposition:

Suppose $M \neq$ crit value of F

$F_n \longrightarrow F \implies E_n^M \longrightarrow E^M$



Pf: $F_n \longrightarrow F \implies G_n \longrightarrow G$

$\dim(E_n)$ constant k up to subseq. $\quad \left(\begin{array}{c} \text{spec discrete} \\ \Downarrow \\ \dim(E_n) \text{ bdd} \end{array} \right)$

$E_n \longrightarrow E^*$ up to further subseq.

Lemma $\implies E^* =$ mzr of G on dim k subspaces

ISTS $\dim(E) = k$

If $\dim(E^*) > \dim(E) \implies G(E^*) \geq G(E)$

if $=$ then E not unique mzr $\rightarrow\leftarrow$

if $>$ then $G_n(E) < G_n(E_n)$ large n $\rightarrow\leftarrow$

If $\dim(E^*) < \dim(E) \implies \exists\, v \in E \setminus E^* \implies \exists\, v \in E \setminus E_n$ large n

$F(v) < M - \varepsilon$

$\implies F_n(v) < M - \varepsilon/2$ large n. $\rightarrow\leftarrow$

Coro: $N(M) = \#\{\mu \in \text{spec} : |\mu| < M\}$ is cts

$R_\theta = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}$ unitary

Example: (Rellich 1937)

$$O(n) \curvearrowright \mathrm{Sym}(n) = \{n \times n \text{ symmetric matrices}\}$$

$$A \longmapsto U^{-1} \cdot A \cdot U \qquad \text{conjugation}$$

$$\{cI : c \in \mathbb{R}\} = \text{fixed pt. set}$$

Action cts $\Rightarrow$ if $A \sim I$, then $U^{-1} A U \sim I$

$$A_t = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$= t \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

Choose $\theta = 1/t \quad \Rightarrow \quad t \mapsto A_t$ continuous at $t = 0$

$$\left.\begin{aligned} \ker(A - tI) &= \left\langle \begin{pmatrix} \cos 1/t \\ -\sin(1/t) \end{pmatrix} \right\rangle \\ \ker(A + tI) &= \left\langle \begin{pmatrix} \sin(1/t) \\ \cos(1/t) \end{pmatrix} \right\rangle \end{aligned}\right\} \text{ do not converge as } t \to 0$$

Moral: Sum of eigenspaces converge but individual eigenspaces may not.

$t \mapsto A_t$ can be made smooth with $\theta = e^{-t^2}$

So differentiability does not help.

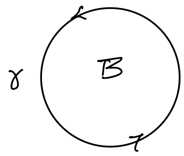
Riesz projector:

$$A : V \longrightarrow V$$

Fact: $z \longmapsto (A - zI)^{-1}$ holomorphic

$B = \text{disk} \subset \mathbb{C}$

$\gamma = \partial B$



Assume: $z \in \gamma \implies A - zI$ invertible

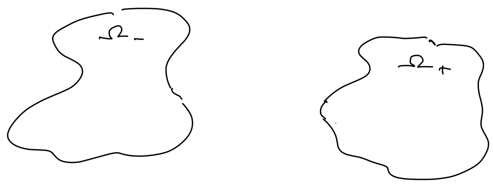
Define: $$P_\gamma^A(u) := \frac{-1}{2\pi i} \int_\gamma (A - zI)^{-1} dz$$

Fact: P_γ^A projection onto $\bigoplus_{\mu \in B} \ker(A - \mu I)$

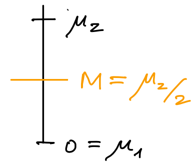
$$(A_n - zI)^{-1} \longrightarrow (A - zI)^{-1} \implies im(P_\gamma^{A_n}) \longrightarrow im(P_\gamma^A)$$

See e.g. Rauch-Taylor "... wildly perturbed domains"

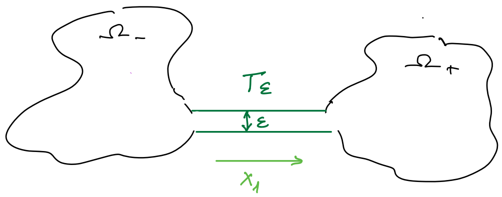
<u>Dumbbells:</u>



$$\Omega_0 = \Omega_- \amalg \Omega_+$$



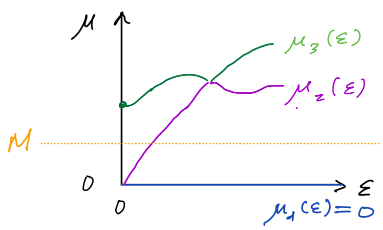
$$E^M = \langle \mathbb{1}_{\Omega_-} \rangle \oplus \langle \mathbb{1}_{\Omega_+} \rangle$$



$$\Omega_\varepsilon = \Omega_- \cup T_\varepsilon \cup \Omega_+$$

$$E^M_\varepsilon = \langle \mathbb{1}_{\Omega_\varepsilon} \rangle \oplus \langle u_1(\varepsilon) \rangle$$

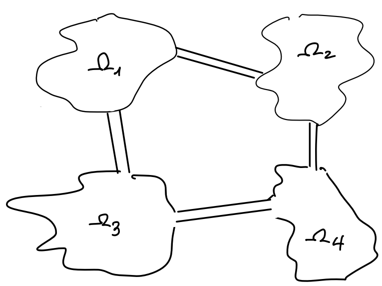
$$u_1(\varepsilon) \sim \begin{cases} +1 & \Omega_+ \\ x_1 & \text{in } T_\varepsilon \\ -1 & \Omega_- \end{cases}$$



<u>Application:</u> (Colin de Verdiere 1986)

> Given $0 = a_1 < a_2 < \cdots < a_K$
>
> $\exists\, \Omega \subset \mathbb{R}^n$ and Λ s.t. $\operatorname{spec}(\Delta) \cap [0, \Lambda] = \{a_1, \ldots, a_K\}$

<u>One idea of the proof:</u>



M small

$$q\big|_{E^M} = \varepsilon^{n-1} q_\Gamma + O(\varepsilon^n)$$

$q_\Gamma \longleftrightarrow$ weighted graph Laplacian.

choose (Γ, wt) s.t. $\operatorname{spec}(q_\Gamma) = \{a_i\}$

<u>Remark</u>: 'Normalization' of vector space.

$$\Omega_n \to \Omega \quad \Longrightarrow \quad \mathrm{spec}(\Delta_n) \longrightarrow \mathrm{spec}(\Delta)$$

$$\mu_n \to \mu \quad \Longrightarrow \quad \ker(\Delta - \mu_n I) \longrightarrow \ker(\Delta - \mu I)$$

Suppose $\exists\ \varphi_n \in C^k(\mathbb{R}^d; \mathbb{R}^d)$ $\quad \varphi_n \to Id \quad \varphi_n(\Omega_n) = \Omega$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\varphi_n^*"] \arrow[dr, dashed, "\tilde{F}_n"'] & H^1(\Omega_n) \arrow[d, "F_n"] \\ & \mathbb{R} \end{tikzcd}$$