Analyticity

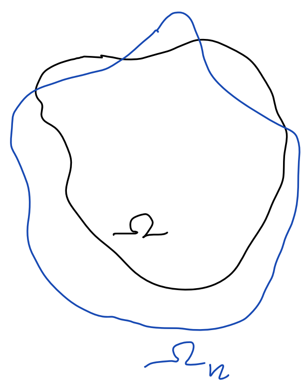
- analyticity for symmetric A_t
- applications of analyticity
  - generic simplicity
  - generic non-existence of Maass forms
- Rellich's pf of analyticity
- two parameter families A_{t_1, t_2}

'Normalization' of vector space

e.g. Suppose $\exists\ \varphi_n \in C^k(\mathbb{R}^d, \mathbb{R}^d)$

$$\varphi_n \xrightarrow{C^k} I$$

$$\varphi_n(\Omega) = \Omega_n$$

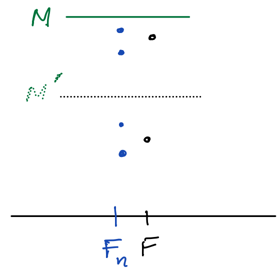


$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\varphi_n^*"] \arrow[dr, dashed, "\tilde{F}_n"'] & H^1(\Omega_n) \arrow[d, "F_n"] \\
 & \mathbb{R}
\end{tikzcd}$$

Recall from last time:

F = Rayleigh quotient

$$E^M = \bigoplus_{|\mu|<M} \ker(A - \mu I)$$



Proposition:

Suppose $M \neq$ crit value of F

$$F_n \longrightarrow F \implies E_n^M \longrightarrow E^M$$

Hausdorff convergence

$A_n \longrightarrow A$ means

(1) $a_n \longrightarrow b \implies b \in A$

(2) $a \in A \implies \exists\, a_n \in A_n$ w/ $a_n \longrightarrow a$ "stability"

$F_t(x) = x^3 - tx$

$\mathrm{crit}(F_{1/n}) = \emptyset$

$\mathrm{crit}(F_0) = \{0\}$

not stable

(Rellich 1937)

$$\begin{pmatrix} \cos 1/t & -\sin 1/t \\ \sin 1/t & \cos 1/t \end{pmatrix} \begin{pmatrix} e^{-1/t^2} & 0 \\ 0 & -e^{-1/t^2} \end{pmatrix} \begin{pmatrix} \cos 1/t & \sin 1/t \\ -\sin 1/t & \cos 1/t \end{pmatrix} = e^{-1/t^2} \begin{pmatrix} \cos(2/t) & \sin(2/t) \\ \sin(2/t) & -\cos(2/t) \end{pmatrix}$$

Theorem: (Rellich, Courant Lecture Notes 1953)

$$\left.\begin{array}{l} t \in (a,b) \\ A_t = n \times n \text{ symmetric} \\ t \longmapsto A_t \in C^1 \end{array}\right\} \implies \begin{array}{l} \nexists\ \lambda_1, \ldots, \lambda_k \in C^1((a,b), \mathbb{R}) \\ \text{such that for each } t \\ \mathrm{spec}(A_t) = \{\lambda_1(t), \ldots, \lambda_k(t)\} \end{array}$$

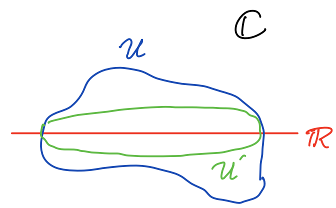
$$A_t = P_{E^M} \Delta_t P_{E^M}$$

ANALYTICITY

V = $\mathbb{C}$-vector space w/ $\langle \cdot , \cdot \rangle$

$t \in \mathbb{C} \quad A_t : V \longrightarrow V$

$t \longmapsto A_t$ holomorphic on $\mathcal{U} \subset \mathbb{C}$ open



Theorem (Rellich, 1937)

Suppose $A_t^* = A_{\bar{t}} \quad \forall\, t \in \mathcal{U}$

Then $\exists$
- $\mathcal{U}' \subset \mathcal{U} \qquad \mathcal{U}' \cap \mathbb{R} = \mathcal{U} \cap \mathbb{R}$
- $\lambda_i : \mathcal{U}' \longrightarrow \mathbb{C}$ holomorphic $\quad i = 1, \ldots n$
- $u_i : \mathcal{U}' \longrightarrow V$ holomorphic $\quad i = 1, \ldots n$

such that
- $A_t u_i(t) = \lambda_i(t) u_i(t)$
- $V = \bigoplus \langle u_i(t) \rangle \quad \forall\, t \in \mathcal{U}'$

orthogonal $\quad t \in \mathbb{R}$

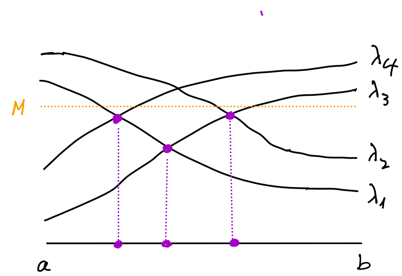
Applications based on:

f analytic $\implies f^{-1}(0)$ discrete unless $f \equiv$

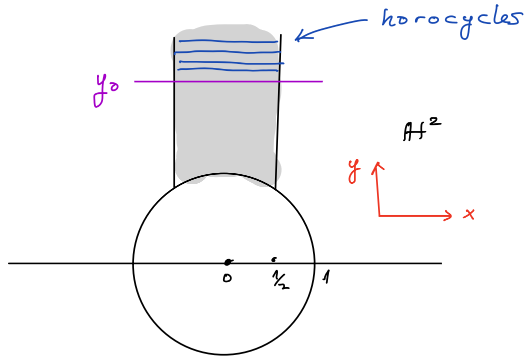
Example $\qquad t \in (a,b) \subset \mathbb{R} \quad t \longmapsto A_t$ analytic

$\exists\, t_0$ s.t. spec(A_{t_0}) simple $\implies$ spec(A_t) simple for $t \notin$ countable set

$f(t) = \lambda_i(t) - \lambda_j(t)$



Maass cusp forms:



X = hyperbolic surface w/ cusps

e.g. $\mathbb{H}^2 / SL_2(\mathbb{Z})$

$$F(u) = \frac{\int u_x^2 + u_y^2}{\int \frac{u^2}{y^2}}$$

Restrict F to u s.t. $\int u(x,y)\,dx = 0 \quad \forall\, y \geq y_0$

$\rightsquigarrow$ "cut-off Laplacian" compact resolvent!

Lax–Phillips 1977 "Scattering Thy for Automorphic Functions"

u "Maass cusp form" $\Leftrightarrow \begin{cases} u \text{ crit pt for } F \\ \& \quad \partial_y \int u(x,y)\,dx = 0 \end{cases}$

$t \mapsto X_t$ analytic $\Rightarrow$ $t \mapsto F_t$ analytic

$\exists\, t_0$ s.t. X_{t_0} has no Maass forms $\Rightarrow$ X_t has no Maass forms for $t \notin$ countable set.

Phillips–Sarnak 1985

<u>Conjecture:</u> (Phillips–Sarnak)

The generic hyperbolic surface has only finitely many Maass forms.
$\mathbb{H}/\Gamma$ w/ cusp (or no) $\lambda > 1/4$

<u>Thm (Hillairet–Judge 2018)</u>

Generic hyperbolic triangle has no Maass forms.
with cusp

Proof of Rellich's thm

V = n-dimensional $\mathbb{C}$-vector space w/ $\langle \cdot , \cdot \rangle$

$t \in \mathbb{C} \quad A_t : V \longrightarrow V$

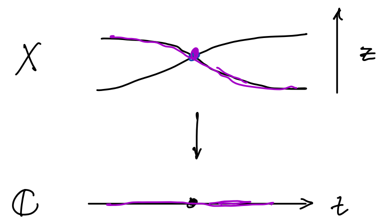
$t \longmapsto A_t$ holomorphic

eigenvalues of A_t = roots of $\det(A_t - zI)$

$$\det(A_t - zI) = W(t,z) = c_0(t) + c_1(t)z + \cdots + c_{n-1}(t)z^{n-1} + z^n$$

Weierstrass polynomial

$X = \{(t,z) \in \mathbb{C}^2 : W(t,z) = 0\}$ Riemann surface



$(t,z) \longmapsto t$

branched covering

eigenvalues are local sections

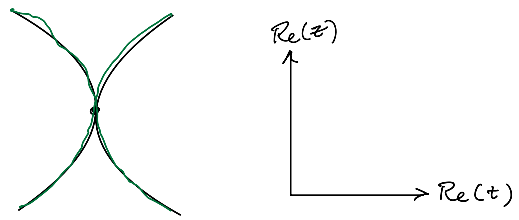
In general, monodromy nontrivial!

Example: $A_t = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix}$

$W(t,z) = z^2 - t$

double covering branched at 0

eigenvalues $\pm\sqrt{t}$ "multivalued"



Lemma: Let $W : \mathcal{U} \subset \mathbb{C}^2 \longrightarrow \mathbb{C}$ Weierstrauss polynomial

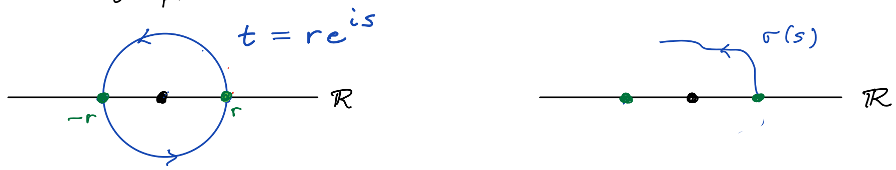
$$\left.\begin{array}{l} (*)\quad W(t,z)=0 \Leftrightarrow W(\bar{t},\bar{z})=0 \\ (**)\quad \left.\begin{array}{l} t\in\mathbb{R} \\ W(t,z)=0 \end{array}\right\} \Rightarrow z\in\mathbb{R} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{For each } t_0\in\mathbb{R}\ \exists\, t_0\in\mathcal{U}'\subset\mathcal{U} \\ \exists\, \lambda_i : \mathcal{U} \longrightarrow \mathbb{C} \text{ analytic} \\ W(t,z) = \prod_{i=1}^{n} (z-\lambda_i(t)) \end{array}\right.$$

"trivial monodromy"

Proof of Lemma

WLOG $t_0 = 0$ otherwise use $t' = t - t_0$.

t - plane z - plane



$W(re^{is}, \sigma(s)) = 0$ $\sigma : \mathbb{R} \longrightarrow \mathbb{C}$ "lift"

$s = k\pi \Rightarrow t = (-1)^k r \in \mathbb{R} \overset{(**)}{\Rightarrow} \bar{\sigma}(k\pi) \in \mathbb{R} \Rightarrow \bar{\sigma}(k\pi) = \sigma(k\pi)$

In particular $\sigma(0) = \bar{\sigma}(0) = \bar{\sigma}(-0)$.

$(*) \Rightarrow W(re^{-is}, \bar{\sigma}(s)) = 0 \ \forall s \Rightarrow W(re^{is'}, \bar{\sigma}(-s')) = 0 \quad \forall s'$

uniqueness of path $\Rightarrow$ $\sigma(s) = \bar{\sigma}(-s) \quad \forall s.$

$\sigma(\pi) = \bar{\sigma}(-\pi) = \sigma(-\pi)$

$\Rightarrow \sigma$ defines global section

$\rightsquigarrow$ holomorphic section λ in punctured nbhd of $t=0$

λ extends to nbhd by removable sing thm.

There are n such lifts λ_i (including multiplicities).

$$\left.\begin{array}{l} W(t,\lambda_i(t)) = 0 \\ W \text{ monic} \end{array}\right\} \Longrightarrow W(t,z) = \prod (z-\lambda_i(t)) \qquad \square$$

Sketch of pf of thm:

$$W(t,z) = \det(A_t - zI) \qquad\qquad Au = zu$$

$$W(\bar t, \bar z) = \det(A_{\bar t} - \bar z I) = \det(A_t^* - \bar z I) \qquad\qquad u^* A^* = \bar z u^*$$

$$t \in \mathbb{R} \Rightarrow A_t = A_t^* \implies \text{roots of } z \mapsto \det(A_t - zI) \text{ real}$$

Consider u_i associated to λ_i

Kato p. 122 "The existence of such an orthonormal basis depending smoothly on t is one of the most remarkable results of the analytic perturbation theory for symmetric operators."

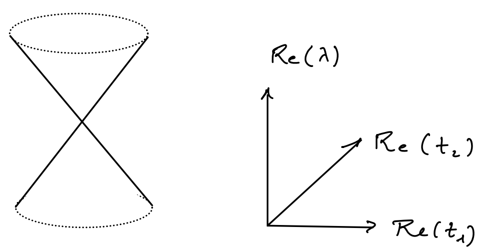
Two parameter families

Example (Rellich 1937)

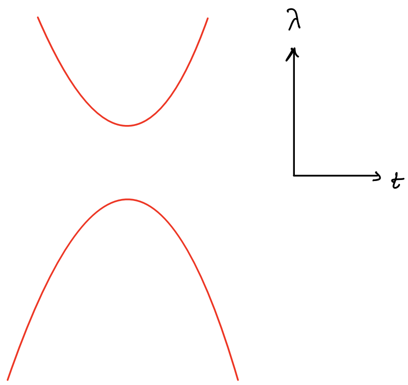
$$A_{t_1,t_2} = \begin{pmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{pmatrix} \qquad \text{symmetric, analytic}$$

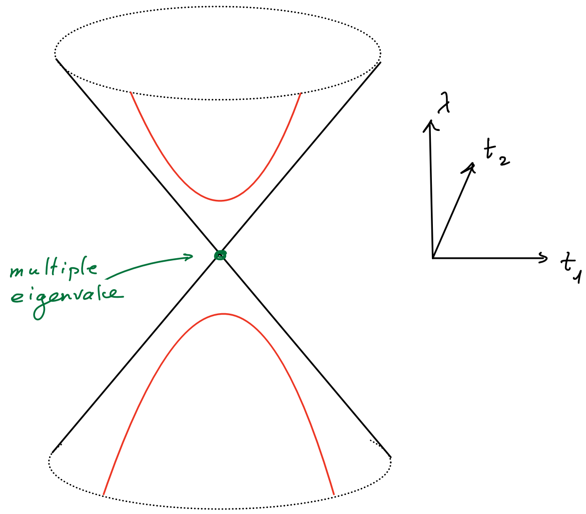
$$\lambda_\pm(t_1,t_2) = \pm(t_1^2 + t_2^2)^{1/2}$$

"Dirac cone"



Crossing avoidance:





Thm: (Wigner & Von Neumann 1929)

Matrices w/ multiple eigenvalues are codim 2 in $Sym(n)$

$\Rightarrow$ generic path A_t has no multiple eigenvalues.

Pf: $O(n)$ acts $Sym(n)$ by conjugation

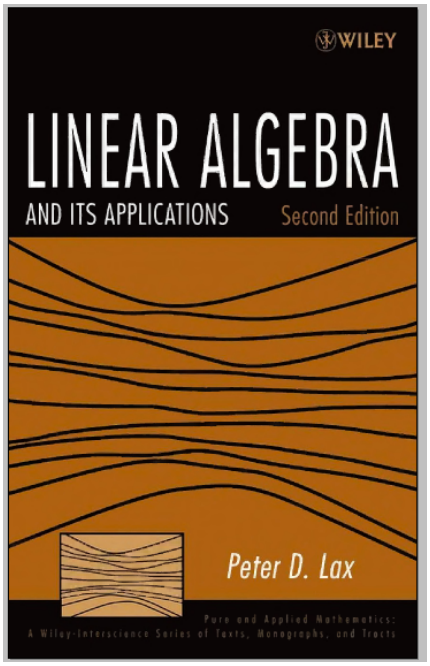
Isotropy of action?

- discrete if A has no multiplicities
- conts if A has multiplicities

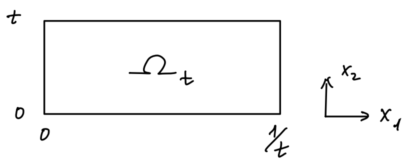
$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

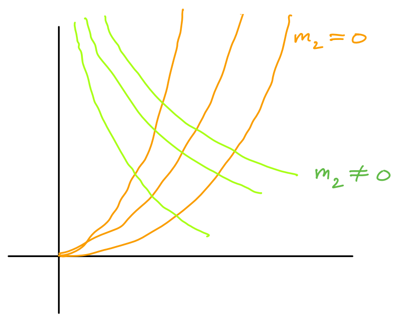
$$\dim(\text{no mult}) = n + \dim(O(n)) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

$$\dim(\text{w/ mult}) = n-1 + \dim\left(O(n)/O(2)\right) = \cdots = \frac{n(n+1)}{2} - 2$$



On the other hand...





$$\cos\left(\frac{\pi m_1}{1/t}\, x_1\right)\cos\left(\frac{\pi m_2}{t}\, x_2\right)$$

$$\operatorname{Spec}(\Delta_t) = \left\{ t^2(\pi m_1)^2 + \frac{1}{t^2}(\pi m_2)^2 : \ m_i \in \mathbb{Z} \right\}$$

Neumann