Growth and divisor of complexified horocycle eigenfunctions

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Quantum magnetic particle on $\mathbb{H}$

$\mathbb{H} = \mathbb{C}^+ = \{x + iy : x \in \mathbb{R}, y > 0\}$ — hyperbolic Lobachevsky plane, $ds^2 = (dx^2 + dy^2) \cdot y^{-2}$

$-\Delta_{\mathbb{H}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ is hyperbolic $-$Laplacian

$\tau \in \mathbb{R}$ (large), $D^\tau := -\Delta_{\mathbb{H}} + 2i\tau y \frac{\partial}{\partial x}$ is *magnetic Hamiltonian*

Horocycle (eigen)functions: $D^{\tau_n} u_n = s_n^2 u_n$

$$u = u_n : \mathbb{H} \to \mathbb{C},\ \tau = \tau_n \to +\infty, s = s_n = o(\tau), n = 1, 2, \ldots .$$

If $\hbar = 1/\tau$ then: $\left(-\hbar^2 \Delta_{\mathbb{H}} + 2i\hbar y \frac{\partial}{\partial x} \right) u = cu$, $c \xrightarrow{\hbar \to 0} 0$

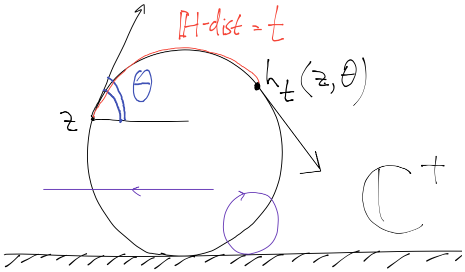
Auxiliary condition: $\sup\limits_{n \in \mathbb{N},\, z \in \mathbb{H}} \|u_n\|_{L^1(\mathcal{B}_{\mathbb{H}}(z,1))} < +\infty.$

Symbol. Classical flow

$$H_1(x, y, \xi_1, \xi_2) := \frac{(y\xi_1 - 1)^2 + (y\xi_2)^2}{2} : T^*\mathbb{H} \to \mathbb{R},$$

$$\frac{1}{\tau^2} \cdot D^\tau = \mathrm{Op}_\hbar(2H_1 - 1)$$

At $\{H_1 = 1/2\} \subset T^*\mathbb{H}$ (shifted circle bundle), H_1 as a *classical* Hamiltonian gives *horocycle* flow.



Quantum Unique Ergodicity. Horocycle case

Null set $\{H_1 = 1/2\}$ is
$\left\{\frac{(1+\cos\theta)\,dx+\sin\theta\,dy}{y} \text{ at } x+iy\colon \theta\in\mathbb{R} \bmod 2\pi,\ x+iy\in\mathbb{C}^+\right\}$.

In this coordinates, $d\mu_L := \dfrac{dx\,dy\,d\theta}{y^2}$ is invariant Liouville measure for $\{H_1 = 1/2\}$.

Definition

$\{u_n\}_{n=1}^{\infty}$ *is* Quantum Uniquely Ergodic (QUE) *sequence if, for any* $a \in C_0^\infty(T^*\mathbb{H})$ *we have*

$$\langle(\mathrm{Op}_{1/\tau_n}\, a)u_n, u_n\rangle_{L^2(\mathbb{H})} \xrightarrow{n\to\infty} \int\limits_{\{H_1=1/2\}} a\, d\mu_L.$$

Horocycle QUE at compact hyperbolic surface

$\Gamma < \mathrm{Isom}^+(\mathbb{H})$ is a discrete torsion-free group with compact fundamental domain $F \subset \mathbb{H}$. $\gamma \in \Gamma$ is $\mathbb{H} \ni z \mapsto \gamma z = \frac{az+b}{cz+d}$, a, b, c, d real with $ad - bc = 1$.

$X = \Gamma \setminus \mathbb{H}$ is compact hyperbolic surface

$u\colon \mathbb{H} \to \mathbb{C}$ is τ-*form* w.r.t. Γ ($\tau \in \mathbb{R}$) if $u(\gamma z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{\tau} u(z)$ for any $z \in \mathbb{H}$ and $\gamma \in \Gamma$. $\mathcal{F}^{\tau}(\Gamma) := \{\tau\text{-forms}\}$

Theorem 1 (S. Zelditch'92, D.'21)

$u_n \in \mathcal{F}^{\tau_n}(\Gamma)$, $D^{\tau_n} u_n = s_n^2 u_n$, $\tau_n \to \infty$, $s_n = o(\tau_n)$, $\int_F |u_n|^2 \, d\mathcal{A}_{\mathbb{H}} = 2\pi \mathcal{A}_{\mathbb{H}}(F)$, $d\mathcal{A}_{\mathbb{H}} = \frac{dx\,dy}{y^2}$.
Then sequence $\{u_n\}_{n=1}^{\infty}$ is QUE.

Proof: pass Furstenberg Theorem on classical unique ergodicity of horocycle flow on X through semiclassical correspondence.

Complexification!

$\mathbb{H}^{\mathbb{C}} = \{(X, Y) \colon X, Y \in \mathbb{C}\}$ is Bruhat–Whitney'59 complexification of $\mathbb{H}$

For $x_1 + iy_1, x_2 + iy_2 \in \mathbb{H}$,

$$\cosh \mathrm{dist}_{\mathbb{H}}(x_1 + iy_1, x_2 + iy_2) := 1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2y_1 y_2}.$$

$u(= u_n)$ as above can be analytically continued to a certain neighborhood $\mathcal{G}_1$ of $\mathbb{H}$ in $\mathbb{H}^{\mathbb{C}}$

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Main results, preliminary form. Growth

Theorem 2 (on growth)

There exist smooth $B_0, b\colon \mathcal{G}_1 \setminus \mathbb{H} \to \mathbb{R}$, $b > 0$ *with:*

$$|\tau_n|^{1/2} \cdot |u_n|^2 \cdot \exp(|\tau_n| B_0) \xrightarrow[\tau_n \to \infty]{}{}^{*} b \text{ in } \mathcal{D}'(\mathcal{G}_1 \setminus \mathbb{H}).$$

$\tilde{\mathcal{Z}}_n = \{u_n = 0\} \subset \mathcal{G}_1 (\subset \mathbb{H}^{\mathbb{C}})$ is regular up to negligible singular set. In any $P \in \tilde{\mathcal{Z}}_n$, $m_n(P)$ is integer multiplicity of divisor of u_n in P. For test 2-forms ω smooth on $\mathcal{G}_1 \setminus \mathbb{H}$, put $\mathcal{Z}_n(\omega) := \int_{\tilde{\mathcal{Z}}_n} m_n \omega$.

Lelong–Poincaré formula: de Rham current $\mathcal{Z}_n$ of degree 2 is

$$\mathcal{Z}_n(\omega) = \frac{i}{\pi} \int_{\mathcal{G}_1} \partial\bar{\partial} \log |u_n| \wedge \omega.$$

Main results, preliminary form. Divisor

Theorem 3 (corollary on divisor)

$$\frac{\mathcal{Z}_n}{|\tau_n|} \xrightarrow{n\to\infty} \frac{1}{2\pi i}\bar{\partial}\partial B_0 \quad \text{in } \mathcal{D}'(\mathcal{G}_1).$$

Proof: in

$$|\tau_n|^{1/2} \cdot |u_n|^2 \cdot \exp(|\tau_n| B_0) \xrightarrow[\tau_n\to\infty]{}^{*} b \text{ in } \mathcal{D}'(\mathcal{G}_1 \setminus \mathbb{H})$$

take logarithm using (pluri)subharmonic dichotomy to arrive at

$$\frac{\log|u_n|}{\tau_n} \xrightarrow{n\to\infty} -\frac{B_0}{2} \text{ in } L^1_{\mathrm{loc}}(\mathcal{G}_1),$$

then use distributional relation $\mathcal{Z}_n = \dfrac{i}{\pi}\partial\bar{\partial}\log|u_n|$, ■

Boutet de Monvel'79 intuition

For, e.g., 2d compact real analytic manifold X consider operators $\exp(-t\sqrt{-\Delta_X})$, $t \in \mathbb{R}^+$, and also $\exp(-t\sqrt{-\Delta_X})u$ for some wave $u\colon X \to \mathbb{C}$. They smoothen u. But u travels to complexified $X^{\mathbb{C}}$ almost unitarily.

Namely, let $g_t\colon T^*X \to X$ be geodesic flow. It possesses an analytic *by time* continuation $g_t\colon T^*X \to X^{\mathbb{C}}$ for $t \in \mathbb{C}$ with $|\Im t|$ small enough.

For $t > 0$ consider 3d hypersurface $\tilde{\Sigma}_t := \{g_{it}(x,\xi)\colon (x,\xi) \in S^*X\} \subset X^{\mathbb{C}}$. Then $\exp(-t\sqrt{-\Delta_X})\colon X \to \tilde{\Sigma}_t$ is almost unitary. Also, for $u \in L^2(X)$, $\exp(-t\sqrt{-\Delta_X})u$ is complex-analytic in $X^{\mathbb{C}}$ near X.

Analytic continuation with an integral operator

For $P = (X, Y) \in \mathbb{H}^{\mathbb{C}}$ put $Z(P) = X + iY$, $\tilde{Z}(P) = X - iY$, the analytic by X and Y continuations of $x \pm iy$ from $\mathbb{H}$ to $\mathbb{H}^{\mathbb{C}}$.

For $z \in \mathbb{H}$, $P = (X, Y) \in \mathbb{H}^{\mathbb{C}}$ and $t > 0$, put $c_t := \frac{4}{4t - t^3}$ and $K_t(z, P) := \left(\frac{z - \tilde{Z}(P)}{\bar{z} - Z(P)}\right) e^{-c_t \cdot \cosh \operatorname{dist}_{\mathbb{H}}(z, P)}$.

Remark. We need an extra mollification of this kernel to arrive to FIO/PDO with non-singular symbol.

Let $D^\tau u = s^2 u$, $v(P) := \int_{\mathbb{H}} u(z) K_t^\tau(z, P)\, d\mathcal{A}_{\mathbb{H}}(z)$. Then, for some $\mathcal{S}(t, \tau, s) \in \mathbb{C}$ not depending on u, function $v(P)/\mathcal{S}(t, \tau, s)$ is an analytic continuation of u to a neighborhood of $\mathbb{H}$ in $\mathbb{H}^{\mathbb{C}}$ ([Fay77]).

Remark. Due to *kernel gauge factor* $\left(\frac{z - \tilde{Z}(P)}{\bar{z} - Z(P)}\right)^\tau$ ($\left(\frac{z_1 - \bar{z}_2}{\bar{z}_1 - z_2}\right)^\tau$ in non-complexified case), such operator acts on $\mathcal{F}^\tau(\Gamma)$ for a $\Gamma < \mathrm{Isom}^+(\mathbb{H})$.

Complexified horocycle flow

(Almost) any horocycle at $\mathbb{H}$ can be parametrized as $\mathbb{R} \ni t \mapsto x_0 + \frac{y_0(t-t_0)}{(t-t_0)^2+1} + i \cdot \frac{y_0}{(t-t_0)^2+1} \in \mathbb{H}$ for some $x_0, t_0 \in \mathbb{R}$, $y_0 > 0$. In both real and imaginary parts we may put $t \in \mathbb{C}$ with $|\Im t| < 1$ to get their analytic (w.r.t. t) continuations.

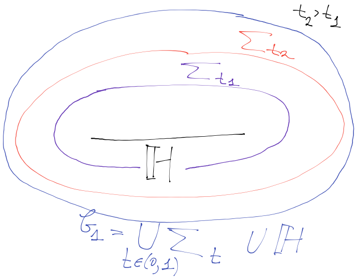
$z \in \mathbb{H}$, $\theta \in \mathbb{R} \bmod 2\pi$, $t \in \mathbb{R}$, let $h_t(z, \theta) \in \mathbb{H}$ be (basepoint of) horocycle starting from z under angle θ to horizontal line $\frac{\partial}{\partial x}$.

Further, for $t \in (0,1)$, consider $h_{-it}(z, \theta) \in \mathbb{H}^{\mathbb{C}}$.

Proposition (on horocycle Grauert tube)

$\mathbb{R} \times \mathbb{R}^+ \times (0,1) \times (\mathbb{R} \bmod 2\pi) \ni (x, y, t, \theta) \mapsto h_{-it}(x+iy, \theta) \in \mathbb{H}^{\mathbb{C}}$ *is a diffeomorphism from its domain to a set of the form* $\mathcal{G}_1 \setminus \mathbb{H}$ *where* $\mathcal{G}_1$ *is some open neighborhood of* $\mathbb{H}$ *in* $\mathbb{H}^{\mathbb{C}}$*. This* $\mathcal{G}_1$ *is called* horocycle Grauert tube*.*

Slices. Graph



In $\mathbb{H}^{\mathbb{C}}$, consider 3d hypersurface

$$\Sigma_t := \{h_{-it}(z, \theta) \colon z \in \mathbb{H}, \theta \in \mathbb{R} \bmod 2\pi\}$$

Define $M_t \colon \{H_1 = 1/2\} \to \mathbb{H}^{\mathbb{C}}$ by

$$M_t \left(\text{covector } \frac{(1 + \cos\theta)\, dx + \sin\theta\, dy}{y} \text{ at } x + iy \right) := h_{-it}(x+iy, \theta)$$

Intuitively, operator with kernel
$K_t^{\tau}(z, P) = \left(\frac{z - \tilde{Z}(P)}{\bar{z} - Z(P)} \right)^{\tau} e^{-\tau c_t \cdot \cosh \operatorname{dist}_{\mathbb{H}^{\mathbb{C}}}(z, P)}$ is a semiclassical
$(\hbar = 1/\tau)$ Fourier Integral Operator $\mathbb{H} \to \Sigma_t$ with a canonical graph

$$\{((z, \xi), (M_t(z, \xi), \text{some covector at } M_t(z, \xi)) : (z, \xi) \in \{H_1 = 1/2\}\} \subset$$
$$\subset \{H_1 = 1/2\} \times T^*\Sigma_t \subset T^*\mathbb{H} \times T^*\Sigma_t.$$

Turn microlocalization to localization

Thus, K_t^τ takes microlocal mass of u on null-set $\{H_1 = 1/2\}$ to local mass of u on Σ_t, up to factors not depending on u.

Proposition

Put $v(P) := Tu(P) = \int_{\mathbb{H}} u(z) K_t^\tau(z, P)\, d\mathcal{A}_{\mathbb{H}}(z)$.
Let $a \in C_0^\infty(\Sigma_t)$. *For some* $\phi(P)$ *smooth, we have:*

$$\int_{\Sigma_t} d\mu_{L,\Sigma_t}(P) a(P) |v(P)|^2 e^{-\tau\phi(P)} = O(\tau^{-4}) + \tau^{-3}\langle Au, u\rangle$$

with $A = \mathrm{Op}_{1/\tau}(b(x,\xi)\cdot a(M_t(x,\xi)))$, $b(x,\xi)$ *not depending on* a *nor on* u, *and* μ_{L,Σ_t} *be a natural Liouville measure on* Σ_t.

Proof. Drop $e^{-\tau\phi(P)}$. Then LHS is $\langle T^* \mathcal{M}_a T u, u\rangle_{L^2(\mathbb{H})}$, $\mathcal{M}_a$ being multiplier by a. Then derive Composition Theorem for semiclassical FIOs with complex phase by hands. ■

A technicality. To apply stationary phase method for FIO Composition Theorem, we need *global maximum property*: for $\theta \in \mathbb{R} \bmod 2\pi$,

$$\arg\max_{z_2 \in \mathbb{H}} |K_t(z_1, h_{-it}(z_2, \theta))| = z_1.$$

$\{(z_1, h_{-it}(z_1, \theta))\colon z_1 \in \mathbb{H}, \theta \in \mathbb{R} \bmod 2\pi\}$ leads to FIO graph.

The answer. Any $P \in \mathcal{G}_1$ is of the form $P = h_{-it}(z, \theta)$ for some $t \in (0, 1)$, $z \in \mathbb{H}$, $\theta \in \mathbb{R} \bmod 2\pi$. Take $B_0(P)$ to have

$$\exp(-B_0) = \left| \frac{z - \tilde{Z}(P)}{\bar{z} - Z(P)} \right|^2 = \frac{2 + (t^2 + 2t) \cdot (1 + \cos\theta)}{2 + (t^2 - 2t) \cdot (1 + \cos\theta)}.$$ Then

$$|\tau_n|^{1/2} \cdot |u_n|^2 \cdot \exp(|\tau_n| B_0) \xrightarrow[\tau_n \to \infty]{}^{*} b \text{ in } \mathcal{D}'(\mathcal{G}_1 \setminus \mathbb{H}), \quad \text{or, since,}$$

$$u(P) = \mathcal{S}^{-1}(t, \tau, s) \int_{\mathbb{H}} \left(\frac{z - \tilde{Z}(P)}{\bar{z} - Z(P)} \right)^{\tau} e^{-\tau c_t \cosh \operatorname{dist}_{\mathbb{H}}(z, P)} u(z)\, d\mathcal{A}(z),$$

Growth of a complexified horocycle eigenfunction is given by the growth of kernel gauge factor restricted to the canonical graph.

A reference request

If $D^\tau u = s^2 u$ then, for $v(P) := \int_{\mathbb{H}} u(z) K_t^\tau(z, P)\, d\mathcal{A}_{\mathbb{H}}(z)$, then $v(z) = \mathcal{S}(t, \tau, s) u(z)$ whenever $z \in \mathbb{H}$. $\mathcal{S}(t, \tau, s) \sim ?$

Put $u(x + iy) := y^{1/2 + i\tilde{s}}$, $\tilde{s} = \sqrt{s^2 - 1/4}$, $u(i) = 1$,

$$\mathcal{S}(t, \tau, s) = \int_{\mathbb{H}} y^{-\frac{3}{2} + i\tilde{s}} \left(\frac{i - x + iy}{-i - x - iy} \right)^\tau \cdot e^{-\tau c_t \cosh \mathrm{dist}_{\mathbb{H}}(i, x+iy)}\, dx \wedge dy.$$

Then $\mathbb{R} \ni x \to X \in \mathbb{C}$, $\mathbb{R} \ni y \to Y \in \mathbb{C}$, and move this 2d contour of integration in 4d $\mathbb{H}^{\mathbb{C}}$ to hit a saddle-point suggested by global maximum property. And then apply machinery from

[Федорюк, *Метод перевала*, 1977]

on higher dimensional saddle point (steepest descent) method — ???

Thank you for attention!