Lambdas, bubbles, and spheres

Iosif Polterovich

(Université de Montréal)

with Mikhail Karpukhin, Nikolai Nadirashvili and Alexei Penskoi

References

References

Based on

References

Based on

[KNPP1] An isoperimetric inequality for Laplace eigenvalues on the sphere, arXiv:1706.05713, 1–18; to appear in J. Diff. Geom.

References

Based on

[KNPP1] An isoperimetric inequality for Laplace eigenvalues on the sphere, arXiv:1706.05713, 1–18; to appear in J. Diff. Geom.

[KNPP2] Conformally maximal metrics for Laplace eigenvalues on surfaces , arXiv:2003.02871, 1-52.

Eigenvalues of the Laplacian

Consider the eigenvalue problem:

$$\Delta f = \lambda f$$

on a compact Riemannian manifold (M, g) without boundary.

Eigenvalues of the Laplacian

Consider the eigenvalue problem:

$$\Delta f = \lambda f$$

on a compact Riemannian manifold (M, g) without boundary.

The spectrum is discrete, and the eigenvalues form a sequence

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \cdots \nearrow +\infty$$

Eigenvalues of the Laplacian

Consider the eigenvalue problem:

$$\Delta f = \lambda f$$

on a compact Riemannian manifold (M, g) without boundary.

The spectrum is discrete, and the eigenvalues form a sequence

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leqslant \lambda_2(M, g) \leqslant \cdots \nearrow +\infty$$

Set

$$\bar{\lambda}_k(M, g) = \lambda_k(M, g) \operatorname{Vol}(M, g)^{2/d},$$

where $d = \dim M$. This quantity is invariant under rescaling.

Geometric optimization of eigenvalues

Consider $\bar{\lambda}_k(M, g)$ as a *functional* on the space of Riemannian metrics on M.

$$g \longmapsto \bar{\lambda}_k(M, g)$$

Geometric optimization of eigenvalues

Consider $\bar{\lambda}_k(M, g)$ as a *functional* on the space of Riemannian metrics on M.

$$g \longmapsto \bar{\lambda}_k(M, g)$$

A natural **geometric optimization** problem: find

$$\Lambda_k(M) = \sup_g \bar{\lambda}_k(M, g),$$

Colbois-Dodziuk (1994): If dim $M \geq 3$, $\Lambda_k(M) = +\infty$.

Geometric optimization of eigenvalues

Consider $\bar{\lambda}_k(M, g)$ as a *functional* on the space of Riemannian metrics on M.

$$g \longmapsto \bar{\lambda}_k(M, g)$$

A natural **geometric optimization** problem: find

$$\Lambda_k(M) = \sup_g \bar{\lambda}_k(M, g),$$

Colbois-Dodziuk (1994): If $\dim M \geq 3$, $\Lambda_k(M) = +\infty$.

Need further restrictions, such as fixed conformal class.

From now on, assume that M is a surface.

Geometric optimization of eigenvalues

Consider $\bar{\lambda}_k(M,g)$ as a *functional* on the space of Riemannian metrics on M.

$$g \longmapsto \bar{\lambda}_k(M,g)$$

A natural **geometric optimization** problem: find

$$\Lambda_k(M) = \sup_g \bar{\lambda}_k(M,g),$$

Colbois-Dodziuk (1994): If $\dim M \geq 3$, $\Lambda_k(M) = +\infty$.

Need further restrictions, such as fixed conformal class.

From now on, assume that M is a surface. A metric realizing the supremum (if exists!) is called a **maximal metric**.

Topological upper bounds for $\bar{\lambda}_1$

- Yang–Yau (1980), El Soufi–Ilias (1984): for an orientable surface M of genus γ we have

$$\bar{\lambda}_1(M,g) \leqslant 8\pi \left[\frac{\gamma+3}{2}\right].$$

Topological upper bounds for $\bar{\lambda}_1$

- Yang–Yau (1980), El Soufi–Ilias (1984): for an orientable surface M of genus γ we have

$$\bar{\lambda}_1(M, g) \leqslant 8\pi \left[\frac{\gamma + 3}{2}\right].$$

Karpukhin (2019): strict inequality for $\gamma \neq 0, 2$.

Topological upper bounds for $\bar{\lambda}_1$

- Yang–Yau (1980), El Soufi–Ilias (1984): for an orientable surface M of genus γ we have

$$\bar{\lambda}_1(M,g) \leqslant 8\pi \left[\frac{\gamma+3}{2}\right].$$

  Karpukhin (2019): strict inequality for $\gamma \neq 0, 2$.

- Karpukhin (2016): for a non-orientable surface M of genus γ we have

$$\bar{\lambda}_1(M,g) \leqslant 16\pi \left[\frac{\gamma+3}{2}\right].$$

  Here γ is the genus of the orientable double cover.

Topological upper bounds for $\bar{\lambda}_k$

- Korevaar (1993): there exists a constant $C > 0$ such that on any (orientable) surface M of genus γ,

$$\bar{\lambda}_k(M) \leq Ck(\gamma + 1).$$

Topological upper bounds for $\bar{\lambda}_k$

- Korevaar (1993): there exists a constant $C > 0$ such that on any (orientable) surface M of genus γ,

$$\bar{\lambda}_k(M) \leq Ck(\gamma + 1).$$

Conjectured by Yau (1982); discussed by Gromov (1993), generalized by Grigor'yan–Netrusov–Yau (1999, 2004).

Topological upper bounds for $\bar{\lambda}_k$

- Korevaar (1993): there exists a constant $C > 0$ such that on any (orientable) surface M of genus γ,

  $$\bar{\lambda}_k(M) \leq Ck(\gamma + 1).$$

  Conjectured by Yau (1982); discussed by Gromov (1993), generalized by Grigor'yan–Netrusov–Yau (1999, 2004).

- Hassannezhad (2011):

  $$\bar{\lambda}_k(M) \leq C(k + \gamma).$$

Topological upper bounds for $\bar{\lambda}_k$

- Korevaar (1993): there exists a constant $C > 0$ such that on any (orientable) surface M of genus γ,

$$\bar{\lambda}_k(M) \leq Ck(\gamma + 1).$$

  Conjectured by Yau (1982); discussed by Gromov (1993), generalized by Grigor'yan–Netrusov–Yau (1999, 2004).

- Hassannezhad (2011):

$$\bar{\lambda}_k(M) \leq C(k + \gamma).$$

Question: Can C be made explicit?

Maximal metrics for λ_1: examples

Maximal metrics for λ_1: examples

- Hersch (1970): $\Lambda_1(\mathbb{S}^2) = 8\pi$ and the maximum is achieved on the *standard metric* on $\mathbb{S}^2$.

Maximal metrics for λ_1: examples

- Hersch (1970): $\Lambda_1(\mathbb{S}^2) = 8\pi$ and the maximum is achieved on the *standard metric* on $\mathbb{S}^2$.
- Li–Yau (1982): $\Lambda_1(\mathbb{R}P^2) = 12\pi$ and the maximum is achieved on the *standard metric* on $\mathbb{R}P^2$.

Maximal metrics for λ_1: examples

- Hersch (1970): $\Lambda_1(\mathbb{S}^2) = 8\pi$ and the maximum is achieved on the *standard metric* on $\mathbb{S}^2$.
- Li–Yau (1982): $\Lambda_1(\mathbb{R}P^2) = 12\pi$ and the maximum is achieved on the *standard metric* on $\mathbb{R}P^2$.
- Nadirashvili (1996): $\Lambda_1(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is achieved on the *flat equilateral torus*.

Examples: continued

- Jakobson–Nadirashvili–P. (2006), El Soufi–Giacomini–Jazar (2006), Karpukhin–Cianci–Medvedev (2019):

  $\Lambda_1(\mathbb{K}) = \bar{\lambda}_1(\mathbb{K}, g_{\tilde{\tau}_{3,1}}) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where $\tau_{3,1}$ is a *Lawson bipolar surface* (a Klein bottle of revolution), and E is a complete elliptic integral of the second kind.

Examples: continued

- Jakobson–Nadirashvili–P. (2006), El Soufi–Giacomini–Jazar (2006), Karpukhin–Cianci–Medvedev (2019):

  $\Lambda_1(\mathbb{K}) = \bar{\lambda}_1(\mathbb{K}, g_{\tilde{\tau}_{3,1}}) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where $\tau_{3,1}$ is a *Lawson bipolar surface* (a Klein bottle of revolution), and E is a complete elliptic integral of the second kind.

- Jakobson–Levitin–Nadirashvili–Nigam–P. (2005), Nayatani–Shoda (2017): $\Lambda_1(\Sigma_2) = 16\pi$.

Examples: continued

- Jakobson–Nadirashvili–P. (2006), El Soufi–Giacomini–Jazar (2006), Karpukhin–Cianci–Medvedev (2019):

  $\Lambda_1(\mathbb{K}) = \bar{\lambda}_1(\mathbb{K}, g_{\tilde{\tau}_{3,1}}) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where $\tau_{3,1}$ is a *Lawson bipolar surface* (a Klein bottle of revolution), and E is a complete elliptic integral of the second kind.

- Jakobson–Levitin–Nadirashvili–Nigam–P. (2005), Nayatani–Shoda (2017): $\Lambda_1(\Sigma_2) = 16\pi$. A maximal metric for the first eigenvalue on the surface of genus two Σ_2 is a metric with conical singularities on the *Bolza surface* induced from the canonical metric on the sphere using the standard branched double covering.

Examples: continued

- Jakobson–Nadirashvili–P. (2006), El Soufi–Giacomini–Jazar (2006), Karpukhin–Cianci–Medvedev (2019):

  $\Lambda_1(\mathbb{K}) = \bar{\lambda}_1(\mathbb{K}, g_{\tilde{\tau}_{3,1}}) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where $\tau_{3,1}$ is a *Lawson bipolar surface* (a Klein bottle of revolution), and E is a complete elliptic integral of the second kind.

- Jakobson–Levitin–Nadirashvili–Nigam–P. (2005), Nayatani–Shoda (2017): $\Lambda_1(\Sigma_2) = 16\pi$. A maximal metric for the first eigenvalue on the surface of genus two Σ_2 is a metric with conical singularities on the *Bolza surface* induced from the canonical metric on the sphere using the standard branched double covering. Maximal metric is not unique.

Maximization of eigenvalues in a conformal class: first eigenvalue

Maximization of eigenvalues in a conformal class: first eigenvalue

Given a conformal class $\mathcal{C}$ on a surface M, set

$$\Lambda_k(M, \mathcal{C}) = \sup_{g \in \mathcal{C}} \bar{\lambda}_k(g).$$

Maximization of eigenvalues in a conformal class: first eigenvalue

Given a conformal class $\mathcal{C}$ on a surface M, set

$$\Lambda_k(M, \mathcal{C}) = \sup_{g \in \mathcal{C}} \bar{\lambda}_k(g).$$

Theorem (Nadirashvili-Sire, Petrides, 2010s)

Maximization of eigenvalues in a conformal class: first eigenvalue

Given a conformal class $\mathcal{C}$ on a surface M, set

$$\Lambda_k(M, \mathcal{C}) = \sup_{g \in \mathcal{C}} \bar{\lambda}_k(g).$$

Theorem (Nadirashvili-Sire, Petrides, 2010s)

(i) For any conformal class $\mathcal{C}$ of Riemannian metrics on a closed surface M, there exists a metric $g \in \mathcal{C}$, possibly with a finite number of conical singularities, such that

$$\Lambda_1(M, \mathcal{C}) = \bar{\lambda}_1(M, g).$$

Remarks

- Metrics with conical singularities arising in this context: $g = \alpha(x) g_0$, where g_0 is a constant curvature metric, and $\alpha(x) \geq 0$ is a smooth function with finitely many zeros.

Remarks

- Metrics with conical singularities arising in this context: $g = \alpha(x)g_0$, where g_0 is a constant curvature metric, and $\alpha(x) \geq 0$ is a smooth function with finitely many zeros.
- The cone angle is equal to $2\pi(r+1)$, where r is the order of vanishing at the zero point.

Remarks

- Metrics with conical singularities arising in this context: $g = \alpha(x) g_0$, where g_0 is a constant curvature metric, and $\alpha(x) \geq 0$ is a smooth function with finitely many zeros.
- The cone angle is equal to $2\pi(r+1)$, where r is the order of vanishing at the zero point.
- The example of the genus 2 surface shows that conical singularities may indeed occur.

Remarks

- Metrics with conical singularities arising in this context: $g = \alpha(x) g_0$, where g_0 is a constant curvature metric, and $\alpha(x) \geq 0$ is a smooth function with finitely many zeros.
- The cone angle is equal to $2\pi(r+1)$, where r is the order of vanishing at the zero point.
- The example of the genus 2 surface shows that conical singularities may indeed occur.
- New proof by Karpukhin–Stern (2020) using min-max energy for harmonic maps.

Remarks

- Metrics with conical singularities arising in this context: $g = \alpha(x) g_0$, where g_0 is a constant curvature metric, and $\alpha(x) \geq 0$ is a smooth function with finitely many zeros.
- The cone angle is equal to $2\pi(r+1)$, where r is the order of vanishing at the zero point.
- The example of the genus 2 surface shows that conical singularities may indeed occur.
- New proof by Karpukhin–Stern (2020) using min-max energy for harmonic maps.
- Matthiesen–Siffert (2019): On any surface M there exists a globally maximizing metric g, smooth outside a finite number of conical singularities , such that $\Lambda_1(M) = \bar{\lambda}_1(M, g)$.

Maximization of eigenvalues in a conformal class: higher eigenvalues

Maximization of eigenvalues in a conformal class: higher eigenvalues

(ii) For any conformal class $\mathcal{C}$ of Riemannian metrics on M and for any $k > 1$, either one has

$$\Lambda_k(M, \mathcal{C}) = \Lambda_{k-1}(M, \mathcal{C}) + 8\pi,$$

Maximization of eigenvalues in a conformal class: higher eigenvalues

(ii) For any conformal class $\mathcal{C}$ of Riemannian metrics on M and for any $k > 1$, either one has

$$\Lambda_k(M, \mathcal{C}) = \Lambda_{k-1}(M, \mathcal{C}) + 8\pi,$$

or there exists a metric $g \in \mathcal{C}$, possibly with a finite number of conical singularities, such that

$$\Lambda_k(M, \mathcal{C}) = \bar{\lambda}_k(M, g) > \Lambda_{k-1}(M) + 8\pi.$$

Existence vs bubbling

Existence vs bubbling

The previous result can be interpreted as follows:

Existence vs bubbling

The previous result can be interpreted as follows: for a given k, either

- there exists a maximal metric for λ_k which is smooth outside a finite number of conical singularities,

Existence vs bubbling

The previous result can be interpreted as follows: for a given k, either

- there exists a maximal metric for λ_k which is smooth outside a finite number of conical singularities, or
- the maximum of λ_k is attained in the limit by a sequence of metrics exhibiting **bubbling**, that is, concentration of measure at certain points. Bubbles can be viewed as spheres blown out of some points of the original surface.

Existence vs bubbling

The previous result can be interpreted as follows: for a given k, either

- there exists a maximal metric for λ_k which is smooth outside a finite number of conical singularities, or
- the maximum of λ_k is attained in the limit by a sequence of metrics exhibiting **bubbling**, that is, concentration of measure at certain points. Bubbles can be viewed as spheres blown out of some points of the original surface.

It is easy to see from the variational principle that the number of bubbles is at most k.

Maximization of eigenvalues on the sphere

Maximization of eigenvalues on the sphere

Theorem [KNPP1] Let $(\mathbb{S}^2, g)$ be the sphere endowed with a metric g which is smooth outside a finite number of conical singularities. Then

$$\bar{\lambda}_k(\mathbb{S}^2, g) \leq 8\pi k, \quad k \geqslant 1.$$

Maximization of eigenvalues on the sphere

Theorem [KNPP1] Let $(\mathbb{S}^2, g)$ be the sphere endowed with a metric g which is smooth outside a finite number of conical singularities. Then

$$\bar{\lambda}_k(\mathbb{S}^2, g) \leq 8\pi k, \quad k \geqslant 1.$$

For $k \geqslant 2$ the inequality is strict, and the equality is attained in the limit by a sequence of metrics degenerating to a union of k touching identical round spheres.

Maximization of eigenvalues on the sphere

Theorem [KNPP1] Let $(\mathbb{S}^2, g)$ be the sphere endowed with a metric g which is smooth outside a finite number of conical singularities. Then

$$\bar{\lambda}_k(\mathbb{S}^2, g) \leq 8\pi k, \quad k \geqslant 1.$$

For $k \geqslant 2$ the inequality is strict, and the equality is attained in the limit by a sequence of metrics degenerating to a union of k touching identical round spheres.

In particular,

$$\Lambda_k(\mathbb{S}^2) = 8\pi k, \quad k \geqslant 1.$$

Historical remarks

Historical remarks

- This result was conjectured by Nadirashvili (2002).

Historical remarks

- This result was conjectured by Nadirashvili (2002).
- $k = 2$: Nadirashvili (2002), Petrides (2014).

Historical remarks

- This result was conjectured by Nadirashvili (2002).
- $k = 2$: Nadirashvili (2002), Petrides (2014).
- $k = 3$: Nadirashvili–Sire (2017).

Historical remarks

- This result was conjectured by Nadirashvili (2002).
- $k = 2$: Nadirashvili (2002), Petrides (2014).
- $k = 3$: Nadirashvili–Sire (2017).
- Numerical evidence: Kao–Lai–Osting (2017).

Maximization of eigenvalues on the projective plane

Theorem (Karpukhin, 2019) The equality

$$\Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$$

holds for any $k \geqslant 1$. For $k \geqslant 2$ the supremum can not be attained on a smooth metric, and is realized in the limit by a sequence of metrics degenerating to a union of $k-1$ identical round spheres and a standard projective plane touching each other, such that the ratio of the areas of the projective plane and the spheres is $3:2$.

Maximization of eigenvalues on the projective plane

Theorem (Karpukhin, 2019) The equality

$$\Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$$

holds for any $k \geqslant 1$. For $k \geqslant 2$ the supremum can not be attained on a smooth metric, and is realized in the limit by a sequence of metrics degenerating to a union of $k-1$ identical round spheres and a standard projective plane touching each other, such that the ratio of the areas of the projective plane and the spheres is 3 : 2.

Proved for $k = 2$ by Nadirashvili–Penskoi (2018). Conjectured for all k in [KNPP1].

Explicit Korevaar-type inequality for higher eigenvalues

Explicit Korevaar-type inequality for higher eigenvalues

Theorem [KNPP2] (i) Let M be an orientable surface of genus γ. Then

$$\Lambda_k(M) \leqslant 8\pi k \left[\frac{\gamma+3}{2}\right], \quad k \geqslant 1.$$

Explicit Korevaar-type inequality for higher eigenvalues

Theorem [KNPP2] (i) Let M be an orientable surface of genus γ. Then

$$\Lambda_k(M) \leqslant 8\pi k \left[\frac{\gamma+3}{2}\right], \;\; k \geqslant 1.$$

(ii) Let M be an non-orientable surface, and let γ be the genus of its orientable double cover. Then

$$\Lambda_k(M) \leqslant 16\pi k \left[\frac{\gamma+3}{2}\right], \;\; k \geqslant 1.$$

Explicit Korevaar-type inequality for higher eigenvalues

Theorem [KNPP2] (i) Let M be an orientable surface of genus γ. Then

$$\Lambda_k(M) \leqslant 8\pi k \left[\frac{\gamma+3}{2}\right], \quad k \geqslant 1.$$

(ii) Let M be an non-orientable surface, and let γ be the genus of its orientable double cover. Then

$$\Lambda_k(M) \leqslant 16\pi k \left[\frac{\gamma+3}{2}\right], \quad k \geqslant 1.$$

Proof is based on Yang–Yau method and its adaptation by Karpukhin to the nonorientable case combined with the explicit bounds for the sphere and the projective plane.

Maximization on spheres: ideas of the proof

Maximization on spheres: ideas of the proof

By gluing techniques (see Colbois-El Soufi, 2003)

$$\Lambda_k(\mathbb{S}^2) \geq 8\pi k, \;\; k \geqslant 1.$$

Maximization on spheres: ideas of the proof

By gluing techniques (see Colbois-El Soufi, 2003)

$$\Lambda_k(\mathbb{S}^2) \geq 8\pi k, \quad k \geqslant 1.$$

Suppose for some $k > 1$ we have a strict inequality

$$\Lambda_k(\mathbb{S}^2) > 8\pi k$$

Maximization on spheres: ideas of the proof

By gluing techniques (see Colbois-El Soufi, 2003)

$$\Lambda_k(\mathbb{S}^2) \geq 8\pi k, \;\; k \geqslant 1.$$

Suppose for some $k > 1$ we have a strict inequality

$$\Lambda_k(\mathbb{S}^2) > 8\pi k$$

From the existence vs bubbling argument, it means that for the smallest k with this property there exists a maximal metric g (possibly with conical singularities) such that

$$\bar{\lambda}_k(\mathbb{S}^2, g) > 8\pi k.$$

Maximization on spheres: ideas of the proof

By gluing techniques (see Colbois-El Soufi, 2003)

$$\Lambda_k(\mathbb{S}^2) \geq 8\pi k, \;\; k \geqslant 1.$$

Suppose for some $k > 1$ we have a strict inequality

$$\Lambda_k(\mathbb{S}^2) > 8\pi k$$

From the existence vs bubbling argument, it means that for the smallest k with this property there exists a maximal metric g (possibly with conical singularities) such that

$$\bar{\lambda}_k(\mathbb{S}^2, g) > 8\pi k.$$

Let us show this is impossible.

Minimal immersions

Let M and N be smooth manifolds and h be a Riemannian metric on N. An immersion $f : M \looparrowright N$ is called *minimal f* if it is extremal for the volume functional

$$V[f] = \int_M dVol_{f^*h}.$$

The manifold M is endowed with a Riemannian metric f^*h and is referred to as *(immersed) minimal submanifold*.

Extremality condition

Extremality condition

Takahashi's theorem (1966): an isometric immersion $f : M \looparrowright \mathbb{R}^{n+1}, \quad f = (f^1, \ldots, f^{n+1})$, by Laplace eigenfunctions f^i with a common eigenvalue yields a *minimal* immersion into an n-dimensional sphere.

Extremality condition

Takahashi's theorem (1966): an isometric immersion $f : M \looparrowright \mathbb{R}^{n+1}, \quad f = (f^1, \ldots, f^{n+1})$, by Laplace eigenfunctions f^i with a common eigenvalue yields a *minimal* immersion into an n-dimensional sphere.

Theorem (Nadirashvili, 1996; El Soufi-Ilias, 2008) If a metric g on a compact surface M is extremal for the eigenvalue λ_k then there exists an isometric minimal immersion $M \looparrowright \mathbb{S}^n_R$ to the sphere of some dimension $n \geq 2$ of radius $R = \sqrt{2/\lambda_k(M, g)}$ by the corresponding eigenfunctions.

Harmonic maps

Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $f : M \longrightarrow N$ is called harmonic if f is extremal for the energy functional

$$E[f] = \frac{1}{2} \int_M \mathrm{trace}_g f^* h \, dVol_g,$$

Harmonic maps

Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $f : M \longrightarrow N$ is called harmonic if f is extremal for the energy functional

$$E[f] = \frac{1}{2} \int_M \mathrm{trace}_g f^* h \, dVol_g,$$

In local coordinates

$$\mathrm{trace}_g f^* h = g^{kl} \frac{\partial f^i}{\partial x^k} \frac{\partial f^j}{\partial x^l} h_{ij}.$$

Minimal immersions and harmonic maps

Minimal immersions and harmonic maps

Minimal immersions and harmonic maps are closely related.

Minimal immersions and harmonic maps

Minimal immersions and harmonic maps are closely related.

Example A submanifold $M \looparrowright \mathbb{R}^n$ is minimal if and only if the coordinate functions x^i are harmonic (in the usual sense) with respect to the Laplace-Beltrami operator on M.

Minimal immersions and harmonic maps

Minimal immersions and harmonic maps are closely related.

Example A submanifold $M \looparrowright \mathbb{R}^n$ is minimal if and only if the coordinate functions x^i are harmonic (in the usual sense) with respect to the Laplace-Beltrami operator on M.

In general, the following result holds:

Minimal immersions and harmonic maps

Minimal immersions and harmonic maps are closely related.

Example A submanifold $M \looparrowright \mathbb{R}^n$ is minimal if and only if the coordinate functions x^i are harmonic (in the usual sense) with respect to the Laplace-Beltrami operator on M.

In general, the following result holds:

Theorem Let $f : (M, g) \looparrowright (N, h)$ be an isometric immersion. Then f is minimal if and only if it is harmonic.

Minimal immersions and harmonic maps

Minimal immersions and harmonic maps are closely related.

Example A submanifold $M \looparrowright \mathbb{R}^n$ is minimal if and only if the coordinate functions x^i are harmonic (in the usual sense) with respect to the Laplace-Beltrami operator on M.

In general, the following result holds:

Theorem Let $f : (M, g) \looparrowright (N, h)$ be an isometric immersion. Then f is minimal if and only if it is harmonic.

Branched harmonic immersions give rise to metrics with conical singularities, which occur at points where $df = 0$.

Harmonic degree and small eigenvalues

Harmonic degree and small eigenvalues

Theorem (Calabi, Barbosa) Let $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a harmonic immersion (possibly, with branch points).

Harmonic degree and small eigenvalues

Theorem (Calabi, Barbosa) Let $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a harmonic immersion (possibly, with branch points). Then

$$\mathrm{Area}(\mathbb{S}^2, f^* g_{\mathbb{S}^n}) = 4\pi d$$

for some $d \in \mathbb{N}$.

Harmonic degree and small eigenvalues

Theorem (Calabi, Barbosa) Let $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a harmonic immersion (possibly, with branch points). Then

$$\mathrm{Area}(\mathbb{S}^2, f^* g_{\mathbb{S}^n}) = 4\pi d$$

for some $d \in \mathbb{N}$.

The integer d is called the harmonic degree of f.

Harmonic degree and small eigenvalues

Theorem (Calabi, Barbosa) Let $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a harmonic immersion (possibly, with branch points). Then

$$\mathrm{Area}(\mathbb{S}^2, f^* g_{\mathbb{S}^n}) = 4\pi d$$

for some $d \in \mathbb{N}$.

The integer d is called the harmonic degree of f.

Key observation: (Ejiri, 1998; case $n = 2$ due to Montiel–Ros and Nayatani): large harmonic degree implies many small eigenvalues.

Harmonic degree and small eigenvalues

Theorem (Calabi, Barbosa) Let $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a harmonic immersion (possibly, with branch points). Then

$$\mathrm{Area}(\mathbb{S}^2, f^* g_{\mathbb{S}^n}) = 4\pi d$$

for some $d \in \mathbb{N}$.

The integer d is called the harmonic degree of f.

Key observation: (Ejiri, 1998; case $n = 2$ due to Montiel–Ros and Nayatani): large harmonic degree implies many small eigenvalues.

In particular, if $k > 1$, for any λ_k-extremal metric

$$\lambda_k(\mathbb{S}^2, g)\mathrm{area}(\mathbb{S}^2, g) < 8\pi k. \qquad \square$$

Nadirashvili–Sire–Petrides theorem: ideas of the proof following [KNPP2]

- From eigenvalues of metrics to eigenvalues of Radon measures (cf. Kokarev, 2014).

Nadirashvili–Sire–Petrides theorem: ideas of the proof following [KNPP2]

- From eigenvalues of metrics to eigenvalues of Radon measures (cf. Kokarev, 2014).

  $\Delta u = \lambda V u$, g -fixed background Riemannian metric, $V \geq 0$, $V \in L^1(M, g)$.

Nadirashvili–Sire–Petrides theorem: ideas of the proof following [KNPP2]

- From eigenvalues of metrics to eigenvalues of Radon measures (cf. Kokarev, 2014).

  $\Delta u = \lambda V u$, g -fixed background Riemannian metric, $V \geq 0$, $V \in L^1(M, g)$.

- Auxiliary optimization problem in terms of potentials of Schrödinger operators.

Nadirashvili–Sire–Petrides theorem: ideas of the proof following [KNPP2]

- From eigenvalues of metrics to eigenvalues of Radon measures (cf. Kokarev, 2014).

  $\Delta u = \lambda V u$, g -fixed background Riemannian metric, $V \geq 0$, $V \in L^1(M, g)$.

- Auxiliary optimization problem in terms of potentials of Schrödinger operators. Potentials are allowed to take negative values to have more freedom for perturbations.

Nadirashvili–Sire–Petrides theorem: ideas of the proof following [KNPP2]

- From eigenvalues of metrics to eigenvalues of Radon measures (cf. Kokarev, 2014).

  $\Delta u = \lambda V u$, g -fixed background Riemannian metric, $V \geq 0$, $V \in L^1(M, g)$.

- Auxiliary optimization problem in terms of potentials of Schrödinger operators. Potentials are allowed to take negative values to have more freedom for perturbations. L^∞-norm of the potentials bounded by $N > 0$.

Nadirashvili–Sire–Petrides theorem: ideas of the proof following [KNPP2]

- From eigenvalues of metrics to eigenvalues of Radon measures (cf. Kokarev, 2014).

  $\Delta u = \lambda V u$, g -fixed background Riemannian metric, $V \geq 0$, $V \in L^1(M, g)$.

- Auxiliary optimization problem in terms of potentials of Schrödinger operators. Potentials are allowed to take negative values to have more freedom for perturbations. L^∞-norm of the potentials bounded by $N > 0$. We are interested in the limit as $N \to \infty$.

Nadirashvili–Sire–Petrides theorem: ideas of the proof following [KNPP2]

- From eigenvalues of metrics to eigenvalues of Radon measures (cf. Kokarev, 2014).

  $\Delta u = \lambda V u$, g -fixed background Riemannian metric, $V \geq 0$, $V \in L^1(M, g)$.

- Auxiliary optimization problem in terms of potentials of Schrödinger operators. Potentials are allowed to take negative values to have more freedom for perturbations. L^∞-norm of the potentials bounded by $N > 0$. We are interested in the limit as $N \to \infty$.

  Grigor'yan-Nadirashvili-Sire (2016) : maximizing potentials turn out to be non-negative.

Construction of the maximizing sequence

For each $k \geq 1$, there exist maps

$$\phi_{m,k} = (u^1_{m,k}, \ldots, u^d_{m,k}) : M \to \mathbb{R}^d$$

for some $d \in \mathbb{N}$, such that as $m \to \infty$:

Construction of the maximizing sequence

For each $k \geq 1$, there exist maps

$$\phi_{m,k} = (u^1_{m,k}, \ldots, u^d_{m,k}) : M \to \mathbb{R}^d$$

for some $d \in \mathbb{N}$, such that as $m \to \infty$:

(1) $\Delta \phi_{m,k} = \Lambda^m_k \, V_{m,k} \, \phi_{m,k}, \quad \Lambda^m_k \to \Lambda_k := \Lambda_k(M, \mathcal{C}).$

Construction of the maximizing sequence

For each $k \geq 1$, there exist maps

$$\phi_{m,k} = (u^1_{m,k}, \ldots, u^d_{m,k}) : M \to \mathbb{R}^d$$

for some $d \in \mathbb{N}$, such that as $m \to \infty$:

(1) $\Delta \phi_{m,k} = \Lambda_k^m \, V_{m,k} \, \phi_{m,k}, \quad \Lambda_k^m \to \Lambda_k := \Lambda_k(M, \mathcal{C})$.

(2) There exists a weak limit $\phi_{m,k} \rightharpoonup \phi_k = (u^1_k, \ldots, u^d_k)$ in H^1 and $\phi_{m,k} \to \phi_k$ in L^2.

Construction of the maximizing sequence

For each $k \geq 1$, there exist maps

$$\phi_{m,k} = (u^1_{m,k}, \dots, u^d_{m,k}) : M \to \mathbb{R}^d$$

for some $d \in \mathbb{N}$, such that as $m \to \infty$:

(1) $\Delta \phi_{m,k} = \Lambda^m_k \, V_{m,k} \, \phi_{m,k}, \quad \Lambda^m_k \to \Lambda_k := \Lambda_k(M, \mathcal{C})$.

(2) There exists a weak limit $\phi_{m,k} \rightharpoonup \phi_k = (u^1_k, \dots, u^d_k)$ in H^1 and $\phi_{m,k} \to \phi_k$ in L^2.

(3) $|\phi_k| = 1$ dv_g-a.e. (essentially, ϕ_k is a map to a sphere!)

Construction of the maximizing sequence

For each $k \geq 1$, there exist maps

$$\phi_{m,k} = (u^1_{m,k}, \ldots, u^d_{m,k}) : M \to \mathbb{R}^d$$

for some $d \in \mathbb{N}$, such that as $m \to \infty$:

(1) $\Delta \phi_{m,k} = \Lambda_k^m \, V_{m,k} \, \phi_{m,k}$, $\quad \Lambda_k^m \to \Lambda_k := \Lambda_k(M, \mathcal{C})$.

(2) There exists a weak limit $\phi_{m,k} \rightharpoonup \phi_k = (u^1_k, \ldots, u^d_k)$ in H^1 and $\phi_{m,k} \to \phi_k$ in L^2.

(3) $|\phi_k| = 1$ dv_g-a.e. (essentially, ϕ_k is a map to a sphere!)

(4) $V_{m,k} \, dv_g \rightharpoonup^* d\mu_k$ for some Radon measure $d\mu_k$.

Why is this not good enough?

Why is this not good enough?

Problem: weak convergence in H^1 does not imply that the limiting functions ϕ_k are eigenfunctions of the limiting problem.

Why is this not good enough?

Problem: weak convergence in H^1 does not imply that the limiting functions ϕ_k are eigenfunctions of the limiting problem.

$$
\begin{array}{ccc}
\displaystyle\int \nabla\phi_{m,k}\nabla\psi\, dv_g & = & \displaystyle\Lambda_k^m \int \phi_{m,k}\,\psi\, V_{m,k} dv_g \\
\Big\downarrow{\scriptstyle m\to\infty} & & ?\Big\downarrow{\scriptstyle m\to\infty} \\
\displaystyle\int \nabla\phi_k\nabla\psi\, dv_g & \overset{?}{=} & \displaystyle\Lambda_k \int \phi_k\,\psi\, d\mu_k
\end{array}
$$

Why is this not good enough?

Problem: weak convergence in H^1 does not imply that the limiting functions ϕ_k are eigenfunctions of the limiting problem.

$$\begin{array}{ccc}
\displaystyle\int \nabla\phi_{m,k}\nabla\psi\, dv_g & = & \displaystyle\Lambda_k^m \int \phi_{m,k}\,\psi\, V_{m,k} dv_g \\
\Big\downarrow{\scriptstyle m\to\infty} & & ?\Big\downarrow{\scriptstyle m\to\infty} \\
\displaystyle\int \nabla\phi_k\nabla\psi\, dv_g & \overset{?}{=} & \displaystyle\Lambda_k \int \phi_k\,\psi\, d\mu_k
\end{array}$$

On the right-hand side we need stronger convergence, like uniform.

Strong convergence of the maximizing sequence

- Good and bad points.

Strong convergence of the maximizing sequence

- Good and bad points. Bad points have arbitrary small neighborhoods Ω with $\lambda_1^D(\Omega, V_{m,k})$ not large enough.

Strong convergence of the maximizing sequence

- Good and bad points. Bad points have arbitrary small neighborhoods Ω with $\lambda_1^D(\Omega, V_{m,k})$ not large enough.

  There are at most k bad points.

Strong convergence of the maximizing sequence

- Good and bad points. Bad points have arbitrary small neighborhoods Ω with $\lambda_1^D(\Omega, V_{m,k})$ not large enough.

  There are at most k bad points. As it turns out, these are precisely the points admitting concentration of measure in arbitrary small neighborhoods (bubbles).

Strong convergence of the maximizing sequence

- Good and bad points. Bad points have arbitrary small neighborhoods Ω with $\lambda_1^D(\Omega, V_{m,k})$ not large enough.

  There are at most k bad points. As it turns out, these are precisely the points admitting concentration of measure in arbitrary small neighborhoods (bubbles).

- **Proposition** Given a good point p, there exists a neighborhood $\Omega \ni p$ such that $\phi_{m,k} \to \phi_k$ in $H^1(\Omega)$.

Strong convergence of the maximizing sequence

- Good and bad points. Bad points have arbitrary small neighborhoods Ω with $\lambda_1^D(\Omega, V_{m,k})$ not large enough.

  There are at most k bad points. As it turns out, these are precisely the points admitting concentration of measure in arbitrary small neighborhoods (bubbles).

- **Proposition** Given a good point p, there exists a neighborhood $\Omega \ni p$ such that $\phi_{m,k} \to \phi_k$ in $H^1(\Omega)$.

  This could be viewed as an ε-*regularity* type statement: weak convergence implies strong convergence, provided certain parameter is small.

Strong convergence of the maximizing sequence

- Good and bad points. Bad points have arbitrary small neighborhoods Ω with $\lambda_1^D(\Omega, V_{m,k})$ not large enough.

  There are at most k bad points. As it turns out, these are precisely the points admitting concentration of measure in arbitrary small neighborhoods (bubbles).

- **Proposition** Given a good point p, there exists a neighborhood $\Omega \ni p$ such that $\phi_{m,k} \to \phi_k$ in $H^1(\Omega)$.

  This could be viewed as an ε-*regularity* type statement: weak convergence implies strong convergence, provided certain parameter is small. Small parameter: $1/\lambda_1^D(\Omega, V_{m,k})$.

Behaviour away from bad points

Corollary In a neighborhood of a good point, $\phi_{m,k} \rightrightarrows \phi_k$ except on a set of arbitrary small capacity.

Behaviour away from bad points

Corollary In a neighborhood of a good point, $\phi_{m,k} \rightrightarrows \phi_k$ except on a set of arbitrary small capacity.

Combining this with the results (1) and (4) of Part I:

$$\Delta\phi_{m,k} = \Lambda_k^m \, V_{m,k} \, \phi_{m,k}, \;\; |\phi_k| = 1, \; dv_g - a.e.$$

Behaviour away from bad points

Corollary In a neighborhood of a good point, $\phi_{m,k} \rightrightarrows \phi_k$ except on a set of arbitrary small capacity.

Combining this with the results (1) and (4) of Part I:

$$\Delta\phi_{m,k} = \Lambda_k^m \, V_{m,k} \, \phi_{m,k}, \;\; |\phi_k| = 1, \; dv_g - a.e.$$

we obtain that on the set $G \subset M$ of good points

$$d\mu_k = \frac{|\nabla\phi_k|^2}{\Lambda_k} \, dv_g$$

Behaviour away from bad points

Corollary In a neighborhood of a good point, $\phi_{m,k} \rightrightarrows \phi_k$ except on a set of arbitrary small capacity.

Combining this with the results (1) and (4) of Part I:

$$\Delta \phi_{m,k} = \Lambda_k^m \, V_{m,k} \, \phi_{m,k}, \ \ |\phi_k| = 1, \ dv_g - a.e.$$

we obtain that on the set $G \subset M$ of good points

$$d\mu_k = \frac{|\nabla \phi_k|^2}{\Lambda_k} \, dv_g$$

and ϕ_k is a weak solution of

$$\Delta \phi_k = \frac{|\nabla \phi_k|^2}{\Lambda_k} \phi_k.$$

Decomposition of the limiting measure

This means that ϕ_k is a weakly harmonic map to $\mathbb{S}^{d-1}$.

Decomposition of the limiting measure

This means that ϕ_k is a weakly harmonic map to $\mathbb{S}^{d-1}$.

Regularity results imply that ϕ_k extends to a smooth harmonic map $M \to \mathbb{S}^{d-1}$.

Decomposition of the limiting measure

This means that ϕ_k is a weakly harmonic map to $\mathbb{S}^{d-1}$.

Regularity results imply that ϕ_k extends to a smooth harmonic map $M \to \mathbb{S}^{d-1}$.

Theorem There exist at most k points $p_1, \ldots, p_l$, $l \leqslant k$ and a harmonic map $\phi_k \colon M \to \mathbb{S}^{d-1}$ such that

$$d\mu_k = \frac{|\nabla \phi_k|^2}{\Lambda_k} dv_g + \sum_{i=1}^{l} w_i \delta_{p_i},$$

where $w_i \geq 0$.

Decomposition of the limiting measure

This means that ϕ_k is a weakly harmonic map to $\mathbb{S}^{d-1}$.

Regularity results imply that ϕ_k extends to a smooth harmonic map $M \to \mathbb{S}^{d-1}$.

Theorem There exist at most k points $p_1, \ldots, p_l$, $l \leqslant k$ and a harmonic map $\phi_k \colon M \to \mathbb{S}^{d-1}$ such that

$$d\mu_k = \frac{|\nabla \phi_k|^2}{\Lambda_k} dv_g + \sum_{i=1}^{l} w_i \delta_{p_i},$$

where $w_i \geq 0$.

The points where $|\nabla \phi_k| = 0$ correspond to the conical singularities of the limiting metric.

Dealing with the bubbles: first eigenvalue

Theorem (Petrides, 2014) $\Lambda_1(M, C) > 8\pi$ on any conformal class on any surface $M \neq \mathbb{S}^2$.

Dealing with the bubbles: first eigenvalue

Theorem (Petrides, 2014) $\Lambda_1(M, C) > 8\pi$ on any conformal class on any surface $M \neq \mathbb{S}^2$.

Proposition (Nadirashvili, 1996; Girouard, 2008; Kokarev, 2014) If a sequence of measures μ_n converges to a Dirac measure, then $\limsup \lambda_1(\mu_n) \leq 8\pi$.

Dealing with the bubbles: first eigenvalue

Theorem (Petrides, 2014) $\Lambda_1(M, C) > 8\pi$ on any conformal class on any surface $M \neq \mathbb{S}^2$.

Proposition (Nadirashvili, 1996; Girouard, 2008; Kokarev, 2014) If a sequence of measures μ_n converges to a Dirac measure, then $\limsup \lambda_1(\mu_n) \leq 8\pi$.

Lemma (Kokarev, 2014) If μ is a discontinuous Radon measure which is not a Dirac measure, then $\lambda_1(\mu) = 0$.

Dealing with the bubbles: first eigenvalue

Theorem (Petrides, 2014) $\Lambda_1(M, C) > 8\pi$ on any conformal class on any surface $M \neq \mathbb{S}^2$.

Proposition (Nadirashvili, 1996; Girouard, 2008; Kokarev, 2014) If a sequence of measures μ_n converges to a Dirac measure, then $\limsup \lambda_1(\mu_n) \leq 8\pi$.

Lemma (Kokarev, 2014) If μ is a discontinuous Radon measure which is not a Dirac measure, then $\lambda_1(\mu) = 0$.

Putting this together implies that the limiting measure has no bubbles unless we are on a sphere, where we know the answer by Hersch's theorem.

Dealing with the bubbles: higher eigenvalues

- Rescaling of the metric near a bubble.

Dealing with the bubbles: higher eigenvalues

- Rescaling of the metric near a bubble. Bubble tree construction (cf. Parker, 1996).

Dealing with the bubbles: higher eigenvalues

- Rescaling of the metric near a bubble. Bubble tree construction (cf. Parker, 1996).

- After an appropriate rescaling, each bubble can be viewed as a sphere, on which we can repeat the previous construction and choose the maximizing sequence of potentials.

Dealing with the bubbles: higher eigenvalues

- Rescaling of the metric near a bubble. Bubble tree construction (cf. Parker, 1996).
- After an appropriate rescaling, each bubble can be viewed as a sphere, on which we can repeat the previous construction and choose the maximizing sequence of potentials.
- If secondary bubbles appear, we apply the process inductively.

Dealing with the bubbles: higher eigenvalues

- Rescaling of the metric near a bubble. Bubble tree construction (cf. Parker, 1996).
- After an appropriate rescaling, each bubble can be viewed as a sphere, on which we can repeat the previous construction and choose the maximizing sequence of potentials.
- If secondary bubbles appear, we apply the process inductively. Since the sequence is maximizing, small bubbles can be ignored, and therefore the process will eventually stop.

Some open questions

- Do smooth metrics with conical singularities ever arise as global maximizers for higher eigenvalues on surfaces?

Some open questions

- Do smooth metrics with conical singularities ever arise as global maximizers for higher eigenvalues on surfaces?

- Which singularities may conformally maximal metrics for the first eigenvalue have in dimensions ≥ 3?

Some open questions

- Do smooth metrics with conical singularities ever arise as global maximizers for higher eigenvalues on surfaces?
- Which singularities may conformally maximal metrics for the first eigenvalue have in dimensions ≥ 3?
- Is there an analogue of the existence vs. bubbling result for conformally maximal metrics for higher eigenvalues in dimensions ≥ 3?

Thank you for your attention!