Asymptotic behaviour of eigenvalues in glueing constructions and new minimal surfaces from shape optimization

(jt. w/ A. Siffert, jt. w/ R. Petrides)

$\uparrow$ closed case $\qquad$ $\uparrow$ $\partial\Sigma \neq \emptyset$

Two problems

1.) Σ closed, consider the functional

$$\underset{\text{metric}}{g} \longmapsto \underset{\text{first eigenvalue of } \Delta}{\lambda_1(\Sigma, g)}\, \text{area}(\Sigma, g)$$

Try to maximize this functional.

2.) if Σ compact with $\partial\Sigma \neq \emptyset$, then consider the same type of problem

$$g \longmapsto \sigma_1(\Sigma, g)\, \text{length}(\partial\Sigma, g)$$

σ_1: first Steklov eigenvalue, i.e. corresponding to Dirichlet-to-Neumann operator

$Tu = \partial_\nu \hat{u}$, $u \in C^\infty(\partial\Sigma)$,

$\hat{u} \in C^\infty(\bar{\Sigma})$ harmonic extension

ν outward pointing normal.

Maximizing metrics correspond to minimal surfaces:

For 1.) minimal surfaces in round spheres $\mathbb{S}^n$ - Nadirashvili, El Soufi - Ilias

2.) free boundary minimal surfaces in Euclidean balls $\mathbb{B}^n$ - Fraser - Schoen

meaning that for 1.) maximizing metric is induced by a minimal immersion

$\Sigma \to \mathbb{S}^n$

and similarly for 2.)

Maximization problem 1.) is classical starting with Hersch's work '70 $\to \mathbb{S}^2$

Nadirashvili '96 : T^2

" "

El Soufi - Giacomini - Jazar '06 K^2 Klein bottle.
Cianci - Karpukhin - Medvedev

Nagatomi - Shoda : Σ_2 genus two surface

Conjectured by Polterovich-Nigam-Jakobson-Levitin-Nadirashvili.

a lot of recent work for corresponding problem on higher eigenvalues: Nadirashvili, Penskoi, Polterovich, Karpukhin, Petrides

For Steklov: Fraser-Schoen (also on higher eigenvalues) on surfaces of genus 0.

Theorem A (K.-Siffert '19) Let Σ be a closed surface then there is a metric g smooth away from at most finitely many conical singularities s.t.

$$\lambda_1(\Sigma,h)\,\mathrm{area}(\Sigma,h) \leq \lambda_1(\Sigma,g)\,\mathrm{area}(\Sigma,g)$$

for any smooth metric h.
In particular, we find (possibly branched) minimal immersion $\Sigma \to \mathbb{S}^N$ (that induces g).

Theorem B (K.-Petrides '20) Let Σ be a compact surface with $\partial\Sigma \neq \emptyset$. Then there is a smooth metric g on Σ s.t.

$$\sigma_1(\Sigma,h)\,\mathrm{length}(\partial\Sigma,h) \leq \sigma_1(\Sigma,g)\,\mathrm{length}(\partial\Sigma,g)$$

for any smooth metric h.

Theorem C (K.-Petrides '20) Let Σ be a compact surface with $\partial\Sigma \neq \emptyset$, then there is a (possibly branched) free boundary minimal immersion $\Sigma \to \mathbb{B}^N$.

First such result without any restriction on topology of Σ.

Many constructions before:

Fraser - Schoen genus 0 case (Steklov)

Folha - Pacard - Zolotareva (perturbation/gluing/desingularization)

Kapouleas - Li

Ketover

Carlotto - Franz - Griado (min-max)

Strategy to prove Theorem A + Theorem B

Consider Thm A (closed case)

Petrides, Nadirashvili - Sire solving maximization problem in a given conformal class.

Allows you to take a maximizing sequence of the following type

$$g_k = \frac{|\nabla \Phi_k|}{\lambda_1(\Sigma, g_k)}$$

where

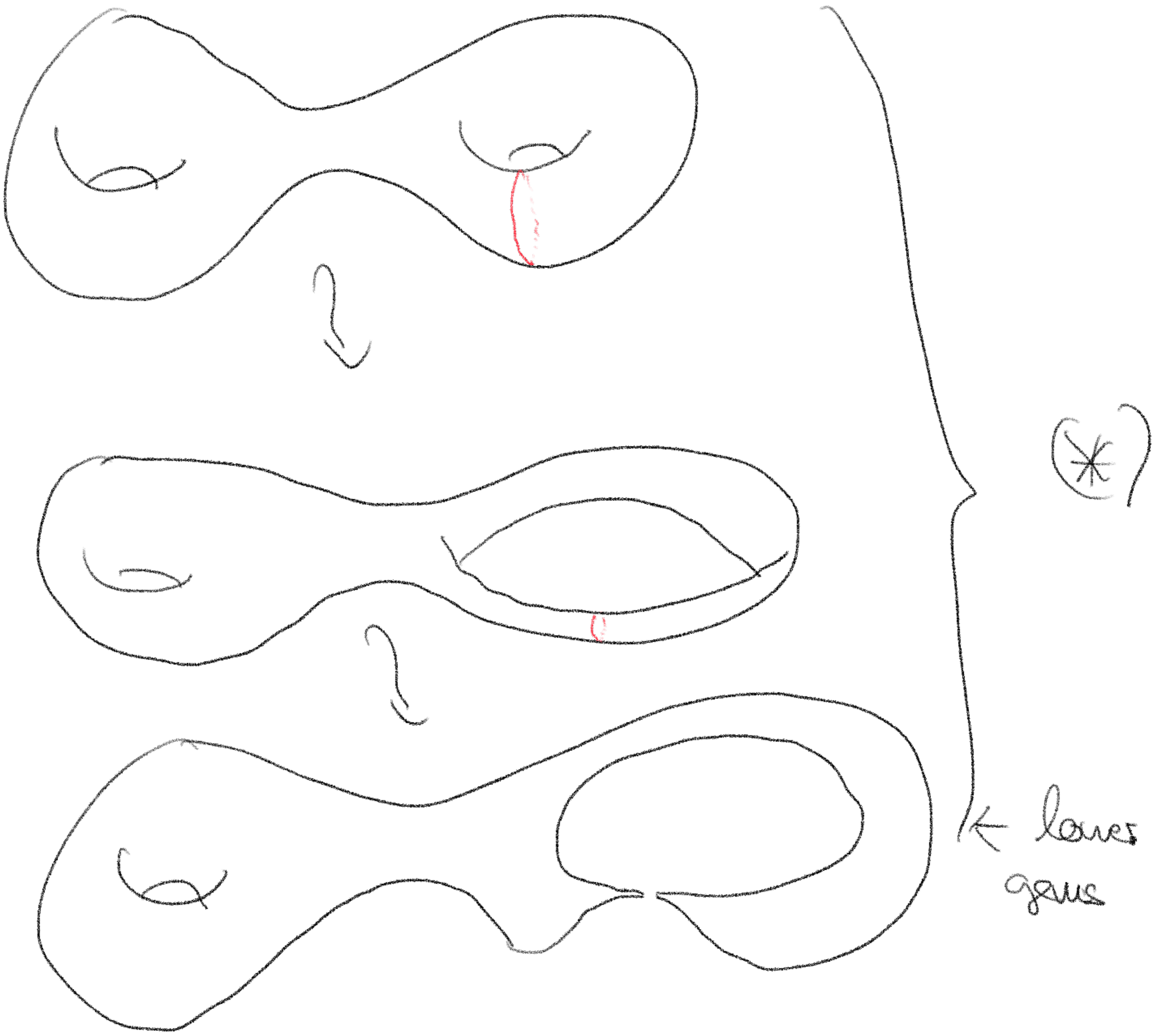
$$\Phi_k : (\Sigma, h_k) \to \mathbb{S}^N \text{ harmonic}$$

h_k: hyperbolic metric conformal to g_k.

Kokarev $\Rightarrow$ Φ_k can not develop bubbles.

$\rightsquigarrow$ compactness in conformal direction.

Only degenerations of η_k are of the type



$$\Lambda_1(\gamma) = \sup_{g} \lambda_1(\Sigma_\gamma, g)\, \mathrm{area}(\Sigma_\gamma, g)$$

closed of genus γ

If we have (*) then one can show

$$\boxed{\Lambda_1(\gamma) \leq \Lambda_1(\gamma - 1).}$$

show that this is impossible.

For Steklov we need to show

$\sigma_1(\gamma, k) > \sigma_1(\gamma-1, k+1)$

$\sigma_1(\gamma, k) > \sigma_1(\gamma, k-1)$

$\uparrow$

sup on surface of genus γ with k boundary components / Romain Petrides

Theorem A' (M.-Siffert '19) Given any closed surface (Σ, g), let Σ' be obtained from Σ by attaching a handle or a Möbius band. Then there is a metric g' on Σ' s.t.

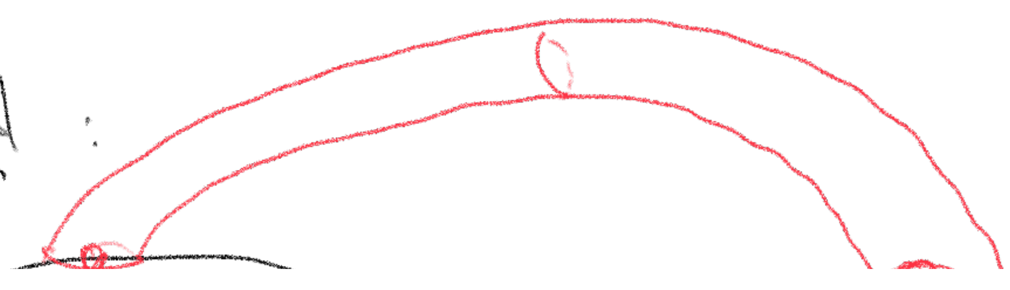
$$\lambda_1(\Sigma, g)\,\text{area}(\Sigma, g) < \lambda_1(\Sigma', g')\,\text{area}(\Sigma', g')$$

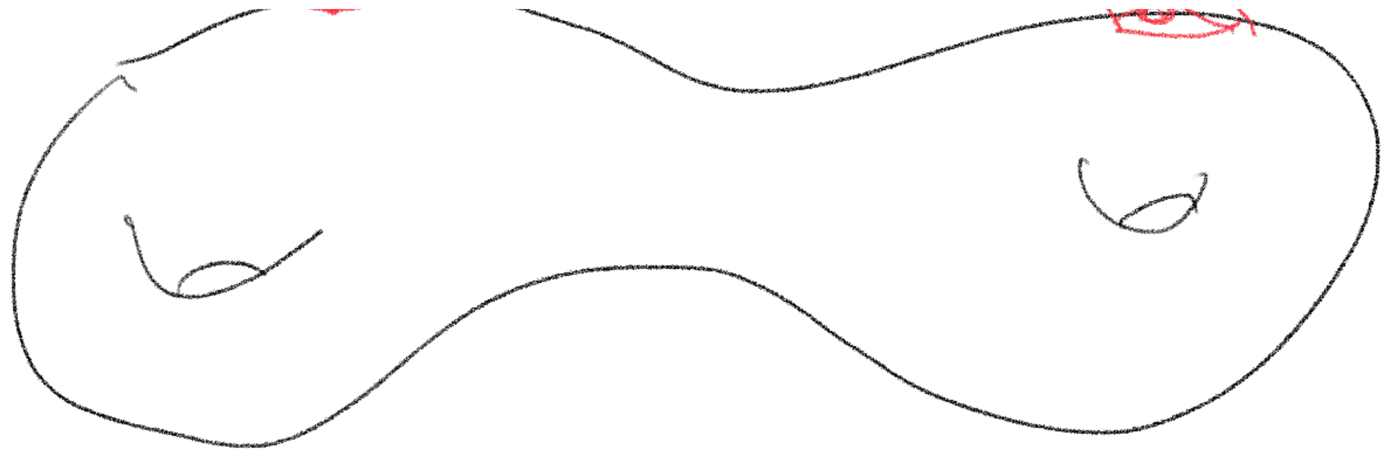
Then Thm A follows from Petrides work combined with induction.

Theorem B' (M.-Petrides '20) Given any compact surface (Σ, g) with $\partial\Sigma \neq \emptyset$ let Σ' be obtained from Σ by attaching a rectangle along two opposite sides of the boundary. Then there is g' on Σ' s.t.

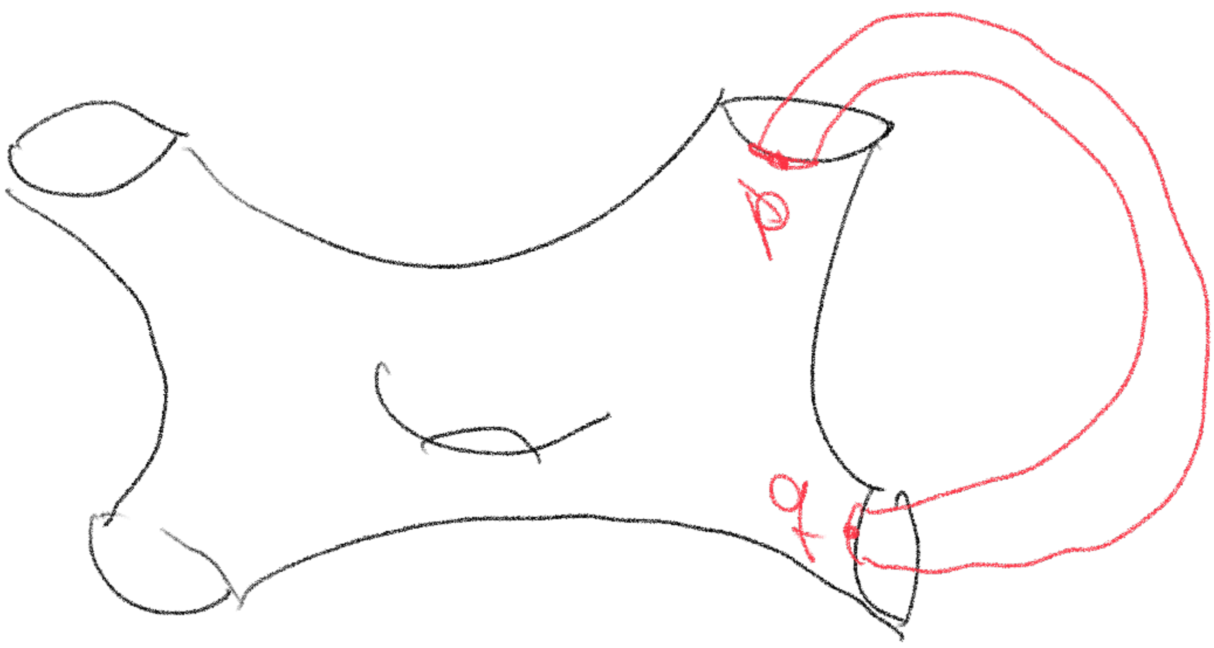
$$\sigma_1(\Sigma, g)\,\text{length}(\partial\Sigma, g) < \sigma_1(\Sigma', g')\,\text{length}(\partial\Sigma', g')$$

Thm A :





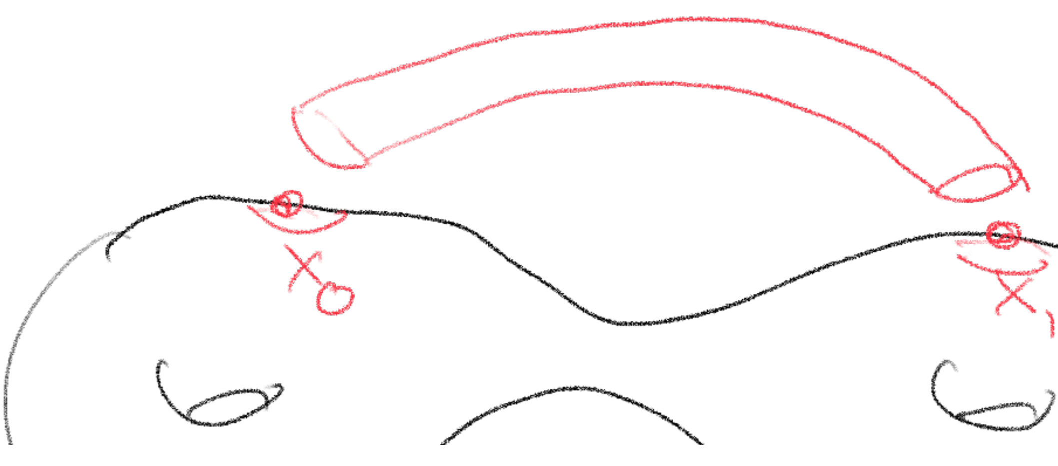
Thm B



Can choose p, q in the same or in different boundary components, can also reverse orientation in glueing.

Recent work related to Thm B' by Girouard–Lagacé, Karpukhin–Stern get some information on asymptotic behavior of $\sigma_1(\gamma, k)$.

How not to prove Thm A' (jt. w/ A. Siffert)



$G_{\varepsilon,h} = S^1_\varepsilon \times [0,h]$ attached along $\partial B_\varepsilon(x_i)$.

Σ

$\rightsquigarrow \Sigma_{\varepsilon,h}$.

$\text{area}(\Sigma_{\varepsilon,h}) \sim \text{area}(\Sigma) + 2\pi h \varepsilon + O(\varepsilon^2)$

<u>Question</u> Asymptotics for $\lambda_1(\Sigma_{\varepsilon,h})$. (and why?)

<u>Case 1</u>

$$\phi(x_0) + \phi(x_1) = 0$$

for any first eigenfunction on Σ,

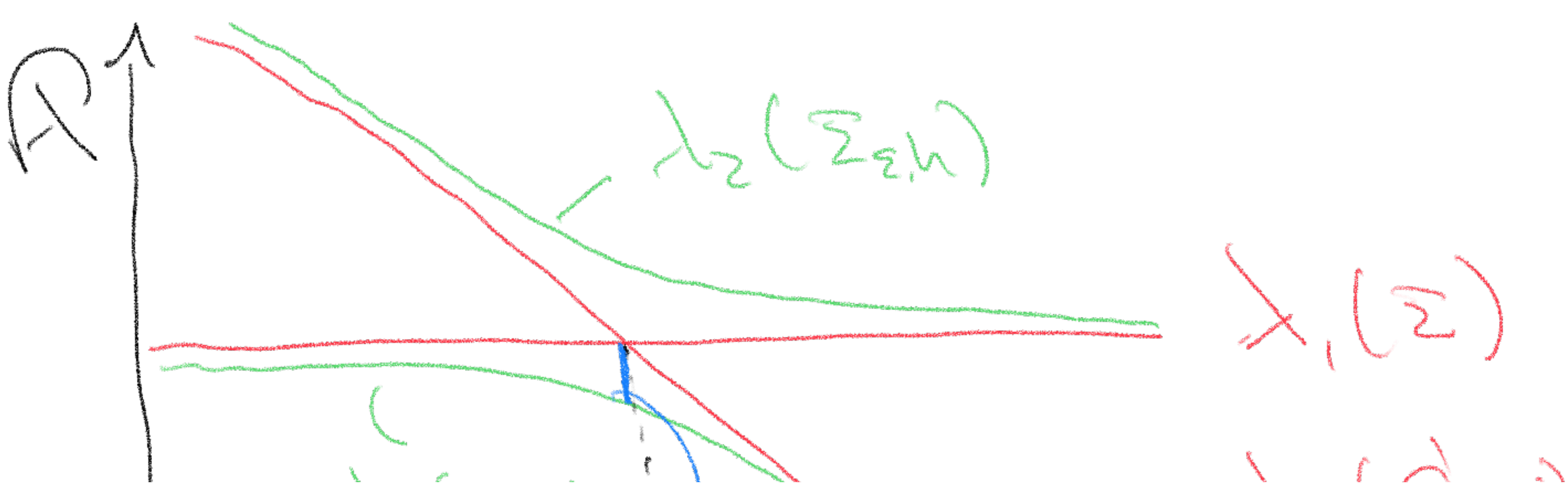
$\longrightarrow$ can use $\Sigma_{\varepsilon,h}$ to obtain Thm 1'.

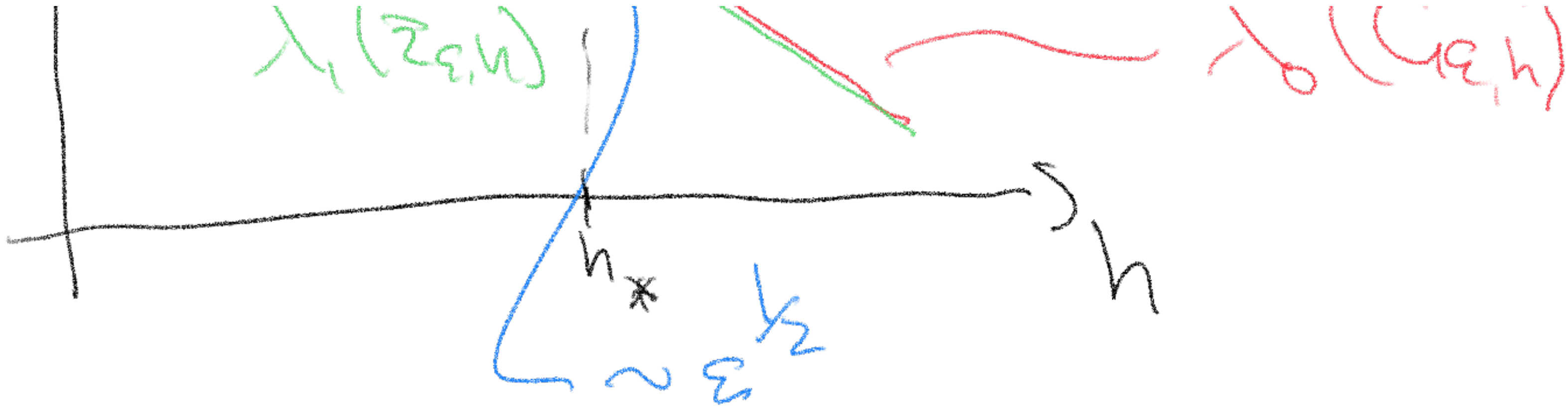
<u>Case 2</u> there is ϕ_0 a $\lambda_1(\Sigma)$-eigenfct s.t.

$$\phi_0(x_0) + \phi_0(x_1) \neq 0.$$

Assume for simplicity $\text{mult}(\lambda_1(\Sigma)) = 1$

<u>Theorem</u> (M.-Siffert '18) In case 2 we have that





$\Downarrow$

$$\lambda_1(\Sigma_{\varepsilon,h})\,\text{area}(\Sigma_{\varepsilon,h}) < \lambda_1(\Sigma)\,\text{area}(\Sigma)$$

- we expect that we can obtain Thm A' only with parameters h near h_*
  ⇝ where we can observe change in topology on a spectral level
- scale $\varepsilon^{1/2}$ originates from

$$\boxed{\int_{\partial C_{\varepsilon,h}} \partial_\nu \psi_{\varepsilon,h}\, d\mathcal{H}^1 \sim \varepsilon^{1/2}} \qquad (**)$$

where $\psi_{\varepsilon,h}$ is a normalized $\lambda_0(C_{\varepsilon,h})$-eigenfunction

In order to have a better scale in (**) we need to attach a degenerating part developing non-discrete spectrum

<u>closed case</u>: attach truncated negatively curved cusp

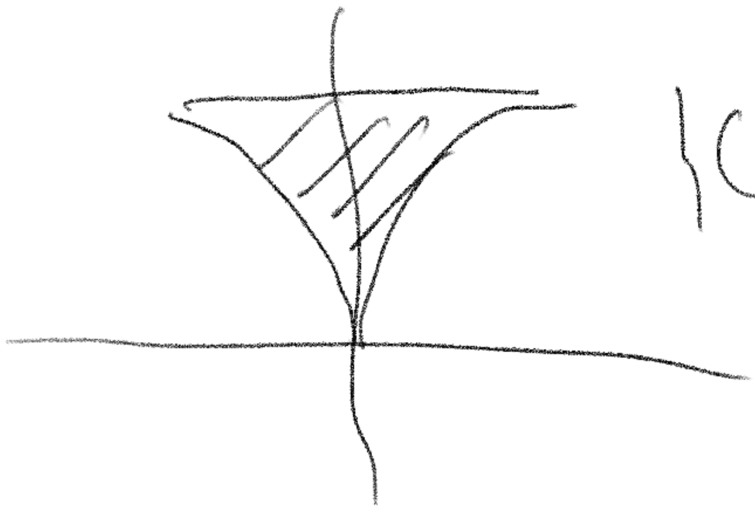
curvature $\to K \longleftrightarrow h$

Note:

$$\{(x,y) \in \mathbb{H}^2 : y \geq 1\} \big/ x \mapsto x+1$$

has continuous spectrum $[\frac{1}{4}, \infty)$

Steklov case attach a cuspidal domain
as truncated



$$\left\{(x,y) \in \mathbb{R}^2 : y \geq 0,\ -\frac{y^2}{2} \leq x \leq \frac{y^2}{2}\right\}$$

Nazarov-Taskinen: non-discrete
spectrum.

How we prove Thm B'

Step 1 Good upper bounds on
eigenvalues (test function est.)

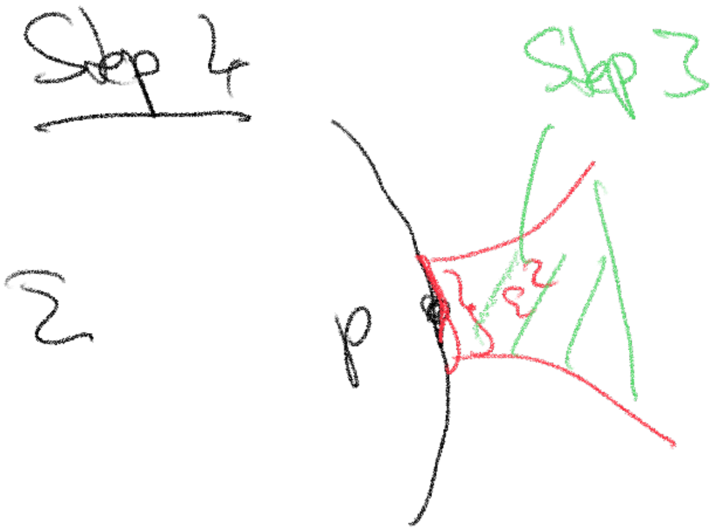
Step 2 pointwise estimates on eigenfcts
in attaching region.

Step 3 using Step 1 + Step 2 to obtain

energy bound control on boundary values

an asymptotic expansion of eigenvalue and eigenfunction in cuspidal domain in a good choice of coordinates

Step 4



→ improve pointwise estimates from Step 4.

Step 5 build a test function out of first few eigenfcts,